Acoustic Diffraction from a Finite Plate

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ABSTRACT

This paper provides a Wiener-Hopf analysis of acoustic diffration by a finite plate focusing on the acquisition of higher order series solution and its physical interpretation to understand the finite diffraction phenomena in the presence of fluid convection. The formulation procedure starts with the use of Prandtl-Glauert transform to elliminate the complexity due to the effect of fluid convection and then a simple Helmholtz equation is derived. On the boundary condition, since Neumann and Dirichlet ones are imposed along the plate line in mixed type, generalized Fourier transform and Wiener-Hopf technique ars used to establish concise and exact integral equations in complex domain. The complete solution is obtained by a series one whose eigenfunctions are generalized gamma functions. Here, we derived a new and exact expression of this special function whose argument is 'integer + 1/2' adequate to mathematical theory of diffraction. Finally, by exact and asymptotic evaluations of inverse Fourier transforms, the scattered and total acoustic fields are visualized in physical domain and each term of the solution is physically interpreted as (i) semi-infinite leading edge scattering, (ii) trailing-edge correction and (iii) interaction between leading and trailing edges, respectively.

INTRODUCTION

Acoustic wave-airfoil interaction has been analytically studied by modeling the airfoil by monochromatic wave and thin/flat plate for mathematical convenience. When the wave is convected in uniform flow, the problem is governed by convective wave equation and the corresponding boundary condition is imposed *on* the plate as an impermeability condition in Neumann type. Since the boundary conditions *ahead of* and *behind of* the plate are given by Dirichlet type, this 3-part mixed boundary value problem could not be solved by conventional methods and thus Wiener-Hopf technique[1] is applied to this problem with integral equation formulation. The current existing formulas for far-field acoustics by Amiet[2] and Martinez and Widnall's[3] have been depending on the idea of Schwarzschild[4] who formulated and solved successive semi-infnite problems with compensation of unsatisfied boundary condition. In this work, a concise and rigorous formulation and solution are introduced based on Wiener-Hopf technique for an accurate solution of low and high frequency waves in the presence of mean flow. In Wiener-Hopf equation, the unknown potential is expanded by a Taylor series in its analytic region and a series solution is obtained with the exact and new formulas for generalized gamma functions[5]. This solution was obtained in series one and we could observe the convergence property of our solution. This series solution is more accurate compared to the currently existing asymptotic ones by Jones[6], Noble[1] and Kobayashi[7]. Finally, the acoustic field and directivity pattern by a single and multiple edges are demonstrated with the inclusion of fluid convection.

PROBLEM DEFINITION

Governing Equation

A plate of length *'2l'(-l<x<l, y=0)* is encountering a small gust within the subsonic uniform flow *U* parallel to the plate as shown in Fig.1. The plate is assumed to be infinitesimally thin and straight and the perturbation to the mean flow is assumed to be a time harmonic plane wave with time factor of exp(-*iωt*). This incoming gust naturally causes that the scattered velocity perturbation has the same time factor as incident one.

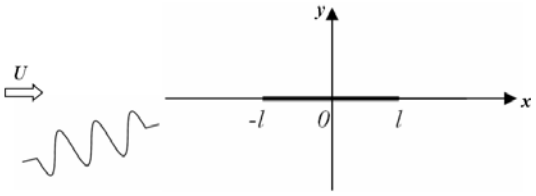

Figure 1. Finite Plate Encountering Incident Wave in Uniform Flow

Then, we can construct a partial differential equation with respect to the scattered potential only as written in Eq.(1).

$$\left((1-M^2)\partial_x^2 + 2ikM\partial_x + \partial_y^2 + k^2\right)\varphi(x,y) = 0 \tag{1}$$

Boundary Conditions

(i) The flow normal to the plate should be impermeable. (ii) The veloctiy of scattered potential should be continuous everywhere along *y=0*. (iii) And, potential itself is continuous out of the plate. Since the fluid is assumed to be inviscid and the uniform flow is parallel to the plate, there is no wake behind the trailing edge thus the Kutta condition is equivalently stated by continuous potential condition.

(i) $\partial_y \varphi = -ik\sin\theta_i (1+M\cos\theta_i)^{-1}\exp[ikx\cos\theta_i(1+M\cos\theta_i)^{-1}]$ for $-l<x<l$ along $y=0$ (2)

(ii) $\varphi_y(x,+0) = \varphi_y(x,-0)$ for all x (3)

(iii) $\varphi(x,+0) = \varphi(x,-0)$ for $x<-l$ or $l<x$ (4)

Since boundary conditions were imposed *on* and *out of* the plate in Neumann type and Dirichlet type, respectively, in this chapter, the Wiener-Hopf technique is applied to solve this kind of mixed boundary value problems.

REFROMULATION AND WIENER-HOPF EQUATION

Coordinate Transform and Wiener-Hopf Procedure

(i) Prandtl-Glauert Transform changes Eq.(1) and boundary conditions Eqs.(2-4) into Helmholtz equation and similar type of boundary conditions with newly defined incidence angle and wavenumber, respectively.

