SIAM J. SCI. COMPUT.
Vol. 28, No. 3, pp. 1133–1152

A NEUMANN–DIRICHLET PRECONDITIONER FOR A FETI-DP FORMULATION OF THE TWO-DIMENSIONAL STOKES PROBLEM WITH MORTAR METHODS*

HYEA HYUN KIM† AND CHANG-OCK LEE†

Abstract. A FETI-DP (dual-primal finite element tearing and interconnecting) formulation for the two-dimensional Stokes problem with mortar methods is considered. Separate sets of unknowns are used for velocity on interfaces, and the mortar constraints are enforced on the velocity unknowns by Lagrange multipliers. Average constraints on edges are further introduced as primal constraints to solve the Stokes problem correctly and to obtain a scalable FETI-DP algorithm. A Neumann–Dirichlet preconditioner is shown to give a condition number bound, $C \max_{i=1,\ldots,N}\{(1+\log(H_i/h_i))^2\}$, where H_i and h_i are the subdomain size and the mesh size, respectively, and the constant C is independent of the mesh parameters H_i and h_i.

Key words. FETI-DP, mortar methods, Neumann–Dirichlet preconditioner, Stokes problem

AMS subject classifications. 65N30, 65N55, 76D07

DOI. 10.1137/030601119

1. Introduction.

In this paper, an iterative substructuring method with Lagrange multipliers is studied for the two-dimensional Stokes problem with nonconforming discretizations. Nonconforming discretizations are important for multiphysics simulations, contact-impact problems, the generation of meshes and partitions aligned with jumps in diffusion coefficients, *hp*-adaptive methods, and special discretizations in the neighborhood of singularities. Of the many methods for nonconforming discretizations, including Galerkin methods [7], we consider the mortar methods [1, 3, 20, 21] originally introduced by Bernardi, Maday, and Patera.

Dual-primal FETI (FETI-DP) methods were introduced in [11] as a generalization of the FETI (finite element tearing and interconnecting) method [12]. Continuity of solutions at subdomain corners is enforced by primal variables to improve the convergence as well as to make local problems nonsingular. Later Mandel and Tezaur [17] proved that these algorithms give the condition number bound, $C(1+\log(H/h))^2$, for both second- and fourth-order elliptic problems in two dimensions. Here, H and h denote the subdomain size and mesh size, respectively. For three-dimensional elliptic problems with heterogeneous coefficients, Klawonn, Widlund, and Dryja [14] introduced averages over individual edges and/or faces as primal constraints to obtain a method as scalable as in two dimensions. Later FETI-DP algorithms were extended to the Stokes problem in both two and three dimensions by Li [15, 16]. Edge and/or face average constraints as well as vertex constraints are selected as a set of primal constraints to enhance the convergence of the FETI-DP algorithms.

Recently, FETI-DP methods have been applied to nonconforming discretizations [8, 9, 10, 13, 18]. For elliptic problems in two dimensions, Dryja and Widlund [9, 10] proposed several preconditioners and showed the condition number bound $C(1+\log(H/h))^2$. However, the constant C depends on the ratio of mesh sizes across

*Received by the editors October 6, 2003; accepted for publication (in revised form) February 2, 2006; published electronically July 17, 2006. This work was partially supported by KRF-2002-070-C00004 and KOSEF R01-2000-00008.

http://www.siam.org/journals/sisc/28-3/60111.html

†Division of Applied Mathematics, KAIST, Daejeon 305-701, Korea (mashy@amath.kaist.ac.kr, colee@amath.kaist.ac.kr).

interfaces. Later, the current authors [13] proposed a Neumann–Dirichlet preconditioner, which gives the same condition number bound without the dependence of C upon the ratio of mesh sizes. The proposed preconditioner is similar to the previous FETI-DP preconditioners except that it solves local problems with Neumann boundary conditions on nonmortar interfaces and with a zero Dirichlet boundary condition on mortar interfaces.

The purpose of this paper is to extend the FETI-DP algorithm developed in [13] to the two-dimensional Stokes problem with mortar discretizations. The inf-sup stable $P_1(h)$-$P_0(2h)$ finite element space is considered in each subdomain. The triangulations are nonmatching across subdomain interfaces. To achieve the optimal approximation, we impose mortar matching conditions on the velocity functions. An optimal approximation of mortar methods for the Stokes problem was proved by Belgacem [2]. The inf-sup constant of the mortar finite element space is important in the analysis. If the constant is independent of mesh size and subdomain size, then the optimal order of approximation follows independent of the number of subdomains and mesh size, as in the case of elliptic problems. In [2], it was shown that the inf-sup constant is independent of mesh size for the Hood–Taylor finite element space, but this was not shown for the subdomain size. For the $P_1(h)$-$P_0(2h)$ mortar finite element space, we compute the inf-sup constant numerically by increasing the number of subdomains and decreasing mesh sizes, and we observe that the constant seems to be independent of these parameters.

We follow the FETI-DP formulation developed in [15]. The fundamental idea of the present paper is the same as one of [15], with additional technical complications caused by nonmatching grids across subdomain interfaces. Mortar matching constraints will be enforced on velocity unknowns across interfaces instead of pointwise matching constraints in conforming discretization. We introduce the primal constraints, i.e., edge average and vertex constraints, to solve the Stokes problem efficiently and correctly. We then propose a Neumann–Dirichlet preconditioner and analyze the condition number bound. The preconditioner consists of local Stokes problems with Neumann boundary conditions on nonmortar edges and a zero Dirichlet boundary condition on the remaining part of the subdomain boundary. The additional complication caused by mortar discretizations can be handled by using this preconditioner. In the analysis, the stability of the mortar projection in the $H_{00}^{1/2}$-norm is used. Our theory can be extended to Lagrange multiplier spaces with this property. Several such Lagrange multiplier spaces have been developed by Wohlmuth [20, 21].

This paper is organized as follows. Section 2 contains a brief introduction to Sobolev spaces and finite elements. In section 3, we derive a FETI-DP formulation of the Stokes problem. Section 4 is devoted to analyzing the condition number bound. Numerical results are included in section 5. Throughout this paper, C denotes a generic constant independent of mesh sizes and subdomain sizes. We will use H_i and h_i to denote the subdomain size and the typical mesh size of each subdomain Ω_i, respectively.

