Proceedings of the Edinburgh Mathematical Society (2012) **55**, 167–179
DOI:10.1017/S0013091510001094

ALGEBRAIC INTEGERS AS SPECIAL VALUES OF MODULAR UNITS

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(Received 23 August 2010)

Abstract Let $\varphi(\tau) = \eta(\frac{1}{2}(\tau+1))^2/\sqrt{2\pi}\exp\{\frac{1}{4}\pi\mathrm{i}\}\eta(\tau+1)$, where $\eta(\tau)$ is the Dedekind eta function. We show that if τ_0 is an imaginary quadratic argument and m is an odd integer, then $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$. This is a generalization of a result of Berndt, Chan and Zhang. On the other hand, when K is an imaginary quadratic field and θ_K is an element of K with $\mathrm{Im}(\theta_K) > 0$ which generates the ring of integers of K over $\mathbb{Z}$, we find a sufficient condition on m which ensures that $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$ is a unit.

Keywords: Dedekind eta function; modular functions; automorphic functions

2010 *Mathematics subject classification:* Primary 11F20
Secondary 11F03

1. Introduction

The *Dedekind eta function* $\eta(\tau)$ is defined to be the infinite product

$$\eta(\tau) = \sqrt{2\pi}\mathrm{e}^{\pi\mathrm{i}/4}q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \tau \in \mathfrak{H}, \tag{1.1}$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ with $\mathrm{i} = \sqrt{-1}$ and $\mathfrak{H} = \{\tau \in \mathbb{C}\colon \mathrm{Im}(\tau) > 0\}$. Define a function

$$\varphi(\tau) = \frac{1}{\sqrt{2\pi}\mathrm{e}^{\pi\mathrm{i}/4}}\frac{\eta((\tau+1)/2)^2}{\eta(\tau+1)} = \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n), \quad \tau \in \mathfrak{H}, \tag{1.2}$$

which is identical to Jacobi's $\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2}$, by Jacobi's triple product identity [**1**, p. 36]. Motivated by Ramanujan's evaluation of $\varphi(m\mathrm{i})/\varphi(\mathrm{i})$ for some positive integers m [**10**] which are algebraic numbers, Berndt *et al*. proved the following theorem.

Theorem 1.1 (Berndt *et al*. [2, Theorem 4.4]). *Let m and n be positive integers. If m is odd, then $\sqrt{2m}\varphi(mn\mathrm{i})/\varphi(n\mathrm{i})$ is an algebraic integer dividing $2\sqrt{m}$, while if m is even, then $2\sqrt{m}\varphi(mn\mathrm{i})/\varphi(n\mathrm{i})$ is an algebraic integer dividing $4\sqrt{m}$.*

In this paper we shall first revisit the theorem and improve it when m is odd, as follows.

Theorem 1.2. *Let m be a positive integer and let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. Then $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $4\sqrt{m}$. In particular, if m is odd, then $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$.*

For $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, the *Siegel function* $g_{(r_1,r_2)}(\tau)$ is defined by

$$g_{(r_1,r_2)}(\tau) = -q^{\boldsymbol{B}_2(r_1)/2} \mathrm{e}^{\pi \mathrm{i} r_2(r_1-1)}(1-q_z) \prod_{n=1}^{\infty}(1-q^n q_z)(1-q^n q_z^{-1}), \quad \tau \in \mathfrak{H}, \tag{1.3}$$

where $\boldsymbol{B}_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial and $q_z = \mathrm{e}^{2\pi \mathrm{i} z}$ with $z = r_1\tau + r_2$. We shall express the function $\varphi(m\tau)/\varphi(\tau)$ as a product of certain eta-quotient and Siegel functions (Lemma 2.6 (i)). Also, we shall prove Theorem 1.2 in § 3 by using integrality of Siegel functions over $\mathbb{Z}[j(\tau)]$ [**6**, § 3], where

$$j(\tau) = \left(\frac{\eta(\tau)^{24} + 2^8\eta(2\tau)^{24}}{\eta(\tau)^{16}\eta(2\tau)^8}\right)^3 = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \cdots$$

is the well-known *modular j-function* [**3**, Theorem 12.17].

On the other hand, let K be an imaginary quadratic field with discriminant d_K, and define

$$\theta_K = \begin{cases} \dfrac{\sqrt{d_K}}{2} & \text{for } d_K \equiv 0 \pmod 4, \\ \dfrac{-1+\sqrt{d_K}}{2} & \text{for } d_K \equiv 1 \pmod 4, \end{cases} \tag{1.4}$$

which generates the ring of integers of K over $\mathbb{Z}$. Ramachandra showed in [**9**, § 6] that if N ($N \geqslant 2$) is an integer with more than one prime ideal factor in K, then $g_{(0,1/N)}(\theta_K)^{12N}$ is a unit. This fact, together with Shimura's Reciprocity Law (Proposition 4.6), enables us to prove the following theorem in § 4.

