확산 모형을 이용한 미국식 옵션의 가치 평가

김 병 천[i] · 오 승 영[ii]

한국과학기술원 테크노경영대학원

서울특별시 동대문구 청량리2동 207-43
전화 : 02-958-3697
팩스 : 02-958-3604

요약문

본 논문은 미국식 옵션의 동적 가치 변화를 연구하고, 확산 모형에 대한 지식을 응용하여 Black-Scholes 모형과 이론적 기반을 공유하면서 미국식 옵션의 분석적 가치 평가 모형을 유도할 수 있는 일반적 방법론을 제시한다. 이 가치평가 모형은 열물리학의 기반 위에서 유럽식 옵션과 조기 행사 프리미엄의 구조적 변화에 대한 분석으로부터 유도되어진다. 이 모형의 유도과정을 통해 미국식 옵션의 전체적인 가치 변화 과정에 대한 이론적 근거를 이해할 수 있다. 또한 조기 행사 경계가 미국식 옵션 가치평가 과정 내에서 유도되어야만 하는 이유에 대한 이론적 근거에 대하여도 접근한다. 마지막으로 몇몇 다른 미국식 옵션의 가치평가에 어떻게 확장될 수 있는가와 그 예를 간단히 소개한다. 이러한 연구는 미국식 옵션의 이론적 기반 및 구조적 변화에 대한 이해에 큰 도움을 준다.

---

[i] 한국과학기술원 테크노경영대학원 교수. E-mail : bckim@kgsm.kaist.ac.kr
[ii] 한국과학기술원 테크노경영대학원 박사과정. E-mail : lessai@kgsm.kaist.ac.kr

American Options Valuation

with Diffusion Equations

Byung Chun Kim and SeungYoung Oh

Graduate School of Management
Korea Advanced Institute of Science and Technology

Abstract

This article explores the nature of the American options dynamics and introduces a general methodology to derive the analytic valuation formula of the American options using the knowledge with diffusion equations. The formula is derived from the structural mechanics between early exercise premium and the European options transformed into the thermal physical environment. The whole information on the dynamics of the American options is provided while deriving the formula. And the reason why the optimal exercise boundary has to be determined in the solution process of the valuation formula is explained. At last, we expand the valuation formula to be applied to several other American options. This will give us the rigorous theoretical framework for the American options and strengthen the understanding of the American option valuation structure.

The theoretical researches on options have been done from 19C, but it is by Black and Scholes in 1973 that the first logical option valuation model has been developed. The landmark publication of Black and Scholes (1973) and Merton (1973) have represented an epoch-making discovery in tackling the option valuation problem. They have derived the model by solving a differential equation that makes use of the knowledge related to stochastic calculus and heat or diffusion equations. Because the Black-Scholes theory delivers a closed form solution to the valuation of European options, we can use the quantitative sensitivity analysis of option to control risk and to establish an effective trading.

But, most options that are publicly traded are American options. The distinctive feature of an American option is its early exercise privilege, that is, the holder can exercise the option prior to the date of expiration. However, the early exercise feature of American options brings with it substantially more complicated payoff structures. The early exercise boundary is not known beforehand but has to be determined as part of the solution in the solution process. For that reason, no general closed-form valuation models for American options that would parallel the black-Scholes model for European options.

The American option valuation problem has been studies over a quarter century. Broadie and Detemple (1996) make a comparison between the existing methods for the valuation of the American options. Huang, Subrahmanyam, and Yu (1996), and Ju (1998) categorize the past research papers into several classes. Among them, the finite difference method of Brennan and Schwartz (1997), the risk neutral valuation approach of Cox and Ross (1976), the binomial method of Cox, Ross and Rubinstein (1979), the analytic valuation model with infinite series of integrals of Geske and Johnson (1984), the quadratic approximation method of Barone-Adesi and Whaley (1987) and MacMillan (1986), the analytic valuation of Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992), the lower and upper bound approximation of Broadie and Detemple (1996), accelerated recursive method of Huang, Subrahmanyam, and Yu (1996), multi-piece exponential function method of Ju (1998), the tangential approximation method of Bunch and Johnson (2000) and so on can be mentioned as the most important works that leave big footprints in American options valuation theory.

Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992) derive the analytic American option valuation formula in different methods, in which they show that the American option value is made up of the corresponding European option value and an integral part representing the early exercise premium. However, the exact formula for optimal exercise boundary is hidden in a veil of mystery as yet.