2. Sobolev spaces and finite elements.

2.1. A model problem. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$, and let $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the norm

$$\|v\|_{0,\Omega}^2 := \int_\Omega v^2 \, dx.$$

The space $L_0^2(\Omega)$ contains functions in $L^2(\Omega)$ with zero average $\int_\Omega v\,dx = 0$. The space $H^1(\Omega)$ consists of functions that are square integrable up to the first weak derivatives with the norm

$$\|v\|_{1,\Omega}^2 := \int \nabla v \cdot \nabla v\,dx + \int v^2\,dx.$$

The space $H_0^1(\Omega)$ is a subspace of $H^1(\Omega)$ with functions having zero trace on the boundary of Ω.

In this paper, we consider the following Stokes problem: For $\mathbf{f} \in [L^2(\Omega)]^2$, find $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ satisfying

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ -\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We then partition Ω into nonoverlapping subdomains $\{\Omega_i\}_{i=1}^N$, which are geometrically conforming; a subdomain intersects its neighbors on a whole edge or at a vertex. For each subdomain, we introduce the space $H_D^1(\Omega_i)$ to be a subspace of $H^1(\Omega_i)$ with zero trace on $\partial\Omega_i \cap \partial\Omega$, the space $L_0^2(\Omega_i)$ to be a subspace of $L^2(\Omega_i)$ with zero average, and the space Π_0, which consists of functions that are constant in each subdomain and have zero average in Ω:

$$\begin{aligned} H_D^1(\Omega_i) &:= \left\{ v \in H^1(\Omega_i) \;:\; v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \right\}, \\ L_0^2(\Omega_i) &:= \left\{ q \in L^2(\Omega_i) \;:\; \int_{\Omega_i} q\,dx = 0 \right\}, \\ \Pi_0 &:= \left\{ q_0 \;:\; q_0|_{\Omega_i} \text{ is constant and } \int_\Omega q_0\,dx = 0 \right\}. \end{aligned} \tag{2.2}$$

The problem (2.1) is then written into an equivalent variational form: Find $(\mathbf{u}, p_I, p_0) \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2 \times \prod_{i=1}^N L_0^2(\Omega_i) \times \Pi_0$ such that

$$\begin{aligned} \sum_{i=1}^N (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i} - \sum_{i=1}^N (p_I + p_0, \nabla \cdot \mathbf{v})_{\Omega_i} &= \sum_{i=1}^N (\mathbf{f}, \mathbf{v})_{\Omega_i} \quad \forall\, \mathbf{v} \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2, \\ -\sum_{i=1}^N (\nabla \cdot \mathbf{u}, q_I)_{\Omega_i} &= 0 \quad \forall\, q_I \in \prod_{i=1}^N L_0^2(\Omega_i), \\ -\sum_{i=1}^N (\nabla \cdot \mathbf{u}, q_0)_{\Omega_i} &= 0 \quad \forall\, q_0 \in \Pi_0, \\ \mathbf{u}|_{\Omega_i} - \mathbf{u}|_{\Omega_j} &= 0 \quad \forall \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j. \end{aligned} \tag{2.3}$$

Here $(\cdot,\cdot)_{\Omega_i}$ denotes the inner product in $[L^2(\Omega_i)]^n$ for $n = 1, 2$. We now introduce a finite element space to approximate the above problem. Triangulations $\Omega_i^{2h_i}$ and $\Omega_i^{h_i}$ for pressure and velocity, respectively, are given in each subdomain. The finer triangulation $\Omega_i^{h_i}$ is obtained from $\Omega_i^{2h_i}$ by connecting midpoints of edges in the triangle $\tau \in \Omega_i^{2h_i}$ so that τ is divided into four triangles. We assume that these triangulations are quasi-uniform and may not match across subdomain interfaces. The finite element space $P_1(h_i) - P_0(2h_i)$ is then associated with each subdomain Ω_i;

we denote by X_i the space of conforming linear finite elements on the triangulation $\Omega_i^{h_i}$ and by Q_i the space of functions constant on each triangles in $\Omega_i^{2h_i}$ with zero average over Ω_i:

$$X_i := \left\{ \mathbf{v}_i \in \left[H_D^1(\Omega_i) \cap C(\Omega_i) \right]^2 \;:\; \mathbf{v}_i|_\tau \text{ is piecewise linear } \forall\, \tau \in \Omega_i^{h_i} \right\},$$
$$Q_i := \left\{ q_i \in L_0^2(\Omega_i) \;:\; q_i|_\tau \text{ is constant } \forall\, \tau \in \Omega_i^{2h_i} \right\}.$$

The inf-sup stability of the space $P_1(h_i) - P_0(2h_i)$ can be shown from the macro element technique in [19] or from the inf-sup stability of the space $P_2(2h_i) - P_0(2h_i)$ in [6]. The space $P_2(2h_i) - P_0(2h_i)$ consists of piecewise quadratic functions for velocity and piecewise constant functions for pressure in the same triangulation $\Omega_i^{2h_i}$. In the proof of the stability of $P_1(h_i) - P_0(2h_i)$, we may regard $P_2(2h_i) - P_0(2h_i)$ as identical to $P_1(h_i) - P_0(2h_i)$.

Our FETI-DP formulation will be described using the space X, an approximate space for velocity, which can be discontinuous across the interfaces except corners; the space Q_I for pressure, which has zero average in each subdomain; and the space W for velocity on the interfaces, which is continuous at corners and can be discontinuous on the remaining part:

$$(2.4)\qquad \begin{aligned} X &:= \left\{ \mathbf{v} \in \prod_{i=1}^{N} X_i \;:\; \mathbf{v} \text{ is continuous at subdomain corners} \right\}, \\ Q_I &:= \prod_{i=1}^{N} Q_i, \\ W_i &:= X_i|_{\partial\Omega_i} \quad \text{for } i = 1, \dots, N, \\ W &:= \left\{ \mathbf{w} \in \prod_{i=1}^{N} W_i \;:\; \mathbf{w} \text{ is continuous at subdomain corners} \right\}. \end{aligned}$$

Throughout this paper, we will use the same notation for a finite element function and the nodal unknowns of the function. For example, $\mathbf{v}_i$ can be used to denote a finite element function or the corresponding nodal unknowns. The same holds for the notations W_i, X, W, etc.