Theorem 1.3. *If m ($m \geqslant 3$) is an odd integer whose prime factors split in K, then $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$ is a unit.*

2. Arithmetic properties of Siegel functions

In this section we shall examine some arithmetic properties of Siegel functions. For the classical theory of modular functions, we refer the reader to [**8**, **11**].

For each positive integer N, let $\zeta_N = \mathrm{e}^{2\pi\mathrm{i}/N}$ and let $\mathcal{F}_N$ be the field of meromorphic modular functions of level N whose Fourier coefficients belong to the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$.

Proposition 2.1. *For each positive integer N, $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$ whose Galois group is isomorphic to*

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} = G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\},$$

where

$$G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

Here, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in G_N$$

acts on $\sum_{n=-\infty}^{\infty} c_n q^{n/N} \in \mathcal{F}_N$ *by*

$$\sum_{n=-\infty}^{\infty} c_n q^{n/N} \mapsto \sum_{n=-\infty}^{\infty} c_n^{\sigma_d} q^{n/N},$$

where σ_d *is the automorphism of* $\mathbb{Q}(\zeta_N)$ *induced by* $\zeta_N \mapsto \zeta_N^d$*. Also, for an element* $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ *let* $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ *be a preimage of* γ *via the natural surjection* $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$*. Then* γ *acts on* $h \in \mathcal{F}_N$ *by composition*

$$h \mapsto h \circ \gamma'$$

as a fractional linear transformation.

Proof. See [**8**, Chapter 6, Theorem 3]. □

Proposition 2.2. *Let* $(r_1, r_2) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ *for some integer* $N \geqslant 2$*.*

(i) $g_{(r_1,r_2)}(\tau)$ *is integral over* $\mathbb{Z}[j(\tau)]$*. Namely,* $g_{(r_1,r_2)}(\tau)$ *is a zero of a monic polynomial whose coefficients are in* $\mathbb{Z}[j(\tau)]$*.*

(ii) *Suppose that* (r_1, r_2) *has the primitive denominator* N *(that is,* N *is the smallest positive integer such that* $(Nr_1, Nr_2) \in \mathbb{Z}^2$*). If* N *is composite (that is,* N *has at least two prime factors), then* $g_{(r_1,r_2)}(\tau)^{-1}$ *is also integral over* $\mathbb{Z}[j(\tau)]$*.*

(iii) $g_{(r_1,r_2)}(\tau)$ *is holomorphic and has no zeros and poles on* $\mathfrak{H}$*. Furthermore,* $g_{(r_1,r_2)}(\tau)$ *(respectively,* $g_{(r_1,r_2)}(\tau)^{12N/\gcd(6,N)}$*) belongs to* $\mathcal{F}_{12N^2}$ *(respectively,* $\mathcal{F}_N$*).*

Proof.

(i) See [**6**, § 3].

(ii) See [**7**, Chapter 2, Theorems 2.2 (i)].

(iii) See [**7**, Chapter 2, Theorem 1.2, and Chapter 3, Theorem 5.2].

□

Remark 2.3. Let $g(\tau)$ be an element of $\mathcal{F}_N$ for some positive integer N. If both $g(\tau)$ and $g(\tau)^{-1}$ are integral over $\mathbb{Q}[j(\tau)]$ (respectively, $\mathbb{Z}[j(\tau)]$), then $g(\tau)$ is called a *modular unit* (respectively, *modular unit over* $\mathbb{Z}$) of level N. As is well known, $g(\tau)$ is a modular unit if and only if it has no zeros or poles on $\mathfrak{H}$ (see [**7**, Chapter 2, § 2] or [**6**, § 2]). Hence, $g_{(r_1,r_2)}(\tau)$ is a modular unit for any $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, by (iii). Moreover, if (r_1, r_2) has a composite primitive denominator, then $g_{(r_1,r_2)}(\tau)$ is a modular unit over $\mathbb{Z}$, by (ii).

We recall the necessary transformation formulae of Siegel functions.