Kim (1990) has divided the time space into a finite number of discrete exercisable points in time. Then he gains the values of the live American call option at the consecutive points in time from the maturity. After finding out the recursive relationship between the value of the live American call option at each point and ignoring a negligible term in equation, he obtains the analytic solution for American call options that can be exercisable at discrete points in time. By taking a limit for the time interval, he finally derives the analytic solution to the valuation problem of American options on assets that pay continuous dividends. This is in accordance with the limit of the formula provided by Geske and Johnson (1984).

Jacka (1991) provides the verification of the essential uniqueness of the solution to the free boundary problem and the identification of the integral equation satisfied by the stopping boundary. He obtains the valuation result using probability theory applied to the optimal stopping problem.

Carr, Jarrow and Myneni (1992) provide another proof of the American Options valuation formula using stochastic calculus and offer intuition on the nature of the early exercise premium. In particular, they show that the early exercise premium is the value of an annuity that pays interest at a certain rate whenever the stock price is low enough so that early exercise is optimal.

The purpose of this article is to provide the information on the structure of American option dynamics and introduce a general methodology to derive the analytic valuation formula of the American options using the knowledge with diffusion equations. This process is sure to be much helpful to understand the mechanism of American options as a function of stock price and time variables. This methodology can be regarded as the expansion of the Black-Scholes model to the American options valuation. The expression will help us to gain rich understanding and intuition for the whole composition and dynamics of American options. And this article examines the reason why the optimal exercise boundary cannot but be determined in the valuation process as well.

This article is organized as follows. In section 1, we transform the American put option into the diffusion equation formulation. In section 2, we examine the meaning of the transformed formulation in the viewpoint of both finance and thermal physics. In section 3, we explain the detail of the American option dynamics as time goes on and derive the American options valuation formula with diffusion equation approach. In section 4, we show that the valuation formula derived in section 3 is expanded to be applied to several other American options. Section 5 is a summary and conclusion.

1. Basic Transformation of American Put Option

We shall first consider an American put option written on an underlying stock price S at time t, with expiration time T and strike price K. The American option valuation problem is to be solved in the domain $D = \{(S,t) \mid 0 \le S < \infty,\ 0 \le t \le T\}$. Assume the stock price dynamics follows a lognormal diffusion process or a geometric Brownian motion

$$dS = (r-q)Sdt + \sigma S dW_t$$

where dW_t is a standard Brownian motion or a Wiener process, r is the risk free interest rate, σ is the volatility, and q is the continuous dividend rate which is less than r. Throughout the article, T, K, r, q and σ are all taken to be constant and greater than or equal to 0, unless otherwise noted.

Black-Scholes Partial Differential Equation for a European put option on an asset paying continuous dividends q with value $P_e(S,t)$ is

$$\frac{1}{2}\frac{\partial^2 P_e}{\partial S^2}(\sigma S)^2 + \frac{\partial P_e}{\partial S}(r-\delta)S + \frac{\partial P_e}{\partial t} - rP_e = 0 \tag{1}$$

with initial and boundary conditions

$$P_e(S,T) = \max(K-S,0), \tag{2}$$

$$P_e(0,t) = Ke^{-r(T-t)} \tag{3}$$

$$P_e(S,t) \approx 0, \quad as \quad S \approx \infty. \tag{4}$$

Equations (1), (2) and (4) are also applied to the American put options case. But we should consider one more condition for American put options for during the options' lifetime.

The characteristic of the early exercise feature of American options leads to the condition that American options must be worth at least their corresponding intrinsic values, namely, $max(S\text{-}K,0)$ for a call and $max(K\text{-}S,0)$ for a put during the life. To represent this constraint, we should introduce a new expression to the original problem instead of Equation (3) as follows,

$$P_a(S,t) \geq P_a(S,T) = \max(K-S,0) \tag{5}$$

where $P_a(S,t)$ is American put option value.

Moreover, the governing equation of Equation (1) is not always true for the American options. We can divide the domain of the stock price and time into two regions, an *exercise region* ($0<S\leq S_c(t)=B(t)$) where the holder of the American option had better exercise it and a *holding region* ($B(t)=S_c(t)<S<\infty$) where the holder of the American option had better hold it rather than exercise it. Here, $S_c(t)$ is commonly referred to as the critical stock price at time t and $B(t)$ is referred to as the optimal exercise boundary at time t. For the exercise region, if we substitute $P_a = K\text{-}S$ into Equation (1), we obtain

$$\frac{\partial P_a}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P_a}{\partial S^2} + (r-\delta) S \frac{\partial P_a}{\partial S} - rP_a < 0. \tag{6}$$