- Governing Equation: $(\partial_\xi^2 + \partial_\eta^2 + \kappa^2)\phi(\xi,\eta) = 0$

  where $\xi = x/l$, $\eta = \beta y/l$, $\beta = \sqrt{1-M^2}$ and $\varphi(x,y) = \phi(\xi,\eta)e^{-i\kappa M\xi}$ where $\kappa = \beta^{-2}kl$

- Boundary Conditions:

  (i) $\phi_\eta(\xi,0) = -i\kappa\sin\theta_e \exp[i\kappa\xi\cos\theta_e]$ on $\eta=0$ for $-1<\xi<1$

  (ii) $\phi_\eta(\xi,+0) = \phi_\eta(\xi,-0)$ for $-\infty<\xi<\infty$

  (iii) $\phi(\xi,+0) = \phi(\xi,-0)$ for $\xi<-1$ or $1<\xi$

(ii) Applying generalized Fourier Transform and Wiener-Hopf technique to Helmholtz equation and corresponding boundary conditions yields a Wiener-Hopf equations in complex(λ) domain as below.

$$e^{-i\lambda}\Phi'_-(\lambda) - \frac{q}{\lambda + p}\left(e^{i(\lambda+p)} - e^{-i(\lambda+p)}\right) + e^{i\lambda}\Phi'_+(\lambda) = -\gamma(\lambda)e^{\pm i\lambda}\Phi^d_\mp(\lambda) \qquad (5.1)\ (5.2)$$

where $p=\kappa cos\theta_e$, $q=\kappa sin\theta_e$ and $\gamma(\lambda) = (\lambda^2 - \kappa^2)^{1/2}$.
Here, we obtained two equations with four unknowns written in the capital Greek letter. This system of equation can be solved not by algebraic procedure but by function theoretic procedure which is at the heart of Wiener-Hopf technique.

(iii) By summation/multiplication decomposition technique of Wiener-Hopf procedrue and mathematical manipulations, we obtain concise and rigorous integral equations as below.

$$\frac{\Psi^{1,2}_+(\lambda)}{\gamma_+(\lambda)} \pm \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} \frac{\Psi^{1,2}_+(\alpha)}{\gamma_-(\alpha)}\frac{e^{2i\alpha}}{\alpha+\lambda}d\alpha = F_a(\lambda) + G_a(\lambda) \pm F_b(\lambda) \pm G_b(\lambda) \qquad (6.1)\ (6.2)$$

where $\Psi^{1,2}_+(\lambda) = \Phi'_+(\lambda) \pm \Phi'_-(-\lambda)$, $F_{a,b}(\lambda) = \frac{qe^{\pm ip}}{\lambda \pm p}\left(\frac{1}{\gamma_+(\lambda)} - \frac{1}{\gamma_+(\mp p)}\right)$, $G_{a,b}(\lambda) = -\frac{qe^{\mp ip}}{\lambda \pm p}\left(W_0(\lambda) - W_0(\mp p)\right)$

$\gamma_-(\lambda) = (\lambda - \kappa)^{1/2}$ $\gamma_+(\lambda) = (\lambda + \kappa)^{1/2}$ $W_0(\lambda) = 2e^{-2i\lambda}\frac{F_c(2\sqrt{\kappa + \lambda/\pi})}{\sqrt{\kappa+\lambda}}$, $F_c(v) = 2^{-1/2}e^{-i\pi/4}\int_v^\infty e^{i\pi t^2/2}dt$

SOLUTION WITH NEW FORMULAS FOR GENERALIZED GAMMA FUNCTIONS

Solution in Complex Domain

Since there has been no report that this specific kind of integral equation having multi-valued kernel is exactly solved, we expanded the unknown by a Taylor series in its analytic region. Then, we could obtain following equation.

$$\Psi^{1,2}_+(\lambda) = \mp\sum_{n=0}^{N}\left(\frac{i}{2}\right)^n \frac{d^n\Psi^{1,2}_+(\kappa)}{n!d\alpha^n}V_n(\lambda) + S^{1,2}(\lambda)$$

where $V_n(\lambda) = \gamma_+(\lambda)W_n(\lambda)$ and $S^{1,2}(\lambda) = \gamma_+(\lambda)[F_a(\lambda) + G_a(\lambda) \pm F_b(\lambda) \pm G_b(\lambda)]$

Here, $W_m(\lambda) = \frac{\sqrt{2}}{\pi}e^{i(2\kappa-\pi/4)}\Gamma_1(m+1/2, -2i(\kappa+\lambda))$ and $\Gamma_1(u,v) = \int_0^\infty \frac{t^{u-1}e^{-t}}{t+v}dt$ is called Kobayashi's generalized gamma function.