Proposition 2.4. *Let $r = (r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$.*

(i) *We have*

$$g_{-r}(\tau) = g_{(-r_1,-r_2)}(\tau) = -g_r(\tau).$$

(ii) *For*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

we get

$$g_r(\tau) \circ S = \zeta_{12}^9 g_{rS}(\tau) = \zeta_{12}^9 g_{(r_2,-r_1)}(\tau),$$
$$g_r(\tau) \circ T = \zeta_{12} g_{rT}(\tau) = \zeta_{12} g_{(r_1,r_1+r_2)}(\tau).$$

Hence, we obtain that, for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$g_r(\tau) \circ \gamma = \varepsilon g_{r\gamma}(\tau)$$

with ε a 12th root of unity (depending on γ).

(iii) *For $s = (s_1, s_2) \in \mathbb{Z}^2$ we have*

$$g_{r+s}(\tau) = g_{(r_1+s_1,r_2+s_2)}(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} \mathrm{e}^{-\pi \mathrm{i}(s_1 r_2 - s_2 r_1)} g_r(\tau).$$

(iv) *Let $r \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ for some integer $N \geqslant 2$. Each element*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{in } \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$$

acts on $g_r(\tau)^{12N/\gcd(6,N)}$ by

$$(g_r(\tau)^{12N/\gcd(6,N)})^\alpha = g_{r\alpha}(\tau)^{12N/\gcd(6,N)} = g_{(r_1 a + r_2 c, r_1 b + r_2 d)}(\tau)^{12N/\gcd(6,N)}.$$

Proof. (i)–(iii) See [**6**, Proposition 2.4].

(iv) See [**7**, Chapter 2, Proposition 1.3]. □

Remark 2.5. The expression $r\alpha$ in (iv) is well defined by (i) and (iii).

Lemma 2.6.

(i) *We can express $\varphi(\tau)$ as*

$$\varphi(\tau) = -\frac{1}{\sqrt{2\pi}} \eta(\tau) g_{(1/2,1/2)}(\tau).$$

(ii) *We get*

$$g_{(0,1/2)}(\tau) g_{(1/2,0)}(\tau) g_{(1/2,1/2)}(\tau) = 2\mathrm{e}^{\pi \mathrm{i}/4}.$$

(iii) *If m ($m \geqslant 3$) is an odd integer, then we have the relation*

$$\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} = (-1)^{(m-1)/2} \prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau).$$

Proof. (i) By the definition (1.3) we have

$$\begin{aligned} g_{(1/2,1/2)}(\tau) &= -q^{\boldsymbol{B}_2(1/2)/2} \mathrm{e}^{-\pi\mathrm{i}/4}(1+q^{1/2}) \prod_{n=1}^{\infty}(1+q^{n+1/2})(1+q^{n-1/2}) \\ &= -\mathrm{e}^{-\pi\mathrm{i}/4} q^{-1/24} \prod_{n=1}^{\infty}(1+q^{n-1/2})^2. \end{aligned}$$

One can then obtain the assertion by the definition (1.1) of $\eta(\tau)$ and the infinite product expansion (1.2) of $\varphi(\tau)$.

(ii) It follows from the definition (1.3) that

$$\begin{aligned} g_{(0,1/2)}(\tau) g_{(1/2,0)}(\tau) g_{(1/2,1/2)}(\tau) &= -2\mathrm{e}^{-3\pi\mathrm{i}/4} \prod_{n=1}^{\infty}(1+q^n)^2(1-q^{n-1/2})^2(1+q^{n-1/2})^2 \\ &= 2\mathrm{e}^{\pi\mathrm{i}/4} \prod_{n=1}^{\infty}(1+q^n)^2(1-q^{2n-1})^2 \\ &= 2\mathrm{e}^{\pi\mathrm{i}/4} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)^2} \cdot \frac{(1-q^n)^2}{(1-q^{2n})^2} \\ &= 2\mathrm{e}^{\pi\mathrm{i}/4}. \end{aligned}$$

(iii) By the definition (1.3) we obtain

$$\begin{aligned} \frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} &= \frac{-q^{m\boldsymbol{B}_2(1/2)/2}\mathrm{e}^{-\pi\mathrm{i}/4}(1+q^{m/2})\prod_{n=1}^{\infty}(1+q^{mn+m/2})(1+q^{mn-m/2})}{-q^{\boldsymbol{B}_2(1/2)/2}\mathrm{e}^{-\pi\mathrm{i}/4}(1+q^{1/2})\prod_{n=1}^{\infty}(1+q^{n+1/2})(1+q^{n-1/2})} \\ &= q^{(1-m)/24} \prod_{n=1}^{\infty} \left(\frac{1+q^{m(n-1/2)}}{1+q^{n-1/2}}\right)^2 \end{aligned}$$