It is more convenient to work in traditional normalized coordinates with the stock price variable normalized by the strike price, the time variable normalized by the volatility and the strike price, the critical stock price variable normalized by the exercise price, and the American options value normalized by the exercise price respectively,

$$x = \ln\frac{S}{K}, \tag{7}$$

$$\tau = \frac{\sigma^2}{2}(T-t),. \tag{8}$$

$$b(\tau) = \ln\frac{B(t)}{K}, \tag{9}$$

$$P_a(S,t) = Ke^{\alpha x+\beta\tau}u(x,\tau). \tag{10}$$

And we introduce a non-dimensional measure of the risk free interest rate and the dividend rate,

$$\theta = \frac{r}{\sigma^2/2}, \tag{11}$$

$$\phi = \frac{\delta}{\sigma^2/2}. \tag{12}$$

If we substitute Equation (10) into Equation (1), with

$$\alpha = -\frac{1}{2}(\theta-\phi-1), \qquad \beta = -\{\frac{1}{4}(\theta-\phi-1)^2 + \theta\}$$

then Equation (1) can be transformed into the basic heat or diffusion equation problem in thermal physics as follows,

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0.$$

We can transform the original problem into the diffusion equation form separately for the exercise region and for the holding region, to understand the structure more clearly.

If we substitute Equations (7) – (12) into Equations (1) – (6), we gain

$$\begin{bmatrix} \dfrac{\partial u}{\partial \tau} - \dfrac{\partial^2 u}{\partial x^2} = f(x,\tau), & -\infty < x \le b(\tau), \quad \tau > 0, \quad f(x,\tau) \ge 0 \\ \dfrac{\partial u}{\partial \tau} - \dfrac{\partial^2 u}{\partial x^2} = 0, & b(\tau) < x < \infty, \quad \tau > 0 \end{bmatrix} \quad \begin{matrix} (13a) \\ (13b) \end{matrix}$$

with initial and boundary conditions

$$u(x,0) = u_o(x) = \max(e^{\frac{1}{2}(\theta-\phi-1)x} - e^{\frac{1}{2}(\theta-\phi+1)x}), \tag{14}$$

$$u(x,\tau) \approx 0 \qquad as \quad x \approx \infty, \tag{15}$$

$$u(x,\tau) = \begin{bmatrix} 0 & if \quad 0 < \theta - \phi < 1 \\ 1 & if \quad \theta - \phi = 1 \\ \infty & if \quad \theta - \phi > 1 \end{bmatrix} \quad as \quad x \approx -\infty, \tag{16}$$

$$u(x,\tau) \ge \max((e^{\frac{1}{2}(\theta-\phi-1)x} - e^{\frac{1}{2}(\theta-\phi+1)x}) \cdot e^{\{\frac{1}{4}(\theta-\phi-1)^2+\theta\}\tau}, 0). \tag{17}$$

Equation (13a) is for the exercise region and Equation (13b) is for the holding region. Equations (13) – (17) are explained in the next section in more detail.

2. The Meaning of the Transformed Model

The Black-Scholes Partial Differential Equation of the American options has minus value at the exercise region, which leads to an opportunity of arbitrage. We might expect that the holder of an American option will choose the exercise policy in such a way that the expected payoff from the option will be maximized. If we could buy a put option and immediately exercise the put option to sell the stock in the market, we could make a riskless profit by the difference. To prohibit this chance, we need to supplement an appropriate amount of value to the governing equation of the American options to make balance. The variable $u(x,\tau)$ expressed in Equations (13a) and (13b) can be interpreted as the American option value in finance and as the temperature profile in thermal physics as well.

Equation (13a) has incorporated with a new term $f(x,\tau)$. Physically speaking, the $f(x,\tau)$ can be interpreted as an internal dimensionless heat generation distribution in the domain. As we can forecast, if the heat is generated inside a conductive material with initial temperature, the temperature profile becomes higher than that of the simple conduction or diffusion of initial temperature with no heat generation. Therefore, we can say that if the supplementary amount of option value $f(x,\tau)$ is added up to the exercise region, it comes to diffuse to the both infinite side ends as τ increases and make an effect of push up the value of the put option all over the stock price domain. We know already that the deep-in-the-money value of the European put option is on a gradual decrease as τ increases. So, we need to add up some amount of value to keep the American option value be equal to the intrinsic value of the put option at the exercise region. $f(x,\tau)$ is to be calculated in the next section.

Equation (14) is the initial condition of partial differential equation problem that represents the initial temperature distribution at τ=0, which corresponds to the terminal value of the American put option at expiration. Equations (15) and (16) are boundary conditions that have been incorporated in American option valuation model. If we solve the diffusion equation problem only with Equation (13b), (14), (15) and (16) for $-\infty<x<\infty$, this process leads us to the Black-Scholes model for the European put option.