We now introduce Sobolev spaces on the subdomain boundaries. The space $H^{1/2}(\partial\Omega_i)$ is the trace space of $H^1(\Omega_i)$ normed by

$$\|w_i\|_{1/2,\partial\Omega_i}^2 := |w_i|_{1/2,\partial\Omega_i}^2 + \frac{1}{H_i}\|w_i\|_{0,\partial\Omega_i}^2,$$

where

$$|w_i|_{1/2,\partial\Omega_i}^2 := \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|w_i(x) - w_i(y)|^2}{|x-y|^2}\, ds(x)\, ds(y).$$

For any $\Gamma_{ij} \subset \partial\Omega_i$, the space $H_{00}^{1/2}(\Gamma_{ij})$ is a set of functions in $L^2(\Gamma_{ij})$ of which zero extension to $\partial\Omega_i$ is contained in $H^{1/2}(\partial\Omega_i)$, and it is equipped with the norm

$$\|v\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 := |v|_{H^{1/2}(\Gamma_{ij})}^2 + \int_{\Gamma_{ij}} \frac{v(x)^2}{\operatorname{dist}(x, \partial\Gamma_{ij})}\, ds.$$

From section 4.1 in [22], the following relation holds for $v \in H_{00}^{1/2}(\Gamma_{ij})$:

$$C_1 \|\widetilde{v}\|_{1/2,\partial\Omega_i} \le \|v\|_{H_{00}^{1/2}(\Gamma_{ij})} \le C_2 \|\widetilde{v}\|_{1/2,\partial\Omega_i}, \tag{2.5}$$

where $\widetilde{v}$ is the zero extension of v to $\partial\Omega_i$. The inequalities in (2.5) also hold for the product spaces $[H^{1/2}(\partial\Omega_i)]^2$ and $[H_{00}^{1/2}(\Gamma_{ij})]^2$ equipped with product norms.

2.2. Mortar methods. We consider the space X for velocity and the space $P = Q_I \times \Pi_0$ for pressure to approximate the Stokes problem (2.3); see (2.4) and (2.2) for the definitions of X, Q_I, and Π_0. We will impose the mortar matching condition on the velocity functions. On the interface $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, two different triangulations are given by the triangulations in Ω_i and Ω_j. We distinguish them by choosing one as a mortar side and the other as a nonmortar side. On each subdomain Ω_i, we define

$$m_i := \{ j \ : \ \Omega_j \text{ is the mortar side of } \Gamma_{ij} \quad \forall\, \Gamma_{ij} \subset \partial\Omega_i \},$$

which is a set of subdomain indices j such that Ω_j intersects Ω_i on the interface Γ_{ij}, where Ω_j is chosen as the mortar side and Ω_i is chosen as the nonmortar side.

On each interface Γ_{ij}, we consider a trace space

$$X_{n(ij)}|_{\Gamma_{ij}}, \tag{2.6}$$

where $X_{n(ij)}$ is the finite element space on the nonmortar side of Γ_{ij}; i.e., $n(ij) = i$ if Ω_i is the nonmortar side of Γ_{ij}. The standard Lagrange multiplier space introduced in [3] is then given by

$$M_{ij} := \{ \boldsymbol{\psi} \in X_{n(ij)}|_{\Gamma_{ij}} \ : \ \boldsymbol{\psi}|_\tau \text{ is constant on } \tau \text{ which intersects } \partial\Gamma_{ij} \}.$$

Here τ is the restriction of a triangle in the triangulation $\Omega_{n(ij)}^{h_{n(ij)}}$ to the interface Γ_{ij}. We then take the Lagrange multiplier space

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}. \tag{2.7}$$

The mortar matching condition on $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_N) \in X$ is

$$\int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds = 0 \quad \forall \boldsymbol{\lambda}_{ij} \in M_{ij}, \ \forall\, i = 1, \dots, N, \ \forall j \in m_i. \tag{2.8}$$

The mortar finite element space of velocity is then given by

$$V := \{ \mathbf{v} \in X \ : \ \mathbf{v} \text{ satisfies the mortar matching condition (2.8)} \}. \tag{2.9}$$

It was shown in [2] that the best approximation property holds for the space $V \times P$, given by the Hood–Taylor finite element space for each subdomain, if it satisfies the inf-sup condition

$$\inf_{q \in P} \sup_{\mathbf{v} \in V} \frac{\int_\Omega \nabla \cdot \mathbf{v} \, q \, dx}{\|v\|_{1,\Omega}^{*} \|q\|_{0,\Omega}} \ge \beta,$$

where the constant β is independent of the mesh size and the subdomain size. Here $\|v\|_{1,\Omega}^*$ denotes the broken H^1-norm

$$\|v\|_{1,\Omega}^* = \left(\sum_{i=1}^{N} \|v\|_{1,\Omega_i}^2 \right)^{1/2} .$$

To our best knowledge, there is no mathematical proof that shows that the constant β is independent of the subdomain size. We compute the inf-sup constant numerically and observe that the constant is independent of the subdomain size as well as the mesh size; see section 5. Therefore we can get a solution as accurate as that in conforming approximations.

3. FETI-DP formulation.

3.1. Notation. We will introduce several notations which will be used in the FETI-DP formulation. We recall X_i, the velocity finite element space in Ω_i; W_i, the trace space of X_i on $\partial\Omega_i$; and X, a subspace of the product space of X_i with functions that are continuous at subdomain corners. The space W is similarly defined as a subspace of the product space of W_i. The definitions of these spaces are given in (2.4).