and

$$\begin{aligned} &\prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau) \\ &\quad = \prod_{k=1}^{m-1} \left(-q^{\boldsymbol{B}_2(1/2)/2} \exp\left\{\pi\mathrm{i}\left(\frac{1}{2}+\frac{k}{m}\right)\left(-\frac{1}{2}\right)\right\} \right. \\ &\qquad\qquad \left. \times (1+q^{1/2}\zeta_m^k) \prod_{n=1}^{\infty}(1+q^{n+1/2}\zeta_m^k)(1+q^{n-1/2}\zeta_m^{-k}) \right) \end{aligned}$$

$$= (-1)^{m-1} e^{\pi i(1-m)/2} q^{(1-m)/24} \prod_{k=1}^{m-1} \prod_{n=1}^{\infty} (1 + q^{n-1/2}\zeta_m^k)(1 + q^{n-1/2}\zeta_m^{-k})$$

$$= (-1)^{(1-m)/2} q^{(1-m)/24} \prod_{n=1}^{\infty} \prod_{k=1}^{m-1} (1 + q^{n-1/2}\zeta_m^k)^2 \quad \text{because } m \text{ is odd}$$

$$= (-1)^{(1-m)/2} q^{(1-m)/24} \prod_{n=1}^{\infty} \left(\frac{1 + q^{m(n-1/2)}}{1 + q^{n-1/2}} \right)^2$$

by the identity

$$\frac{1 + X^m}{1 + X} = \frac{1 - (-X)^m}{1 - (-X)} = \prod_{k=1}^{m-1} (1 - (-X)\zeta_m^k).$$

This proves (iii). □

3. Proof of Theorem 1.2

Let

$$\Delta(\tau) = \eta(\tau)^{24} = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad \tau \in \mathfrak{H},$$

be the *modular discriminant function*.

Proposition 3.1. *Let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic.*

(i) *$j(\tau_0)$ is an algebraic integer.*

(ii) *Let a, b and d be integers with $ad > 0$ and $\gcd(a, b, d) = 1$. Then*

$$\frac{a^{12}\Delta((a\tau_0 + b)/d)}{\Delta(\tau_0)}$$

is an algebraic integer dividing $(ad)^{12}$.

Proof.

(i) See [**8**, Chapter 5, Theorem 4].

(ii) See [**8**, Chapter 12, Theorem 4] or [**4**].

□

Remark 3.2. Case (ii) is the most important special case of the prime factorizations of $\Delta(\alpha\tau_0)/\Delta(\tau_0)$ ($\alpha \in M_2^+(\mathbb{Z})$); this was proved by Hasse for all factorizations.

Proposition 3.3. *Let m be a positive integer and $\tau_0 \in \mathfrak{H}$ be imaginary quadratic.*

(i) *$\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$.*

(ii) *$2g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$ is an algebraic integer dividing 4. In particular, if m is odd, then $g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$ is a unit.*

Proof. (i) Applying Proposition 3.1 (ii) with $(a,b,d)=(m,0,1)$, we see that

$$m^{12}\frac{\Delta(m\tau_0)}{\Delta(\tau_0)}=\left(\sqrt{m}\frac{\eta(m\tau_0)}{\eta(\tau_0)}\right)^{24}$$

is an algebraic integer dividing m^{12}. We then get the assertion by taking the 24th root.

(ii) We obtain by Lemma 2.6 (ii) that

$$\begin{aligned}2\frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)}&=\mathrm{e}^{-\pi\mathrm{i}/4}g_{(0,1/2)}(\tau_0)g_{(1/2,0)}(\tau_0)g_{(1/2,1/2)}(\tau_0)\frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)}\\&=\mathrm{e}^{-\pi\mathrm{i}/4}(g_{(0,1/2)}(\tau_0)g_{(1/2,0)}(\tau_0))g_{(1/2,1/2)}(m\tau_0).\end{aligned}$$

By Propositions 2.2 (i) and 3.1 (i) we know that the values $g_{(0,1/2)}(\tau_0)g_{(1/2,0)}(\tau_0)$, $g_{(1/2,1/2)}(\tau_0)$, $g_{(0,1/2)}(m\tau_0)g_{(1/2,0)}(m\tau_0)$ and $g_{(1/2,1/2)}(m\tau_0)$ are algebraic integers. Moreover, since

$$\begin{aligned}(g_{(0,1/2)}(\tau_0)g_{(1/2,0)}(\tau_0))g_{(1/2,1/2)}(\tau_0)&=(g_{(0,1/2)}(m\tau_0)g_{(1/2,0)}(m\tau_0))g_{(1/2,1/2)}(m\tau_0)\\&=2\mathrm{e}^{\pi\mathrm{i}/4}\end{aligned}$$

by Lemma 2.6 (ii), both $g_{(0,1/2)}(\tau_0)g_{(1/2,0)}(\tau_0)$ and $g_{(1/2,1/2)}(m\tau_0)$ are algebraic integers dividing 2. Hence, the value $2g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$ is an algebraic integer dividing $2\cdot 2=4$.