Equation (17) has been transformed from Equation (5). At the exercise region, the European put option value decreases gradually and is getting lower than the intrinsic value which is identical with the right hand side terms of Equation (17). So, physically speaking, the amount of heat that is needed to complement the decreased temperature should be provided by the internal heat generation, as mentioned earlier. Because the internal heat generation is needed continuously to make up for the loss at the exercise region, the temperature profile should be equal to the right hand side of Equation (17). We can conclude that the following equation is satisfied at the exercise region.

$$u(x,\tau) = (e^{\frac{1}{2}(\theta-\phi-1)x} - e^{\frac{1}{2}(\theta-\phi+1)x}) \cdot e^{\{\frac{1}{4}(\theta-\phi-1)^2+\theta\}\tau}, \quad -\infty < x \le b(\tau). \tag{18}$$

where $b(\tau)$ is the critical point at τ defined in Equation (9).

And, Heat or diffusion equation has two important properties: the one is the *linearity* and the other one is the *uniqueness* of the solution. And these properties lead to the *principle of superposition* that the general solution of a linear homogeneous ordinary differential equation of order n is a linear combination of n linearly independent solutions with n arbitrary constants. These properties are of use in deriving the analytic valuation formula of the American options.

3. Derivation of the American Options Valuation Formula

We begin with the calculation of the amount of heat generated internally, $f(x,\tau)$, which implies the additional value of the American option to protect the opportunity of arbitrage. The truth is that we do not need to calculate $f(x,\tau)$ at the exercise region, because we already know the exact temperature profile formula meaning the intrinsic value in Equation (18). But, we still need to obtain $f(x,\tau)$ because the generated heat flows into the holding region to make an effect on the temperature profile that means American options value.

$f(x,\tau)$ can be obtained both by physical method and by mathematical method. But mathematical method is much easier than the physical method. Equation (18) is governed by Equation (13a) at the exercise region. So, we can calculate the first derivative of Equation (18) with respective to τ and the second derivative of it with respective to x, and put into Equation (13a). With an easy mathematics, we can obtain

$$f(x,\tau) = (\theta \cdot e^{\frac{1}{2}(\theta-\phi-1)x} - \phi \cdot e^{\frac{1}{2}(\theta-\phi+1)x}) \cdot e^{\{\frac{1}{4}(\theta-\phi-1)^2+\theta\}\tau} \tag{19}$$

$$-\infty < x \leq b(\tau).$$

Thanks to the property of linearity, the solution of diffusion equation problem of Equations (13) – (17) can be expressed as the sum of the following two problems:

$$\frac{\partial u_{eep}}{\partial \tau} - \frac{\partial^2 u_{eep}}{\partial x^2} = f(x,\tau), \qquad -\infty < x \leq b(\tau), \quad \tau \geq 0, \quad f(x,\tau) \geq 0, \tag{20a}$$

$$u_{eep}(x,0) = u_o(x) = 0, \tag{20b}$$

and

$$\frac{\partial u_{bs}}{\partial \tau} - \frac{\partial^2 u_{bs}}{\partial x^2} = 0, \qquad -\infty < x < \infty, \quad \tau \geq 0, \tag{21a}$$

$$u(x,0)_{bs} = u_o(x) = \max(e^{\frac{1}{2}(\theta-\phi-1)x} - e^{\frac{1}{2}(\theta-\phi+1)x}, 0), \tag{21b}$$

$$u_{bs}(x,\tau) \approx 0 \qquad as \quad x \approx \infty, \tag{21c}$$

$$u(x,\tau) = \begin{bmatrix} 0 & if \quad 0 < \theta - \phi < 1 \\ 1 & if \quad \theta - \phi = 1 \\ \infty & if \quad \theta - \phi > 1 \end{bmatrix} \quad as \quad x \approx -\infty. \tag{21d}$$

We can use the fact that knowing the basic Green's function G(x-ξ, τ-ω) allows us to write the solution of the first problem immediately in terms of a *superposition integral*, which is also implied by linearity.

First, we shall attack the problem of Equation (20). Laplace Transform and Fourier Transform can be used to get the solution to the problem represented in Equation (20). But, to understand the dynamics of heat diffusion in more detail, we shall consider the following problem very similar to Equation (20).