The unknowns $\mathbf{v}_i \in X_i$ and $\mathbf{w}_i \in W_i$ are ordered into

$$\mathbf{v}_i = \begin{pmatrix} \mathbf{v}_I^i \\ \mathbf{v}_r^i \\ \mathbf{v}_c^i \end{pmatrix}, \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_r^i \\ \mathbf{w}_c^i \end{pmatrix},$$

where the subscripts I, r, and c represent the d.o.f. (degrees of freedom) corresponding to the interior, edges, and corners, respectively. We define the following spaces based on the splitting of unknowns: X_I, the space of velocity unknowns at the interior of each subdomain; W_r, the space of velocity unknowns at the interior of both mortar and nonmortar edges; W_c, the space of velocity unknowns at global corners; and W_c^i, the space of velocity unknowns at subdomain corners:

$$\begin{aligned} X_I &= \left\{ \mathbf{v}_I \ : \ \mathbf{v}_I|_{\Omega_i} = \mathbf{v}_I^i, \quad \forall i = 1, \ldots, N \right\}, \\ W_r &= \left\{ \mathbf{w}_r \ : \ \mathbf{w}_r|_{\partial\Omega_i} = \mathbf{w}_r^i, \quad \forall i = 1, \ldots, N \right\}, \\ W_c &= \left\{ \mathbf{w}_c \ : \ \mathbf{w}_c \text{ are velocity unknowns at global corners} \right\}, \\ W_c^i &= \left\{ \mathbf{w}_c^i \ : \ \mathbf{w}_c^i \text{ are velocity unknowns at corners in } \Omega_i \right\}. \end{aligned} \tag{3.1}$$

We further define the restriction map

$$L_c^i \ : \ W_c \to W_c^i$$

that gives

$$L_c^i \mathbf{w}_c = \mathbf{w}_c^i \quad \forall i = 1, \ldots, N, \ \text{for } \mathbf{w}_c \in W_c.$$

We will now express the mortar matching condition (2.8) as a matrix form

$$B\mathbf{w} = \mathbf{0}, \tag{3.2}$$

where

$$\begin{aligned} B &= \begin{pmatrix} B_1 & \cdots & B_N \end{pmatrix}, \\ \mathbf{w} &= \begin{pmatrix} \mathbf{w}_1^t & \cdots & \mathbf{w}_N^t \end{pmatrix}^t, \qquad \mathbf{w}_i = \mathbf{v}_i|_{\partial\Omega_i}. \end{aligned}$$

Since $\mathbf{w}$ is continuous at global corners, the above equation can be written as a different form with unknowns $\mathbf{w}_r \in W_r$ and $\mathbf{w}_c \in W_c$:

$$B_r\mathbf{w}_r + B_c\mathbf{w}_c = \mathbf{0}, \tag{3.3}$$

where B_r and B_c are assembled by local matrices $B_{i,r}$ and $B_{i,c}$,

$$B_r = \begin{pmatrix} B_{1,r} & \cdots & B_{N,r} \end{pmatrix}, \qquad B_c = \sum_{i=1}^{N} B_{i,c}L_c^i.$$

These matrices $B_{i,r}$ and $B_{i,c}$ consist of the columns of B_i corresponding to the unknowns on edges and corners, respectively. We will use any of these two expressions (3.2) and (3.3) from place to place for the sake of convenience.

3.2. FETI-DP formulation with primal constraints. In this section, we formulate a FETI-DP operator with the mortar constraints (3.3). To solve the Stokes problem efficiently, we will consider the following primal constraints:

$$\int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j)\, ds = \mathbf{0} \qquad \forall i = 1, \ldots, N,\ \forall j \in m_i. \tag{3.4}$$

Note that (3.4) holds by replacing $\boldsymbol{\lambda}_{ij} = \binom{1}{0}$ or $\binom{0}{1}$ in (2.8), because constant multipliers belong to the Lagrange multiplier space M_{ij}. These constraints are average matching conditions of each velocity component on subdomain interfaces. These primal constraints were introduced by Li [15, 16] to the Stokes problem with conforming discretization. The primal constraints enlarge the size of the coarse problem so that it may lead to a fast convergence of the FETI-DP iteration.

We write (3.4) as

$$R^t(B_r\mathbf{w}_r + B_c\mathbf{w}_c) = \mathbf{0}, \tag{3.5}$$

where the matrix R is a Boolean matrix that has the number of columns equal to twice the number of interfaces Γ_{ij}, and the number of rows equal to the d.o.f. of the space M. For $\boldsymbol{\lambda} \in M$, at each interior nodal point of Γ_{ij}, $\boldsymbol{\lambda}|_{\Gamma_{ij}}$ has two components corresponding to horizontal and vertical parts of velocity function, and $R^t\boldsymbol{\lambda} = \mathbf{0}$ means that sums of each $\boldsymbol{\lambda}|_{\Gamma_{ij}}$ corresponding to the horizontal and vertical parts of velocity function are zero.

Let U be the Lagrange multiplier space corresponding to the constraints (3.5); for $\boldsymbol{\mu} \in U$, $\boldsymbol{\mu}|_{\Gamma_{ij}}$ has two components that correspond to the constraints for each horizontal velocity and vertical velocity on the interface Γ_{ij}. By introducing Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to enforce the constraints (3.3) and (3.5), we have the following mixed formulation of the problem (2.3): Find $(\mathbf{u}_I, p_I, \mathbf{u}_r, \mathbf{u}_c, p_0, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in X_I \times Q_I \times W_r \times W_c \times \Pi_0 \times U \times M$ such that

$$\begin{pmatrix} A_{II} & G_{II} & A_{Ir} & A_{Ic} & G_{I0} & \mathbf{0} & \mathbf{0} \\ G_{II}^t & \mathbf{0} & G_{rI}^t & G_{cI}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{rI} & G_{rI} & A_{rr} & A_{rc} & G_{r0} & B_r^tR & B_r^t \\ A_{cI} & G_{cI} & A_{cr} & A_{cc} & G_{c0} & B_c^tR & B_c^t \\ G_{I0}^t & \mathbf{0} & G_{r0}^t & G_{c0}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R^tB_r & R^tB_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_r & B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \\ \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{0} \\ \mathbf{f}_r \\ \mathbf{f}_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{3.6}$$