Generalized Gamma Functions Occuring in Diffraction Theory

Since generalized gamma function is not in close form but in integral form, it cannot be directly applied to the calculation. And, since currently existing formulas are restricted to asymptotic ones and recurrence relations, we needed to find a better formula for our problem. In our solution procedure, we could observe that the argument of generalized gamma function is "integer+1/2".

$$\Gamma_1(n+1/2,\lambda) = \int_0^\infty \frac{e^{-t}t^{n-1/2}}{t+\lambda}dt = 2\int_0^\infty \frac{s^{2n}e^{-s^2}}{s^2+\lambda}ds$$

And, author could derive an exact and handy formula of generalized gamma function as below.

$$\Gamma_1(n+1/2,\lambda) = \sum_{k=0}^{n-1}(-)^{n-1-k}\Gamma(k+1/2)\lambda^{n-1-k} + (-)^n\pi\lambda^{n-1/2}e^{\lambda}erfc(\sqrt{\lambda})$$

Then, we could finally obtain the complete solution in complex domain as below.

$$\Phi(\lambda,\eta)=\begin{cases} A(\lambda)e^{-\gamma(\lambda)\eta}, \eta \geq 0 \\ -A(\lambda)e^{\gamma(\lambda)\eta}, \eta \leq 0 \end{cases}$$

where $A(\lambda)=A_1(\lambda)+A_2(\lambda)+A_3(\lambda)$, $A_1(\lambda)=-\dfrac{iqe^{-i(\lambda+p)}}{(\lambda+p)\sqrt{\kappa+\lambda}\sqrt{\kappa+p}}$, $A_2(\lambda)=\dfrac{iqe^{i(\lambda+p)}}{(\lambda+p)\sqrt{\kappa-\lambda}\sqrt{\kappa-p}}$,

$$A_3(\lambda)=-\frac{ie^{i\lambda}}{\sqrt{\kappa-\lambda}}\left(G_a(\lambda)-\sum_{n=0}^{N}\left(\frac{i}{2}\right)^n\frac{1}{n!}\frac{x_n^1-x_n^2}{2}W_n(\lambda)\right)-\frac{ie^{-i\lambda}}{\sqrt{\kappa+\lambda}}\left(G_b(-\lambda)-\sum_{n=0}^{N}\left(\frac{i}{2}\right)^n\frac{1}{n!}\frac{x_n^1+x_n^2}{2}W_n(-\lambda)\right)$$

Figure 2 shows the converging history of $A(\lambda)$ as the number of series terms are increasing. And we could observe that the higher the wavenumber, the less the terms required.

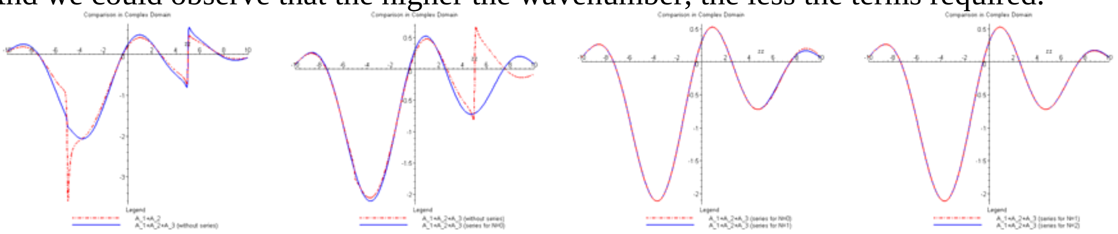

Figure 2. Converging History of Series Solution along the Path of Integration

Figure 3 shows the total acoustic field and the directivity pattern of scattered field, respectively. The contour clearly demonstates the finite diffraction and the direcvity shows the effect of fluid convection. Here, the magnitude of wave propagating downstream decreased whereas the amplitude and the wavenumber of diffracted wave propagating upstream are increased due to the compression by flow direction opposite to the wave's.

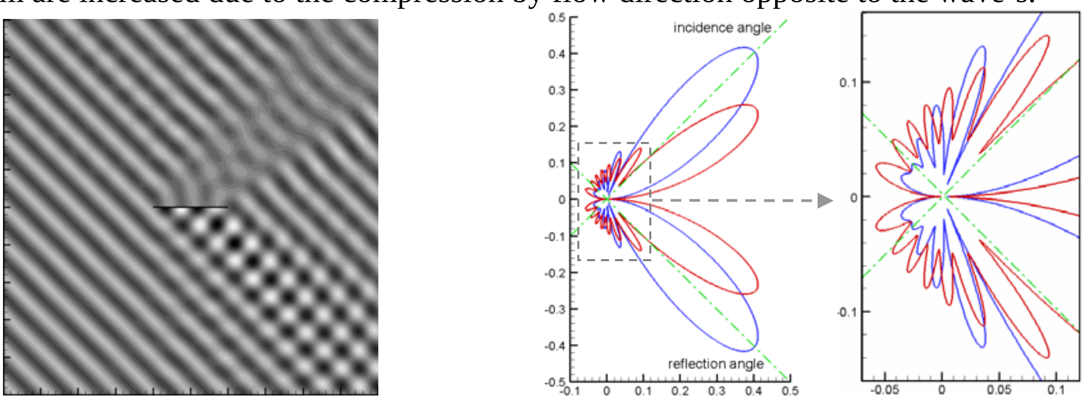

Figure 3. Contour and Directivity (kl=20, θ_i=45deg, Blue: M=0, Red: M=0.5)

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