Next, suppose that m ($m\geqslant 3$) is odd. Recall the relation

$$\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)}=(-1)^{(m-1)/2}\prod_{k=1}^{m-1}g_{(1/2,1/2+k/m)}(\tau)$$

given in Lemma 2.6 (iii). Since each vector

$$\left(\frac{1}{2},\frac{1}{2}+\frac{k}{m}\right)$$

has a composite primitive denominator, $g_{(1/2,1/2+k/m)}(\tau)$ is a modular unit over $\mathbb{Z}$ by Proposition 2.2 (ii); hence, so is $g_{(1/2,1/2)}(m\tau)/g_{(1/2,1/2)}(\tau)$. Therefore, the value $g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$ is a unit by Proposition 3.1 (i). □

Now we are ready to prove Theorem 1.2. Let m be a positive integer and let $\tau_0\in\mathfrak{H}$ be imaginary quadratic. By Lemma 2.6 (i) we have

$$2\sqrt{m}\frac{\varphi(m\tau_0)}{\varphi(\tau_0)}=\sqrt{m}\frac{\eta(m\tau_0)}{\eta(\tau_0)}\cdot 2\frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)}.$$

Thus, it follows from Proposition 3.3 (i) and (ii) that $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $4\sqrt{m}$. Likewise, if m is odd, then

$$\sqrt{m}\frac{\varphi(m\tau_0)}{\varphi(\tau_0)}=\sqrt{m}\frac{\eta(m\tau_0)}{\eta(\tau_0)}\cdot\frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)}\tag{3.1}$$

is an algebraic integer dividing $\sqrt{m}$. This completes the proof of Theorem 1.2.

On the other hand, when $\tau_0 = n\mathrm{i}$ we are able to improve Theorem 1.1 as a corollary.

Corollary 3.4. *Let m and n be positive integers. If m is odd, then $\sqrt{m}\varphi(mn\mathrm{i})/\varphi(n\mathrm{i})$ is an algebraic integer dividing $\sqrt{m}$, while if m is even, then $2\sqrt{m}\varphi(mn\mathrm{i})/\varphi(n\mathrm{i})$ is an algebraic integer dividing $4\sqrt{m}$.*

Remark 3.5. Berndt *et al.* [**2**] used only the argument of Proposition 3.1 (ii) in order to achieve Theorem 1.1.

4. Proof of Theorem 1.3

Lemma 4.1. *Let m ($m \geqslant 2$) be an integer. Then we have the following identities.*

(i)

$$\prod_{\substack{a,b\in\mathbb{Z},\\ 0\leqslant a,b<m,\ (a,b)\neq(0,0)}} g_{(a/m,b/m)}(\tau)^{12m} = m^{12m}.$$

(ii)

$$\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau) = \mathrm{i}^{m-1}\left(\sqrt{m}\frac{\eta(m\tau)}{\eta(\tau)}\right)^2.$$

Proof. (i) See [**7**, Example, p. 45].

(ii) We deduce from the definition (1.3) that

$$\begin{aligned}
&\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau)\\
&\qquad = \prod_{k=1}^{m-1}(-q^{\boldsymbol{B}_2(0)/2}\zeta_{2m}^{-k}(1-\zeta_m^k)\prod_{n=1}^{\infty}(1-q^n\zeta_m^k)(1-q^n\zeta_m^{-k}))\\
&\qquad = \mathrm{i}^{m-1}mq^{(m-1)/12}\prod_{n=1}^{\infty}\left(\frac{1-q^{mn}}{1-q^n}\right)^2\\
&\qquad\qquad \text{by the identity } \frac{1-X^m}{1-X} = 1+X+\cdots+X^{m-1} = \prod_{k=1}^{m-1}(1-X\zeta_m^k)\\
&\qquad = \mathrm{i}^{m-1}\left(\sqrt{m}\frac{\eta(m\tau)}{\eta(\tau)}\right)^2 \quad \text{by the definition (1.1).}
\end{aligned}$$

□

Remark 4.2. Let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. By Propositions 2.2 (i), 3.1 (i) and Lemma 4.1 (i) we see that $\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau_0)$ is an algebraic integer dividing m. It then follows from Lemma 4.1 (ii) that $\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$. This gives another proof of Proposition 3.3 (i) without using the usual argument of Hasse (namely, Proposition 3.1 (ii)).

