혼합경계조건을 수반한 산란문제의 이론적 연구

Theoretical study of scattering problem with mixed boundary conditions

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1. Introduction

One of the remarkable features of the mathematical description by means of partial differential equations is the comparative ease with which solutions can be obtained for certain geometrical shapes, such as circles and infinite strips, by the method of separation of variables. In contrast, considerable difficulty is usually encountered in finding solutions for shapes not covered by the method of separation of variables.

Scattering by rigid screen like this problem can be described by partial differential equation having mixed boundary conditions. Mixed boundary conditions mean that some parts of the boundary of the region are prescribed by function itself and the rests of the boundary of the region are prescribed by the normal derivative of the function. In other words, mixed boundary value problem is a partial differential equation having both the Dirichlet and Neumann boundary conditions. Many methods are available for mixed boundary value problem. The method used in this thesis is Wiener-Hopf technique using the property of analytic continuation in complex domain.

But, when we solve the wave equation for the scattered potential, unlike the case of diffraction by semi-infinite geometry, mathematical difficulties arises due to the finite geometry such as a blade. The finiteness of geometry in our problem brings a serious mathematical difficulty which is the appearance of simultaneous integral equations resulted from the three-part mixed boundary value problem. There has been no general method for the simultaneous integral equations having multi-valued kernel function because two integral equations are strongly coupled. So, this study provides a formulation of independent integral equations by decoupling the simultaneous equations. The brief procedure is explained in the following chapter.

2. Formulation

2.1 Governing equation

Consider a blade of finite length submerged in subsonic uniform flow, as illustrated in Figure 1. It is assumed that the blade is infinitesimally thin and the mean flow is parallel to the blade surface with velocity U in the x-direction. There are steady-state waves with time factor e^{-iwt} incoming on and these cause velocity perturbation to the mean flow. Denote the total velocity potential by the summation of the incident and the scattered potential.

$$\phi_t(x,y,t) = \phi_i(x,y,t) + \phi_s(x,y,t)$$

$$\phi_i(x,y,t) = e^{iwt - k\cos\theta x - k\sin\theta y}$$

The governing equation will be setup for the scattered potential and this is convective wave equation from the linearized Euler equations. Two-dimensional convective wave equation is written as below and the scattered potential has the time factor, e^{-iwt}, which is the same as the factor of incident wave

$$\nabla^2\phi_s(x,y,t)-\frac{1}{c^2}\frac{D^2}{Dt^2}\phi_s(x,y,t)=0$$

$$\phi_s(x,y,t)=e^{-iwt}\phi(x,y)$$

Now the equation is rewritten with the definition of wave number and Mach Number.

$$\frac{\partial^2}{\partial y^2}\phi(x,y)+(1-M^2)\frac{\partial^2}{\partial x^2}\phi(x,y)+2ikM\frac{\partial}{\partial x}\phi(x,y)+k^2\phi(x,y)=0 \quad (1)$$

Here, k is assumed to have a positive imaginary part k_2, that is, $k=k_1+ik_2$ for the use of Wiener-Hopf technique. We will work with finite k_2 and after obtaining the solution it is set to be zero.

2.2 Boundary conditions

The flow normal to the blade will be zero and so the incident velocity perturbation will induce a scattered field which must satisfy the boundary condition which can be stated that the normal derivative of the total potential on the blade surface is zero. Also the normal derivative of the total potential and therefore of the scattered potential will be continuous everywhere on y=0. But potential itself is continuous except for the region of the blade. In addition to these condition, it is worthy to consider the behavior of the scattered potential at infinity.

Boundary conditions are specified as follow:

(i) $\partial\phi_t/\partial y=0$ on $y=0,\ p\le x\le q$ so that $\partial\phi/\partial y=ik\sin\theta e^{-ikx\cos\theta}$ on $y=0,\ p\le x\le q$

(ii) $\partial\phi_t/\partial y$ and therefore $\partial\phi/\partial y$ are continuous on $y=0,\ -\infty\le x\le\infty$

(iii) ϕ_t and therefore ϕ are continuous on $y=0,\ -\infty\le x\le p$ and $q\le x\le\infty$

(iv) For any fixed y, $y\ge 0$ or $y\le 0$

$$|\phi|<D_1\exp[-\frac{k_2}{1+M}x] \quad \text{as } x\to\infty$$

$$|\phi|<D_2\exp[\frac{k_2}{1-M}x] \quad \text{as } x\to-\infty$$

2.3 Transformed potential

Now we introduce the Fourier transform of potential.