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = \delta(x-\xi)\delta(\tau-\omega), \qquad -\infty < x < \infty, \quad \tau \geq 0, \tag{22a}$$

$$u(x,0) = 0, \tag{22b}$$

$$u(x,\tau) \approx 0, \qquad as \quad |x| \approx \infty, \tag{22c}$$

where ξ and ω are fixed constants, $-\infty < \xi < \infty$, $0 < \omega < \infty$ and δ denotes the Dirac delta function. We may interpret Equation (22) physically as the equation governing the temperature in an infinite conductor that is subjected to a concentrated unit source of heat at the point $x=\xi$. This source of heat is turned on only for the instant $\tau=\omega$ and is absent for all other times. And the conductor has a constant temperature that we normalize to equal zero, which represent the initial condition, Equation (22b), and boundary condition, Equation (22c).

The solution of the inhomogeneous diffusion Equation (22) is the following Green's function, which is a function of $x=\xi$ and $\tau=\omega$:

$$G(x-\xi,\tau-\omega)=\frac{1}{2\sqrt{\pi(\tau-\omega)}}e^{-(x-\xi)^2/4(\tau-\omega)}.$$

At $\tau=\omega$, only the position of $x=\xi$ has the temperature and others has 0. The initially sharply peaked profile is gradually smoothed out as τ increases under the action of diffusion. These are remarkable features for diffusion phenomena.

Since we get the solution of the diffusion equation with $\delta(x\text{-}\xi)$ and $\delta(\tau\text{-}\omega)$, linearity implies that the solution of the inhomogeneous diffusion Equation (20) due to the $f(\xi,\omega)$ is just

$$u_{eep}(x,\tau)=\int_{\xi=-\infty}^{\infty}\int_{\omega=-\infty}^{\infty} f(\xi,\omega)\cdot G(x-\xi,\tau-\omega)\ \ d\omega d\xi\,.$$

Here, $f(x,\tau)$ is defined in Equation (19). The substitutions give

$$u_{eep}(x,\tau)=\int_{\xi=-\infty}^{b(\tau)}\int_{\omega=-\infty}^{\infty}(\theta\cdot e^{\frac{1}{2}(\theta-\phi-1)x}-\phi\cdot e^{\frac{1}{2}(\theta-\phi+1)x})\cdot e^{\{\frac{1}{4}(\theta-\phi-1)^2+\theta\}\tau}$$
$$\times\ \frac{1}{2\sqrt{\pi(\tau-\omega)}}e^{-(x-\xi)^2/4(\tau-\omega)}\quad d\omega d\xi.$$

With some mathematical treatment, we obtain

$$u_{eep}(x,\tau) = \theta \cdot e^{\frac{1}{2}(\theta-\phi-1)x} e^{\{\frac{1}{4}(\theta-\phi-1)^2+\theta\}\tau} \int_{\omega=0}^{\tau} e^{-\theta(\tau-\omega)} N(-\hat{d}_{12}) d\omega$$
$$- \phi \cdot e^{\frac{1}{2}(\theta-\phi+1)x} e^{\{\frac{1}{4}(\theta-\phi+1)^2+\phi\}\tau} \int_{\omega=0}^{\tau} e^{-\phi(\tau-\omega)} N(-\hat{d}_{11}) d\omega. \qquad (23)$$

where

$$N(d_{11}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_{11}} e^{-\frac{1}{2}y^2} dy,$$
$$\hat{d}_{11} = \frac{b(\tau)-x}{\sqrt{2(\tau-\omega)}} - \frac{(\theta-\phi+1)\sqrt{2(\tau-\omega)}}{2},$$
$$\hat{d}_{12} = d_{11} - \sqrt{2\tau}.$$

N stands for the standard Gaussian cumulative distribution function.

As mentioned earlier, $f(x,\tau)$ is the additional value of the American option to protect the opportunity of arbitrage or the amount of heat generated internally at the exercise region. And the temperature is apt to get down below the $u(x,\tau)$ of Equation (18) at the exercise region because the heat tends to flow from high temperature to low temperature. The temperature of the exercise region is always higher than that of the holding region. So, the heat at the exercise region flows into the holding region as τ increases and this makes the temperature of the holding region be on a gradual increase. Once the temperature of a point in x gets higher than the initial temperature, the point becomes to be classified into the holding region.

At every time, we should determine the critical point $b(\tau)$, meaning the optimal exercise boundary[1]. This is the right end of the region where we should generate heat to complement the decreased value. At the critical point, the temperature is no longer lower than the initial temperature even without heat generation. The critical point plays a role of a point of inflection mathematically as well.