From now on, we let K be an imaginary quadratic field and θ_K be as in (1.4). We denote by H_K and $K_{(N)}$ the Hilbert class field and the ray class field modulo N ($N \geqslant 1$) of K, respectively.

Proposition 4.3 (main theorem of complex multiplication).

$$K_{(N)} = K\mathcal{F}_N(\theta_K) = K(h(\theta_K)\colon h \in \mathcal{F}_N \text{ is defined and finite at } \theta_K).$$

Proof. See [**8**, Chapter 10, Corollary to Theorem 2] or [**11**, Chapter 6]. □

Proposition 4.4.

(i) *If N ($N \geqslant 2$) is an integer with more than one prime ideal factor in K, then $g_{(0,1/N)}(\theta_K)^{12N}$ is a unit in $K_{(N)}$.*

(ii) *If m ($m \geqslant 3$) is an odd integer, then $(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$ is an algebraic integer in $K_{(48m^2)}$.*

Proof. (i) See [**9**, § 6].

(ii) We see that

$$\begin{aligned}\left(\sqrt{m}\frac{\varphi(m\tau)}{\varphi(\tau)}\right)^2 &= \left(\sqrt{m}\frac{\eta(m\tau)}{\eta(\tau)}\right)^2\left(\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)}\right)^2 \quad \text{by Lemma 2.6 (i)}\\ &= (-1)^{(1-m)/2}\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau)g_{(1/2,1/2+k/m)}(\tau)^2\\ &\qquad \text{by Lemmas 4.1 (ii) and 2.6 (iii).}\end{aligned} \tag{4.1}$$

And $(\sqrt{m}\varphi(m\tau)/\varphi(\tau))^2$ belongs to $\mathcal{F}_{48m^2}$ by Proposition 2.2 (iii). Therefore,

$$(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$$

lies in $K_{(48m^2)}$ by Proposition 4.3, which is an algebraic integer by Theorem 1.2. □

Remark 4.5. In [**5**] Jung *et al.* showed that if K is an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, then the singular value $g_{(0,1/N)}(\theta_K)^{12N}$ in Proposition 4.4 (i) is in fact a primitive generator of $K_{(N)}$ over K, which is called a *Siegel–Ramachandra invariant* (see [**7**, Chapter 11, § 1] or [**9**]).

On the other hand, we have the following explicit description of Shimura's reciprocity law, due to Stevenhagen [**12**], which connects the class field theory with the theory of modular functions.

Proposition 4.6 (Shimura's Reciprocity Law). *Let* $\min(\theta_K, \mathbb{Q}) = X^2+BX+C \in \mathbb{Z}[X]$. *For every positive integer N the matrix group*

$$W_{K,N} = \left\{\begin{pmatrix} t-Bs & -Cs \\ s & t\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\colon t,s \in \mathbb{Z}/N\mathbb{Z}\right\}$$

gives rise to the surjection

$$\begin{aligned} W_{K,N} &\to \mathrm{Gal}(K_{(N)}/H_K), \\ \alpha &\mapsto (h(\theta_K) \mapsto h^{\alpha}(\theta_K)), \end{aligned}$$

where $h \in \mathcal{F}_N$ is defined and finite at θ_K. Its kernel is given by

$$\mathrm{Ker}_{K,N} = \begin{cases} \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} & \textit{if } K = \mathbb{Q}(\sqrt{-1}), \\ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} & \textit{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & \textit{otherwise.} \end{cases}$$

Proof. See [**12**, § 3]. □

Proposition 4.7. *If m $(m \geqslant 2)$ is an integer whose prime factors split in K, then $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$ is a unit.*

Proof. We get from Lemma 4.1 (ii) that

$$\left(\sqrt{m}\frac{\eta(m\theta_K)}{\eta(\theta_K)} \right)^{24m} = \prod_{k=1}^{m-1} g_{(0,k/m)}(\theta_K)^{12m}. \tag{4.2}$$

For each $1 \leqslant k \leqslant m-1$, let us write

$$\frac{k}{m} = \frac{a}{b} \quad \text{with relatively prime positive integers } a \text{ and } b.$$

Note that b has more than one prime ideal factor in K, by the assumption on m. Thus, $g_{(0,1/b)}(\theta_K)^{12b}$ is a unit in $K_{(b)}$, by Proposition 4.4 (i). On the other hand, since