Here

$$\begin{pmatrix} A_{II} & A_{Ir} & A_{Ic} \\ A_{rI} & A_{rr} & A_{rc} \\ A_{cI} & A_{cr} & A_{cc} \end{pmatrix} \quad \text{is a stiffness matrix given by} \quad \sum_{i=1}^{N} (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i},$$

$$\begin{pmatrix} G_{II}^t & G_{rI}^t & G_{cI}^t \end{pmatrix} \quad \text{is a matrix given by} \quad \sum_{i=1}^{N} (-\nabla \cdot \mathbf{v}, p_I)_{\Omega_i},$$

$$\begin{pmatrix} G_{I0}^t & G_{r0}^t & G_{c0}^t \end{pmatrix} \quad \text{is a matrix given by} \quad \sum_{i=1}^{N} (-\nabla \cdot \mathbf{v}, p_0)_{\Omega_i}.$$

The velocity spaces X_I, W_r, and W_c are defined in (3.1). For the definitions of spaces Q_I, Π_0, and M, see (2.4), (2.2), and (2.7), respectively. Since $p_0|_{\Omega_i}$ is constant and $\mathbf{v}_I|_{\partial\Omega_i} = \mathbf{0}$ for $\mathbf{v}_I \in X_I$, the divergence theorem gives $G_{I0} = \mathbf{0}$.

Let

$$\mathbf{z}_r = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \end{pmatrix}, \quad \mathbf{z}_c = \begin{pmatrix} \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \end{pmatrix}.$$

We regard $\mathbf{z}_c$ as a primal variable in the FETI-DP formulation and then write (3.6) into

$$\begin{pmatrix} K_{rr} & K_{rc} & \widetilde{B}_r^t \\ K_{rc}^t & K_{cc} & \widetilde{B}_c^t \\ \widetilde{B}_r & \widetilde{B}_c & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_r \\ \mathbf{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{f}}_r \\ \widetilde{\mathbf{f}}_c \\ \mathbf{0} \end{pmatrix}.$$

In section 4, we will show that the matrix K_{rr} is invertible; see Lemma 4.1. After eliminating $\mathbf{z}_r$, we obtain the following equation for $\mathbf{z}_c$ and $\boldsymbol{\lambda}$:

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{cl}^t & F_{ll} \end{pmatrix} \begin{pmatrix} \mathbf{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}_c \\ \mathbf{d}_l \end{pmatrix},$$

where

$$\begin{aligned} F_{ll} &= \widetilde{B}_r K_{rr}^{-1} \widetilde{B}_r^t, \\ F_{cl} &= K_{rc}^t K_{rr}^{-1} \widetilde{B}_r^t - \widetilde{B}_c^t, \\ F_{cc} &= K_{cc} - K_{rc}^t K_{rr}^{-1} K_{rc}, \\ \mathbf{d}_l &= \widetilde{B}_r K_{rr}^{-1} \widetilde{\mathbf{f}}_r, \\ \mathbf{d}_c &= \widetilde{\mathbf{f}}_c - K_{rc}^t K_{rr}^{-1} \widetilde{f}_r. \end{aligned}$$

The matrix F_{cc}, a coarse problem in the FETI-DP formulation, is invertible; see Lemma 4.2. By eliminating $\mathbf{z}_c$, we then obtain the following equation for $\boldsymbol{\lambda}$:

$$F_{DP} \boldsymbol{\lambda} = \mathbf{d}_l - F_{cl}^t F_{cc}^{-1} \mathbf{d}_c, \tag{3.7}$$

where

$$F_{DP} = F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl}.$$

Since the primal constraints (3.4) are selected from the mortar matching condition, the solution $\boldsymbol{\lambda}$ is not uniquely determined in the space M. We define a subspace

$$\widetilde{M} = \left\{ \boldsymbol{\lambda} \in M \ : \ R^t \boldsymbol{\lambda} = \mathbf{0} \right\}, \tag{3.8}$$

where the matrix R is given in (3.5). The matrix F_{DP} is symmetric and positive definite on $\widetilde{M}$; see Remark 4.3. Hence the solution $\boldsymbol{\lambda}$ of (3.7) is uniquely determined in $\widetilde{M}$.

3.3. Preconditioner. We will define several norms of the finite element function spaces given on the interfaces and propose a preconditioner for the operator F_{DP}.

First, we define a norm for the velocity space W on the interfaces. For $\mathbf{w}_i \in W_i$, we define $S_i \mathbf{w}_i$ by

$$\begin{pmatrix} A^i_{II} & G^i_{II} & A^i_{Ir} & A^i_{Ic} \\ (G^i_{II})^t & \mathbf{0} & (G^i_{rI})^t & (G^i_{cI})^t \\ A^i_{rI} & G^i_{rI} & A^i_{rr} & A^i_{rc} \\ A^i_{cI} & G^i_{cI} & A^i_{cr} & A^i_{cc} \end{pmatrix} \begin{pmatrix} \mathbf{u}^i_I \\ p^i_I \\ \mathbf{w}^i_r \\ \mathbf{w}^i_c \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ S_i \begin{pmatrix} \mathbf{w}^i_r \\ \mathbf{w}^i_c \end{pmatrix} \end{pmatrix},$$

where the superscript i for a matrix denotes the submatrix corresponding to the subdomain Ω_i. Since the upper left 2×2 matrix

$$\begin{pmatrix} A^i_{II} & G^i_{II} \\ (G^i_{II})^t & \mathbf{0} \end{pmatrix}$$

represents the local Stokes problem with a Dirichlet boundary condition, it is invertible so that the Schur complement S_i is well defined. We then assemble local Schur complement matrices and define

$$S := \operatorname{diag}(S_1, \ldots, S_N).$$

It can be seen easily that S is symmetric and positive definite on W.