The critical point plays a great role in option valuation theory. Two factors make influence on the temperature of the holding region. The one is the length of the region where heat generates internally. And the other one is the amount of heat which is generated at each point of the domain internally. The critical point moves left as τ increases so that the length of heat generation is on a gradual decrease. And the amount of the generated heat differs as τ increases. As these two factors vary, the amount of heat that flows into the holding region differs as τ increases. Consequently, the temperature of the holding region affects from them. And then we should adjust the critical point because the temperature of the holding region moves higher than ever. This means critical point affect the temperature of the holding region and the temperature of the holding region affect the critical point recursively. They are mutually connected very closely. For this reason, it is impossible to extract only the optimal exercise boundary from the analytic valuation formula or to make the valuation formula with no optimal exercise boundary.

Next, we shall deal with the problem of Equation (21). To solve the

---

[1] The free boundary problem associated with the optimal stopping problem for American put options was studied by McKean (1965) and van Moerbeke (1976).

homogeneous diffusion equation (21), we note that it is equivalent to

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = \delta(\tau) u_o(x), \qquad -\infty < x < \infty, \ \ \tau \geq 0, \tag{24a}$$

$$u(x,\tau) = 0, \qquad \tau \geq 0, \tag{24b}$$

As can be verified by noting that integrating the inhomogeneous diffusion Equation (24a) with respect to *τ* from $\tau=0^-$ to $\tau=0^+$ gives $u(x,\tau)=u_0(x)$. Since the right hand side of (24a) vanishes when $\tau>0$, Equations (21) and (24) are equivalent.

The solution of the inhomogeneous diffusion Equation (24) is the following Green's function, which is a function of $x=\xi$.

$$G(x-\xi,\tau) = \frac{1}{2\sqrt{\pi(\tau-\omega)}} e^{-(x-\xi)^2/4\tau}$$

To solve (21), we set $f(\xi,\omega)$ in Equation (23) equal to $\delta(\omega)\cdot u_0(\xi)$ and obtain

$$\begin{aligned} u_{bs}(x,\tau) &= \int_{\xi=-\infty}^{\infty} u_0(\xi)\cdot G(x-\xi,\tau) \ \ d\xi \\ &= \frac{1}{2\sqrt{\pi\tau}} \int_{\xi=-\infty}^{0} \left(e^{\frac{1}{2}(\theta-\phi-1)x} - e^{\frac{1}{2}(\theta-\phi+1)x} \right) e^{-(x-\xi)^2/4\tau} d\xi \\ &= e^{\frac{1}{2}(\theta-\phi-1)x+\frac{1}{4}(\theta-\phi-1)^2\tau} N(-\hat{d}_2) - e^{\frac{1}{2}(\theta-\phi+1)x+\frac{1}{4}(\theta-\phi+1)^2\tau} N(-\hat{d}_1) . \end{aligned} \tag{25}$$

where

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}y^2} dy,$$
$$\hat{d}_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(\theta - \phi + 1)\sqrt{2\tau},$$
$$\hat{d}_2 = d_1 - \sqrt{2\tau}$$

This corresponds to the Black-Scholes Model for European put option valuation.

Now, it is the right time to sum up the two results, Equations (23) and (25) so that we can get the solution for the original diffusion equation problem of Equations (13) – (17),

$$u(x,\tau) = u_{bs}(x,\tau) + u_{eep}(x,\tau).$$

If we retrace our steps back, writing

$$P_a(S,t) = Ke^{\alpha x + \beta\tau} u(x,\tau) \quad (11)$$

and making change of variables back with Equation (7)-(12), then we gain the final American put options valuation Formula as follows.

$$P_a(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$$
$$+ rK\int_{u=0}^{T-t} e^{-ru}N(-d_{12})du - \delta S\int_{u=0}^{T-t} e^{-\delta u}N(-d_{11})du \qquad (26)$$

where

$$d_1 = \frac{\log\frac{S}{K} + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$
$$d_{11} = \frac{\log\frac{S}{B(u+t)} + (r-\delta+\frac{1}{2}\sigma^2)(u)}{\sigma\sqrt{u}}, \qquad d_{12} = d_1 - \sigma\sqrt{u}$$

The first term in Equation (26) is the European put option price while the second term represents the early exercise premium[2]. The early exercise premium has originally been created from the additional heat generation $f(x,\tau)$ in Equation (20a) and its distribution in the rage of $0<S\leq S_c(t)$ as time passes, so the integrand function in the second term is always positive.

Unfortunately, the solution to the integral equation of the early exercise premium representation cannot be explicitly know because of the interaction between the optimal exercise boundary and the American options value, and thus it needs to be solved numerically.