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in W_{K,b}/\mathrm{Ker}_{K,b} \simeq \mathrm{Gal}(K_{(b)}/H_K),$$

we deduce that

$$\begin{aligned} (g_{(0,1/b)}(\theta_K)^{12b})^{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)} &= (g_{(0,1/b)}(\tau)^{12b})^{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)}(\theta_K) && \text{by Proposition 4.6} \\ &= \left(g_{(0,1/b)\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)}(\tau)^{12b} \right)(\theta_K) && \text{by Proposition 2.4 (iv)} \\ &= g_{(0,a/b)}(\theta_K)^{12b}, \end{aligned}$$

which is also a unit. Therefore, $\sqrt{m}\eta(m\theta_K)\eta(\theta_K)$ becomes a unit, by the relation (4.2). □

Now, we can prove Theorem 1.3. Let m ($m \geqslant 3$) be an odd integer whose prime factors split in K. Since both $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$ and $g_{(1/2,1/2)}(m\theta_K)/g_{(1/2,1/2)}(\theta_K)$ are units by Propositions 4.7 and 3.3 (ii), the conclusion follows from (3.1) with $\tau_0 = \theta_K$.

Corollary 4.8. *Let m ($m \geqslant 3$) be an odd integer whose prime factors p satisfy $p \equiv 1 \pmod 4$. Then $\sqrt{m}\varphi(m\mathrm{i})/\varphi(\mathrm{i})$ is a unit.*

Proof. If we take $K = \mathbb{Q}(\sqrt{-1})$, then $\theta_K = \mathrm{i}$. For each prime factor p of m, the fact that $p \equiv 1 \pmod 4$ implies that p splits in K [**3**, Corollary 5.17]. Therefore, we get the assertion by applying Theorem 1.3. $\square$

We close this section by evaluating $\sqrt{m}\varphi(m\mathrm{i})/\varphi(\mathrm{i})$ when $m = 3$ and 5, explicitly.

Example 4.9. We shall first estimate $\sqrt{3}\varphi(3\mathrm{i})/\varphi(\mathrm{i})$. If $K = \mathbb{Q}(\sqrt{-1})$, then $\theta_K = \mathrm{i}$ and $H_K = K$ [**3**, Theorem 12.34]. By Proposition 4.6 we have

$$\begin{aligned}\mathrm{Gal}(K_{(6)}/K) &\simeq W_{K,6}/\,\mathrm{Ker}_{K,6}\\ &= \left\{\alpha_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \alpha_2 = \begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix},\ \alpha_3 = \begin{pmatrix}1 & -4\\ 4 & 1\end{pmatrix},\ \alpha_4 = \begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix}\right\}.\end{aligned}$$

Since

$$\begin{aligned} x = \left(\sqrt{3}\frac{\varphi(3\mathrm{i})}{\varphi(\mathrm{i})}\right)^{24} &= g_{(0,1/3)}(\mathrm{i})^{12} g_{(0,2/3)}(\mathrm{i})^{12} g_{(1/2,5/6)}(\mathrm{i})^{24} g_{(1/2,7/6)}(\mathrm{i})^{24} \quad \text{by (4.1)}\\ &= g_{(0,1/3)}(\mathrm{i})^{24} g_{(1/2,1/6)}(\mathrm{i})^{48} \quad \text{by Proposition 2.4 (i) and (iii)}\\ &\approx 72\,954, \end{aligned}$$

x lies in $K_{(6)}$ by Propositions 2.2 (iii) and 4.3. Hence, its conjugates $x_k = x^{\alpha_k}$ ($1 \leqslant k \leqslant 4$) over K are

$$\begin{aligned} x_1 &= g_{(0,1/3)}(\mathrm{i})^{24} g_{(1/2,1/6)}(\mathrm{i})^{48},\\ x_2 &= g_{(2/3,1/3)}(\mathrm{i})^{24} g_{(5/6,1/6)}(\mathrm{i})^{48},\\ x_3 &= g_{(1/3,1/3)}(\mathrm{i})^{24} g_{(1/6,1/6)}(\mathrm{i})^{48},\\ x_4 &= g_{(2/3,0)}(\mathrm{i})^{24} g_{(5/6,1/2)}(\mathrm{i})^{48} \end{aligned}$$

with some multiplicity by Propositions 4.6 and 2.4 (iv). We claim that the minimal polynomial of x over K has integer coefficients. Indeed, since x is a real algebraic integer by definition (1.2) and Theorem 1.2, we have

$$[\mathbb{Q}(x):\mathbb{Q}] = \frac{[K(x):K]\cdot[K:\mathbb{Q}]}{[K(x):\mathbb{Q}(x)]} = \frac{[K(x):K]\cdot 2}{2} = [K(x):K],$$

from which the claim follows. Thus, x is a zero of the polynomial

$$(X - x_1)(X - x_2)(X - x_3)(X - x_4) = (X^2 - 72\,954X + 729)^2$$

whose coefficients can be determined by numerical approximation with the aid of a computer. Therefore, we obtain

$$\sqrt{3}\frac{\varphi(3\mathrm{i})}{\varphi(\mathrm{i})} = \sqrt[24]{x} = \sqrt[24]{36\,477+21\,060\sqrt{3}} = \sqrt[4]{3+2\sqrt{3}}.$$