$$\Phi(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x,y)e^{i\xi x}dx=\Phi_l(\xi,y)+\Phi_c(\xi,y)+\Phi_r(\xi,y)$$

where

$$\Phi_l(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{p}\phi(x,y)e^{i\xi x}dx=\frac{1}{\sqrt{2\pi}}e^{i\xi p}\int_{-\infty}^{p}\phi(x,y)e^{i\xi(x-p)}dx=e^{i\xi p}\Phi_-(\xi,y)$$

$$\Phi_c(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{p}^{q}\phi(x,y)e^{i\xi x}dx=e^{i\xi p}\Phi_c^+(\xi,y)=e^{i\xi q}\Phi_c^-(\xi,y)$$

$$\Phi_r(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{q}^{\infty}\phi(x,y)e^{i\xi x}dx=\frac{1}{\sqrt{2\pi}}e^{i\xi q}\int_{q}^{\infty}\phi(x,y)e^{i\xi(x-q)}dx=e^{i\xi q}\Phi_+(\xi,y)$$

From the boundary condition (iv), for a given y, $|\phi|<D_1\exp[-\frac{k_2}{1+M}x]$ as x goes to plus infinity and $|\phi|<D_2\exp[\frac{k_2}{1-M}x]$ as x goes to minus infinity where D_1 & D_2 are constants. Therefore and $\Phi_+(\xi)$ is analytic for $\mathrm{Im}(\xi)>-\frac{k_2}{1+M}$ and $\Phi_-(\xi)$ is analytic for $\mathrm{Im}(\xi)<\frac{k_2}{1-M}$. Therefore $\Phi(\xi)$ is analytic in the strip $-\frac{k_2}{1+M}<\mathrm{Im}(\xi)<\frac{k_2}{1-M}$.

2.4 Transformed equation and its solution

If now we apply a Fourier transform in x to Eq.(1) we find that

$$\frac{\partial^2}{\partial y^2}\Phi(\xi,y)-\left[(1-M^2)\xi^2-2kM\xi-k^2\right]\Phi(\xi,y)=0$$

But there are two forms of solution since ϕ and hence Φ is discontinuous across y=0.

$$\Phi(\xi,y)=\begin{cases}A_1(\xi)e^{-\gamma(\xi)y}+A_2(\xi)e^{\gamma(\xi)y} & , \quad y\ge 0\\ B_1(\xi)e^{-\gamma(\xi)y}+B_2(\xi)e^{\gamma(\xi)y} & , \quad y\le 0\end{cases}$$

where

$$\gamma(\xi)=\left[(1-M)^2\xi^2-2kM\xi-k^2\right]^{1/2}$$

The real part of $\gamma(\xi)$ is always positive when the imaginary part of ξ is in $-\frac{k_2}{1+M}<\mathrm{Im}(\xi)<\frac{k_2}{1-M}$, thus we must take $A_2(\xi)=B_1(\xi)=0$. This states that potential at infinity is physically reasonable, i.e., non-diverging. Then

$$\Phi(\xi,y)=\begin{cases}A_1(\xi)e^{-\gamma(\xi)y} & , \quad y\ge 0\\ B_2(\xi)e^{\gamma(\xi)y} & , \quad y\le 0\end{cases} \qquad (2)$$

From the boundary condition (ii), $\partial\Phi/\partial y$ is continuous across y=0.

Then $-\gamma(\xi)A_1(\xi)=\gamma(\xi)B_2(\xi)$, i.e., we can set $A_1(\xi)=-B_2(\xi)=A(\xi)$. Hence, the solution can be written as

$$\Phi(\xi,y)=\begin{cases}A(\xi)e^{-\gamma(\xi)y} & , \quad y\ge 0\\ -A(\xi)e^{\gamma(\xi)y} & , \quad y\le 0\end{cases}$$

The function A is an arbitrary function determined from the boundary condition on y=0. When a transform is discontinuous across y=0 we extend the notation:

$$\Phi_l(\xi,\pm 0)=e^{i\xi p}\Phi_-(\xi,\pm 0), \Phi_r(\xi,\pm 0)=e^{i\xi q}\Phi_+(\xi,\pm 0)$$

where in the usual way ± 0 means the limit as y tends to zero approached from positive and negative values of y respectively. Using the boundary condition (iii)

$$\Phi_l(\xi,+0)=\Phi_l(\xi,-0)=\Phi_l(\xi,0)$$
$$\Phi_r(\xi,+0)=\Phi_r(\xi,-0)=\Phi_r(\xi,0)$$

We also write for the transformed potential of $\partial\phi/\partial y$.

$$\Phi_l^{'}(\xi,\pm 0)=e^{i\xi p}\Phi_-^{'}(\xi,\pm 0), \Phi_r^{'}(\xi,\pm 0)=e^{i\xi q}\Phi_+^{'}(\xi,\pm 0)$$