Following the same solution processes as the American put option valuation formula, we can obtain the analytic valuation formula for American call option

[2] Jamshidian (1992) has interpreted the early exercise premium as the delay exercise compensation

$C_a(S,t)$ as follows:

$$C_a(S,t) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) + \delta S\int_{u=0}^{T-t} e^{-\delta u}N(d_{11})du - rK\int_{u=0}^{T-t} e^{-ru}N(d_{12})du \quad (27)$$

Like the American put option case, the first term in Equation (27) is the European call option value while the second term represents the early exercise premium.

Equations (26) and (27) give us the information that the American option valuation formula consist of the European option valuation formula and early exercise premium. So, we can conclude that the existence of the analytic European option valuation formula is the necessary condition for the existence of the analytic American option valuation formula. If we confirm the valuation formula for a European-style option contract is such a shape as Black-Scholes model, we can easily obtain the analytic American option valuation formula for the contract with Equations (26) and (27). Some examples are given in the next section. Even if the option contract is far from the shape of plain vanilla option such as exotic options, we can obtain the analytic American option valuation formula by following the whole solution process explained in section 3.

With Equations (26) and (27) we can confirm that the decision of early exercise depends on the competition between the time value of the strike price K and the loss of insurance value associated with the holding of the option. The early exercise of an American put on a non-dividend paying asset may become preferable when the gain in the time value of K – receiving amount of cash K now rather than at the time of expiration – exceeds the insurance value of the put. [Kim (1990), Carr, Jarrow and Myneni (1992)]

4. Application of the Analytic American option valuation formula

We shall deal with the problem of valuing American options on stock index, foreign currency, futures and exchange contracts in this section[3].

Under the condition that we assume the same stochastic process for stock, stock index, foreign currency and futures, it is well known that the options on stock index, foreign currency, and futures are very similar with stock paying continuous dividends. This means they have very similar partial differential equation one another. It is also known that the dividend yields (q_i) in stock index option, the foreign currency risk-free interest rate (r_f) in currency option, risk-free interest rate (r) in futures options act like the dividend yields (q) in the option on stock paying continuous dividend for the European style option. We can conclude that the American option valuation formula on them is analogous to Equation (26) for put option case and Equation (27) for call option case because of the necessary condition between them.

[3] Most traded stock and futures options are American style, but most index options are European style.

Let $P_a(S,K,r,q,\sigma,t,T)$ denote the value of the American put option, $P_a(S,t)$ of Equation (26) and $C_a(S,K,r,q,\sigma,t,T)$ denote the value of the American call option, $C_a(S,t)$ of Equation (27), where all parameters are included.

For the stock index options, we obtain the value, P_a, of the American put and the value, C_a, of the American call options by replacing q with q_i and S with I meaning of stock index in the American option valuation formula as

$$P_a = P_a(I,K,r,q_i,\sigma,t,T)$$
$$C_a = C_a(I,K,r,q_i,\sigma,t,T).$$

For the foreign currency options, by replacing q with r_f and S with E meaning of the foreign currency exchange rate in the American option valuation formula, we obtain the put and the call valuation formula as

$$P_a = P_a(E,K,r,r_f,\sigma,t,T)$$

$$C_a = C_a(E,K,r,r_f,\sigma,t,T).$$

The identical process is applied for the futures options case. We obtain the value, P_a, of the American put and the value, C_a, of the American call options by replacing q with r and S with F meaning of futures price in the American option valuation formula as

$$P_a = P_a(F, K, r, r, \sigma, t, T)$$

$$C_a = C_a(F, K, r, r, \sigma, t, T).$$

This is in accordance with the valuation model for the American futures options of Kim (1994).

Lastly, the American option valuation formula can be extended to the valuation of the American options to exchange one asset S_2 for another S_1, whose payoff at expiration is

$$\max(a_1S_1 - a_2S_2, 0)$$

where a_1 and a_2 are the constant quantity of asset S_1 and S_2 which pay continuous dividend at rate q_1 and q_2 and has the volatility of σ_1 and σ_2 respectively.

Margrabe (1978) developed the pricing equation for a European Exchange option. Because the necessary condition between European and American option valuation formula, we can easily extend the equation into the analytic American Exchange option valuation formula as

$$C_a = C_a(a_1S_1, a_2S_2, q_1, q_2, \hat{\sigma}, t, T)$$

where

$$\hat{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_1} \ .$$

Here, ρ is the correlation between the two assets. This is in accordance with the concept of Bjerksund and Stensland (1993).