Example 4.10. Next, we consider $\sqrt{5}\varphi(\sqrt{5}\mathrm{i})/\varphi(\mathrm{i})$. Let $K=\mathbb{Q}(\sqrt{-1})$. By Proposition 4.6 we have

$$\begin{aligned}
&\mathrm{Gal}(K_{(10)}/K)\\
&\qquad \simeq W_{K,10}/\operatorname{Ker}_{K,10}\\
&\qquad = \left\{\alpha_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \alpha_2=\begin{pmatrix}1&-4\\4&1\end{pmatrix},\ \alpha_3=\begin{pmatrix}1&-6\\6&1\end{pmatrix},\ \alpha_4=\begin{pmatrix}2&-3\\3&2\end{pmatrix},\right.\\
&\qquad\qquad \left.\alpha_5=\begin{pmatrix}2&-5\\5&2\end{pmatrix},\ \alpha_6=\begin{pmatrix}2&-7\\7&2\end{pmatrix},\ \alpha_7=\begin{pmatrix}3&0\\0&3\end{pmatrix},\ \alpha_8=\begin{pmatrix}4&-5\\5&4\end{pmatrix}\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
x=\left(\sqrt{5}\frac{\varphi(5\mathrm{i})}{\varphi(\mathrm{i})}\right)^{120} &= g_{(0,1/5)}(\mathrm{i})^{120}g_{(0,2/5)}(\mathrm{i})^{120}g_{(1/2,1/10)}(\mathrm{i})^{240}g_{(1/2,3/10)}(\mathrm{i})^{240}\\
&\qquad\qquad\text{by Proposition 2.4 (i) and (iii)}\\
&\approx 41\,473\,935\,220\,454\,921\,602\,871\,195\,774\,259\,272\,002,
\end{aligned}$$

x lies in $K_{(10)}$ by Propositions 2.2 (iii) and 4.3. Its conjugates $x_k=x^{\alpha_k}$ $(1\leqslant k\leqslant 8)$ over K are

$$\begin{aligned}
x_1=x_5=x_7=x_8 &= g_{(0,1/5)}(\mathrm{i})^{120}g_{(0,2/5)}(\mathrm{i})^{120}g_{(1/2,1/10)}(\mathrm{i})^{240}g_{(1/2,3/10)}(\mathrm{i})^{240},\\
x_2 &= g_{(4/5,1/5)}(\mathrm{i})^{120}g_{(3/5,2/5)}(\mathrm{i})^{120}g_{(9/10,1/10)}(\mathrm{i})^{240}g_{(7/10,3/10)}(\mathrm{i})^{240},\\
x_3=x_6 &= g_{(1/5,1/5)}(\mathrm{i})^{120}g_{(2/5,2/5)}(\mathrm{i})^{120}g_{(1/10,1/10)}(\mathrm{i})^{240}g_{(3/10,3/10)}(\mathrm{i})^{240},\\
x_4 &= g_{(3/5,2/5)}(\mathrm{i})^{120}g_{(1/5,4/5)}(\mathrm{i})^{120}g_{(3/10,7/10)}(\mathrm{i})^{240}g_{(9/10,1/10)}(\mathrm{i})^{240}
\end{aligned}$$

with some multiplicity by Propositions 4.6 and 2.4 (iv). So x is a zero of the polynomial

$$(X^2-41\,473\,935\,220\,454\,921\,602\,871\,195\,774\,259\,272\,002X+1)^4,$$

which shows that x is a unit. Therefore, we get

$$\begin{aligned}
\sqrt{5}\frac{\varphi(5\mathrm{i})}{\varphi(\mathrm{i})} &= \sqrt[120]{x}\\
&= \sqrt[120]{\begin{aligned}&20\,736\,967\,610\,227\,460\,801\,435\,597\,887\,129\,636\,001\\ &\qquad +9\,273\,853\,844\,735\,993\,106\,095\,069\,260\,699\,853\,880\sqrt{5}\end{aligned}}\\
&= \sqrt[10]{682+305\sqrt{5}}.
\end{aligned}$$

Acknowledgements. This research was partly supported by the Basic Science Research Program through the NRF of Korea funded by MEST (2010-0001654). D.H.S. is partly supported by a TJ Park Postdoctoral Fellowship.

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