We now introduce finite element spaces given on the nonmortar interfaces:

$$\begin{aligned} W_{ij} &:= \left\{ \mathbf{w}_{ij} \in X_{n(ij)}|_{\Gamma_{ij}} \ : \ \mathbf{w}_{ij} \text{ vanishes at the end points of } \Gamma_{ij} \right\}, \\ W_n &:= \prod_{i=1}^{N} \prod_{j \in m_i} W_{ij}. \end{aligned}$$

Here $n(ij)$ denotes the nonmortar subdomain of the interface Γ_{ij}. We define by $E(\mathbf{w}_n)$ the zero extension of the function $\mathbf{w}_n \in W_n$ to the all interfaces, i.e., mortar and nonmortar interfaces. We further define the following subspaces of W and W_n that satisfy a certain set of constraints:

$$\begin{aligned} \widetilde{W} &:= \left\{ \mathbf{w} \in W \ : \ \int_{\Gamma_{ij}} (\mathbf{w}_i - \mathbf{w}_j)\, ds = 0, \quad \forall\, \Gamma_{ij} \right\}, \\ \widetilde{\widetilde{W}} &:= \left\{ \mathbf{w} \in \widetilde{W} \ : \ \int_{\partial\Omega_i} \mathbf{w}_i \cdot \mathbf{n}_i = 0, \quad \forall i \right\}, \\ \widetilde{W}_n &:= \left\{ \mathbf{w}_n \in W_n \ : \ \int_{\Gamma_{ij}} \mathbf{w}\, ds = 0, \quad \forall\, \Gamma_{ij} \right\}, \end{aligned} \tag{3.9}$$

where $\mathbf{n}_i$ denotes the outward unit normal vector on the subdomain boundary $\partial\Omega_i$.

We now introduce the Neumann–Dirichlet preconditioner $\widehat{F}_{DP}^{-1}$ given by

$$\langle \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{W}_n \setminus \{0\}} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle}. \tag{3.10}$$

In the following, we will provide an explicit form of $\widehat{F}_{DP}^{-1}$ in detail. We define the l^2-orthogonal projections P and P_n by

$$P : M \to \widetilde{M}, \qquad P_n : W_n \to \widetilde{W}_n.$$

Since the constraints on the spaces $\widetilde{M}$ and $\widetilde{W}_n$ are given locally on each nonmortar interface, these projections are composed of diagonal blocks of projections defined on each nonmortar interface,

$$P = \operatorname{diag}_{i=1,\dots,N} \left(\operatorname{diag}_{j \in m_i} \left(P^{ij} \right) \right), \quad P_n = \operatorname{diag}_{i=1,\dots,N} \left(\operatorname{diag}_{j \in m_i} \left(P_n^{ij} \right) \right), \tag{3.11}$$

where P^{ij} and P_n^{ij} are l^2-orthogonal projections given by

$$P^{ij} : M|_{\Gamma_{ij}} \to \widetilde{M}|_{\Gamma_{ij}}, \qquad P_n^{ij} : W_n|_{\Gamma_{ij}} \to \widetilde{W}_n|_{\Gamma_{ij}}.$$

We define the restriction R_{ij} and the extension E_{ij}^i by

$$R_{ij} : W_n \to W_{ij}, \quad E_{ij}^i : W_{ij} \to W_i,$$

and then express the zero extension $E(\mathbf{w}_n) = (\mathbf{w}_1, \dots, \mathbf{w}_N)$ as

$$\mathbf{w}_i = E_i \mathbf{w}_n \quad \text{with} \quad E_i = \sum_{j \in m_i} E_{ij}^i R_{ij}. \tag{3.12}$$

Using this notation, the formula (3.10) is written as

$$\langle \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{W}_n \setminus \{0\}} \frac{\langle \widehat{B}\mathbf{w}_n, \boldsymbol{\lambda} \rangle^2}{\langle \widehat{S}\mathbf{w}_n, \mathbf{w}_n \rangle}, \tag{3.13}$$

where

$$\widehat{S} = \sum_{i=1}^{N} E_i^t S_i E_i, \qquad \widehat{B} = \operatorname{diag}_{i=1}^{N} \left(\operatorname{diag}_{j \in m_i} (B_{ij}) \right). \tag{3.14}$$

Here the matrix B_{ij} is a block of B_i corresponding to the unknowns of the nonmortar interface $\Gamma_{ij} \subset \partial\Omega_i$. It is easy to check that

$$\widehat{B} : \widetilde{W}_n \to \widetilde{M}$$

is one-to-one for $\dim(\widetilde{W}_n) = \dim(\widetilde{M})$ and $\widehat{B}(\widetilde{W}_n) \subset \widetilde{M}$, and that $\widehat{S}$ is symmetric and positive definite on $\widetilde{W}_n$; see (3.8) and (3.9). Let

$$\widehat{B}_p = P^t \widehat{B} P_n, \qquad \widehat{S}_p = P_n^t \widehat{S} P_n. \tag{3.15}$$

These operators are invertible, and their inverses are denoted by $\widehat{B}_p^{-1}$ and $\widehat{S}_p^{-1}$, respectively. The maximum in (3.13) occurs when $\mathbf{w}_n \in \widetilde{W}_n$ satisfies $\widehat{S}_p \mathbf{w}_n = \widehat{B}_p^t \boldsymbol{\lambda}$, and it gives

$$\langle \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \langle \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle.$$

Since $\widehat{B}_p$ is block diagonal, $\widehat{B}_p^{-1}$ can be written as

$$\widehat{B}_p^{-1} = \mathrm{diag}_{i=1}^{N}\left(\mathrm{diag}_{j\in m_i}\left((B_p^{ij})^{-1}\right)\right), \qquad B_p^{ij} = (P^{ij})^t B_{ij} P_n^{ij}. \tag{3.16}$$

By using the expressions in (3.11), (3.12), and (3.14)–(3.16), we obtain

$$\widehat{F}_{DP}^{-1} = (\widehat{B}_p^t)^{-1}\widehat{S}_p\widehat{B}_p^{-1} = \sum_{i=1}^{N} B_{i,n}^t S_i B_{i,n},$$

where $B_{i,n}$ is given by

$$B_{i,n} = \begin{pmatrix} \mathrm{diag}_{j\in m_i}\left(P_n^{ij}(B_p^{ij})^{-1}\right) \\ 0 \end{pmatrix} R_i.$$

Here $R_i : M \to \Pi_{j\in m_i} M_{ij}$ is the restriction, and the zero submatrix corresponds to the unknowns of the mortar edges and corners that belong to Ω_i. The matrix $B_{i,n}$ provides each subdomain problem with Neumann boundary conditions on the nonmortar edges and a zero Dirichlet boundary condition on the remaining part of the subdomain boundary. Hence we call it a Neumann–Dirichlet preconditioner.