4. Summary and Conclusion

This article presents a general methodology to derive the analytic valuation formula for the American option. We consider American options in the frame of diffusion equation like Black-Scholes model. Undergoing the derivation process, we can understand the basic structure and dynamics of the American options including the corresponding European options and figure out the close correlation between optimal exercise boundary and American option valuation. This article concludes that optimal exercise boundary and the American valuation formula are so mutually connected that we cannot extract only one of them independently of the other.

We can obtain the analytic American option valuation formula easily if the contract is similar to plain vanilla option like options on stock indices, currency, futures and exchange one asset for another. Even if the option contract is far from the shape of plain vanilla option such as exotic options and corporate securities, we can obtain the analytic American option valuation formula by using the general methodology developed in this article.

References

Barone-Adesi, G. and R. Whaley, 1987, "Efficient Analytical Approximation of American Option Values," *Journal of Finance*, 42, 301-320.

Bjerksund, P. and G. Stensland, 1993, "American Exchange Options and a Put-Call Transformation: A Note," *Journal of Business Finance and Accounting*, 20, 761-764.

Black, F., 1976, "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3, 167-179.

Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.

Brennan, M., and E. Schwartz, 1977, "The Valuation of American Put Options," *Journal of Finance*, 32, 449-462.

Broadie, M., and J. Detemple, 1996, "American Option Valuation : New Bounds, Approximations, and a Comparison of Existing Methods," *Review of Financial Studies*, 9, 1211-1250.

Bunch, D. S., and J. Johnson, 2000, "The American Put Option and Its Critical Stock Price," *Journal of Finance*, 55, 2333-2356.

Carr, P., R. Jarrow, and R. Myneni, 1992, "Alternative Characterizations of American Put Options," *Mathematical Finance*, 2, 87-106.

Cox, J. C., and S. A. Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3, 145-166.

Cox, J. C., S. A. Ross and M. Rubinstein, 1979, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229-264.

Debnath, L., 1997, *Nonlinear Partial Differential Equations for Scientist and Engineers*, Birkhauser Boston.

Garman, M., and S. Kohlhagen, 1983, "Foreign Currency Option Values," *Journal of International Money and Finance*, 2, 231-237.

Geske, R., and H. E. Johnson, 1984, "The American Put Option Valued Analytically," *Journal of Finance*, 39, 1511-1524.

Huang, J., M. G. Subrahmanyam and G. G. Yu, "Pricing and Hedging American Options: A Recursive Integration Method," *Review of Financial Studies*, 9, 277-300.

Hull, J. C., 1997, *Options, Futures, and Other Derivatives*, Prentice-Hall, Inc.

Jacka, S. D., 1991, "Optimal Stopping and the American Put," *Mathematical Finance*, 1, 1-14.

Jamshidian, F., "An Analysis of American Options", *Review of Futures Markets*, 11, 72-82.

Ju, N., 1998, "Pricing an American Option by Approximating Its Early Exercise Boundary as a Multipiece Exponential Function," *Review of Financial Studies*, 11, 627-646.

Kevorkian, J., 2000, *Partial Differential Equations: Analytical Solution Techniques*, Chapman & Hall, Inc.

Kim, I. J., 1990, "The Analytic Valuation of American Options," *Review of Financial Studies*, 3, 547-572.

Kim, I. J., 1994, "Analytic Approximation of the Optimal Exercise Boundaries for American Futures Options," *Journal of Futures Markets*, 14, 1-24.

MacMillan, L. W., 1986, "An Analytic Approximation for the American Put Price," *Advances in Futures and Options Research*, 1, 119-139.

Margrabe, W., 1978, "The Value of an Option to Exchange One Asset for Another," *Journal of Finance*, 33, 177-186.

McKean, H. P., Jr., 1965, “Appendix: A Free Boundary Problem for the Heating Function Arising from a Problem in Mathematical Economics,” *Industrial Management Review*, 6, 32-39.

Merton, R. C., 1973, “Theory of Rational Option Pricing,” *Bell Journal of Economics and Management Science*, 4, 141-183.

Myneni, R., “The Pricing of the American Option,” *Annals of Applied Probability*, 2, 1-23.

Ockendon, J., S. Howison, A. Lacey and A. Movchan, 1999, *Applied Partial Differential Equations*, Oxford University Press.

van Moerbeke, P., 1976, “On optimal stopping and free boundary problem,” *Archive for Rational Mechanics and Analysis*, 60, 101-148.

Wilmott, P., J. Howison and J. Dewynne, 1997, *The Mathematics of Financial Derivatives: A Student Introduction*, Cambridge University Press.

Wilmott, P., 1998, *Derivatives: The Theory and Practice of Financial Engineering*, John Wiley & Sons.