Now from the boundary condition (ii),

$$\Phi_-^{'}(\xi,+0)=\Phi_-^{'}(\xi,-0)=\Phi_-^{'}(\xi,0)$$
$$\Phi_+^{'}(\xi,+0)=\Phi_+^{'}(\xi,-0)=\Phi_+^{'}(\xi,0)$$
$$\Phi_c(\xi,\pm 0)=e^{i\xi p}\Phi_c^{+}(\xi,+0)=e^{i\xi q}\Phi_c^{-}(\xi,-0)$$

2.5 Relation between transformed solution and transformed potential

Matching the expressions in the previous section with Eq.(2), we find

(a) Potential on y=0+

$$e^{i\xi p}\Phi_-(\xi,0)+\Phi_c(\xi,0+)+e^{i\xi q}\Phi_+(\xi,0)=A(\xi)$$

(b) Potential on y=0−

$$e^{i\xi p}\Phi_-(\xi,0)+\Phi_c(\xi,0-)+e^{i\xi q}\Phi_+(\xi,0)=-A(\xi)$$

(c) Normal derivative of potential on y=0

$$e^{i\xi p}\Phi_-^{'}(\xi,0)+\Phi_c^{'}(\xi,0+)+e^{i\xi q}\Phi_+^{'}(\xi,0)=-\gamma(\xi)A(\xi)$$

2.6 Construction of integral equations

After using the Wiener-Hopf technique, two integral equations are obtained.
(Detail mathematical procedure is in [1])

$$\frac{\Phi'_-(\xi,0)}{\gamma_-(\xi)} - \frac{k\sin\theta}{\sqrt{2\pi}} \frac{e^{-ipk\cos\theta}}{(\xi - k\cos\theta)\gamma_-(\xi)} + \left[\frac{k\sin\theta}{\sqrt{2\pi}} \frac{e^{-iqk\cos\theta} e^{i(q-p)\xi}}{(\xi - k\cos\theta)\gamma_-(\xi)} + \frac{e^{i(q-p)\xi}\Phi'_+(\xi,0)}{\gamma_-(\xi)}\right]_- = 0$$

$$\frac{\Phi'_+(\xi,0)}{\gamma_+(\xi)} + \frac{k\sin\theta}{\sqrt{2\pi}} \frac{e^{-iqk\cos\theta}}{\xi - k\cos\theta}\left(\frac{1}{\gamma_+(\xi)} - \frac{1}{\gamma_+(k\cos\theta)}\right) - \left[\frac{k\sin\theta}{\sqrt{2\pi}} \frac{e^{-ipk\cos\theta} e^{i(p-q)\xi}}{(\xi - k\cos\theta)\gamma_+(\xi)} + \frac{e^{i(p-q)\xi}\Phi'_-(\xi,0)}{\gamma_+(\xi)}\right]_+ = 0$$

$$[S(\xi)]_- = -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib} \frac{S(\alpha)}{\alpha - \xi} d\alpha$$

$$[S(\xi)]_+ = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{S(\alpha)}{\alpha - \xi} d\alpha$$

$$-\frac{k_2}{1+M} < a < \operatorname{Im}(\xi) < b < k_2\cos\theta$$

If we use these equations directly, we come to get strongly coupled integral equations which are abstruse to treat. Therefore, in the next section, it is shown that these two coupled equations are divided into two single integral equations.

2.7 Single integral equations

Through some mathematical procedures, new integral equation is obtained.
(Detail mathematical procedure is in [1])

$$\frac{\Sigma_+^*(\zeta)}{\Gamma_+(\zeta)} \pm \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{e^{i(q-p)\frac{\beta}{\sqrt{1-M^2}}}}{(\beta+\zeta)\Gamma_-(\beta)} \Sigma_+^*(\beta) d\beta = R(\zeta)$$

Unknown function : $\Sigma_+^*(\zeta)$

3. Conclusion

Strongly coupled integral equations are converted to two independent integral equations for the evaluation and analysis of scattering by a discontinuous surface of finite length.

4. References

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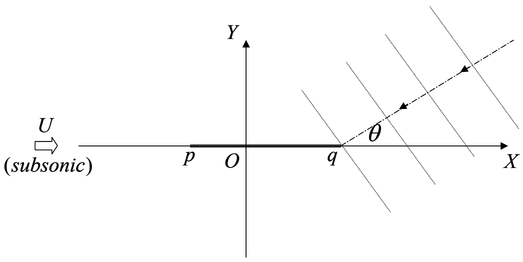

Figure 1