4. Condition number estimation. In this section we analyze the condition number bound of the FETI-DP operator with the Neumann–Dirichlet preconditioner. In advance, we will provide the following two lemmas, which prove that the matrices K_{rr} and F_{cc} are invertible.

LEMMA 4.1. *The matrix K_{rr} is invertible on $X_I \times Q_I \times W_r$.*

Proof. Since the upper left 2×2 matrix of K_{rr} is invertible, it suffices to show that the Schur complement S_{rr} is invertible on W_r:

$$\begin{pmatrix} A_{II} & G_{II} & A_{Ir} \\ G_{II}^t & \mathbf{0} & G_{rI}^t \\ A_{rI} & G_{rI} & A_{rr} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ S_{rr}\mathbf{u}_r \end{pmatrix}. \tag{4.1}$$

For any $\mathbf{u}_r \in W_r$, we consider

$$\begin{aligned} \mathbf{u}_r^t S_{rr} \mathbf{u}_r &= \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \end{pmatrix}^t \begin{pmatrix} A_{II} & G_{II} & A_{Ir} \\ G_{II}^t & \mathbf{0} & G_{rI}^t \\ A_{rI} & G_{rI} & A_{rr} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_r \end{pmatrix}^t \begin{pmatrix} A_{II} & A_{Ir} \\ A_{rI} & A_{rr} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_r \end{pmatrix} + 2p_I^t \begin{pmatrix} G_{II}^t & G_{rI}^t \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_r \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_r \end{pmatrix}^t \begin{pmatrix} A_{II} & A_{Ir} \\ A_{rI} & A_{rr} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_r \end{pmatrix}. \end{aligned}$$

Here $\mathbf{u}_I$ and p_I are the solution of (4.1), and we have used that $G_{II}^t\mathbf{u}_I + G_{rI}^t\mathbf{u}_r = \mathbf{0}$. The last equality in the above equation gives that S_{rr} is symmetric and positive definite on W_r. $\square$

LEMMA 4.2. *Assume that the domain Ω has the triangulation to satisfy that*

$$\begin{aligned} -\sum_i (p_0, \nabla\cdot\mathbf{v}_r)_{\Omega_i} + \sum_{i,j} \boldsymbol{\mu}\cdot\int_{\Gamma_{ij}} (\mathbf{v}_r^i - \mathbf{v}_r^j)\, ds = 0 \quad \forall\, \mathbf{v}_r \in W_r, \\ -\sum_i (p_0, \nabla\cdot\mathbf{v}_c)_{\Omega_i} + \sum_{i,j} \boldsymbol{\mu}\cdot\int_{\Gamma_{ij}} (\mathbf{v}_c^i - \mathbf{v}_c^j)\, ds = 0 \quad \forall\, \mathbf{v}_c \in W_c \end{aligned}$$

give the solution $\binom{p_0}{\boldsymbol{\mu}}$. Then the coarse problem F_{cc} is invertible.

Proof. We define a Schur complement matrix

$$\begin{pmatrix} A_{II} & G_{II} & A_{Ir} & A_{Ic} \\ G_{II}^t & 0 & G_{rI}^t & G_{cI}^t \\ A_{rI} & G_{rI} & A_{rr} & A_{rc} \\ A_{cI} & G_{cI} & A_{cr} & A_{cc} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \\ \mathbf{u}_c \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \begin{pmatrix} S_{rr} & S_{rc} \\ S_{cr} & S_{cc} \end{pmatrix} \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_c \end{pmatrix} \end{pmatrix}.$$

As in the proof of the previous lemma, we can show that the Schur complement

$$\begin{pmatrix} S_{rr} & S_{rc} \\ S_{cr} & S_{cc} \end{pmatrix}$$

is symmetric and positive definite. The matrix F_{cc} is then given by

$$\begin{pmatrix} S_{rr} & S_{rc} & G_{r0} & B_r^t R \\ S_{cr} & S_{cc} & G_{c0} & B_c^t R \\ G_{r0}^t & G_{c0}^t & \mathbf{0} & \mathbf{0} \\ R^t B_r & R^t B_c & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ F_{cc} \begin{pmatrix} \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \end{pmatrix} \end{pmatrix}. \tag{4.2}$$

We assume that $\mathbf{u}_c$, p_0, and $\boldsymbol{\mu}$ give

$$F_{cc} \begin{pmatrix} \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \end{pmatrix} = 0,$$

and we will show that they are zero. Multiplying (4.2) by $\begin{pmatrix} \mathbf{u}_r^t & \mathbf{u}_c^t & -p_0^t & -\boldsymbol{\mu}^t \end{pmatrix}$, we obtain

$$\begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_c \end{pmatrix}^t \begin{pmatrix} S_{rr} & S_{rc} \\ S_{cr} & S_{cc} \end{pmatrix} \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_c \end{pmatrix} + \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_c \end{pmatrix}^t \begin{pmatrix} G_{r0} & B_r^t R \\ G_{c0} & B_c^t R \end{pmatrix} \begin{pmatrix} p_0 \\ \boldsymbol{\mu} \end{pmatrix} \\ - \begin{pmatrix} p_0 \\ \boldsymbol{\mu} \end{pmatrix}^t \begin{pmatrix} G_{r0}^t & G_{c0}^t \\ R^t B_r & R^t B_c \end{pmatrix} \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_c \end{pmatrix} = 0.$$

It follows that $\left(\begin{smallmatrix} \mathbf{u}_r \\ \mathbf{u}_c \end{smallmatrix}\right) = 0$ because the Schur complement matrix is symmetric and positive definite and the last two terms cancel each other. Equation (4.2) then reduces to

$$\begin{pmatrix} G_{r0} & B_r^t R \\ G_{c0} & B_c^t R \end{pmatrix} \begin{pmatrix} p_0 \\ \boldsymbol{\mu} \end{pmatrix} = 0.$$

This is equivalent to the equation in the assumption. Hence we have $\left(\begin{smallmatrix} p_0 \\ \boldsymbol{\mu} \end{smallmatrix}\right) = 0$. ☐

Remark 4.1. Most triangulations of the domain Ω satisfy the assumption of Lemma 4.2 because the number of velocity unknowns $\mathbf{v}_r$ and $\mathbf{v}_c$ is usually grater than the number of unknowns p_0 and $\boldsymbol{\mu}$.

LEMMA 4.3. *We have*

$$B(\widetilde{\widetilde{W}}) = B(\widetilde{W}) = \widetilde{M}.$$

Proof. From $\widetilde{\widetilde{W}} \subset \widetilde{W}$, the inclusion $B(\widetilde{\widetilde{W}}) \subset B(\widetilde{W})$ is obvious.

We will show that

$$B(\widetilde{W}) \subset B(\widetilde{\widetilde{W}}).$$

Let $E(\mathbf{w}_n) = (\mathbf{w}_1, \ldots, \mathbf{w}_N) \in W$ be the zero extension of $\mathbf{w}_n \in W_n$. Since $\mathbf{w}_j|_{\Gamma_{ij}} = 0$ for $j \in m_i$, i.e., for functions on mortar interfaces, and $E(\mathbf{w}_n)$ has zero value at subdomain corners, the identity holds

$$B(E(\mathbf{w}_n)) = \widehat{B}\mathbf{w}_n, \tag{4.3}$$

where the matrix $\widehat{B}$ is defined in (3.14), and it is one-to-one from $\widetilde{W}_n$ onto $\widetilde{M}$,

$$\widehat{B}(\widetilde{W}_n) = \widetilde{M}. \tag{4.4}$$

For $\mathbf{w}_n \in \widetilde{W}_n$, the zero extension $E(\mathbf{w}_n) = (\mathbf{w}_1, \ldots, \mathbf{w}_N)$ satisfies

$$\int_{\Gamma_{ij}} \mathbf{w}_i \, ds = \mathbf{0} \qquad \forall\, i = 1, \ldots, N, \quad \forall \Gamma_{ij} \subset \partial\Omega_i,$$

and then the compatibility condition of the local Stokes problem holds:

$$\int_{\partial\Omega_i} \mathbf{w}_i \cdot \mathbf{n}_i \, ds = 0.$$

This implies that $E(\mathbf{w}_n)$, for $\mathbf{w}_n \in \widetilde{W}_n$, belongs to the space $\widetilde{\widetilde{W}}$. From this fact and (4.3) we obtain

$$\widehat{B}(\widetilde{W}_n) \subset B(\widetilde{\widetilde{W}}). \tag{4.5}$$

From the definition of $\widetilde{W}$ and $\widetilde{M}$, we have

$$B(\widetilde{W}) = \widetilde{M}. \tag{4.6}$$

Combining (4.6), (4.4), and (4.5), we prove $B(\widetilde{W}) \subset B(\widetilde{\widetilde{W}})$. $\square$

Remark 4.2. Lemma 4.3 says that the constraints

$$\int_{\partial\Omega_i} \mathbf{w}_i \cdot \mathbf{n}_i \, ds = 0$$

do not affect the range space $B(\widetilde{W})$.

We now provide the following well-known identity that is useful for the analysis of the condition number bound.

LEMMA 4.4. *For $\boldsymbol{\lambda} \in \widetilde{M}$, we have*

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w} \in \widetilde{\widetilde{W}} \setminus \{0\}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle}.$$

Proof. The problem (3.6) is equivalent to the following min-max problem:

$$\max_{\boldsymbol{\lambda} \in B(\widetilde{\widetilde{W}})} \min_{\mathbf{w} \in \widetilde{\widetilde{W}}} \left\{ \frac{1}{2} \langle S\mathbf{w}, \mathbf{w} \rangle - \langle \mathbf{d}, \mathbf{w} \rangle + \langle B\mathbf{w}, \boldsymbol{\lambda} \rangle \right\}, \tag{4.7}$$

where $\mathbf{d}$ is the Schur complement forcing vector obtained from $\begin{pmatrix} \mathbf{f}_I^t & \mathbf{0}^t & \mathbf{f}_r^t & \mathbf{f}_c^t \end{pmatrix}^t$ after solving local Stokes problems.

Let P_W be the l^2-orthogonal projection

$$P_W : W \to \widetilde{\widetilde{W}}.$$

Note that $B(\widetilde{\widetilde{W}}) = \widetilde{M}$ from Lemma 4.3, and P is the projection from M onto $\widetilde{M}$ introduced in section 3. We consider

$$\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_N) \quad \text{such that} \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{u}_r|_{\partial\Omega_i} \\ L_c^i \mathbf{u}_c \end{pmatrix},$$

where $(\mathbf{u}_r, \mathbf{u}_c)$ is from the solution of (3.6). The fifth and sixth rows in (3.6) give

$$\int_{\partial\Omega_i} \mathbf{w}_i \cdot \mathbf{n}_i \, ds = 0 \quad \forall i = 1, \dots, N,$$
$$\int_{\Gamma_{ij}} (\mathbf{w}_i - \mathbf{w}_j) \, ds = 0 \quad \forall\, \Gamma_{ij}.$$

These equations imply that $\mathbf{w}$ belongs to $\widetilde{\widetilde{W}}$. By taking the Euler–Lagrangian in (4.7), we can see that $(\mathbf{w}, \boldsymbol{\lambda}) \in \widetilde{\widetilde{W}} \times \widetilde{M}$ from the solution of (3.6) satisfies

$$\begin{pmatrix} S_p & B_p^t \\ B_p & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} P_W^t \mathbf{d} \\ \mathbf{0} \end{pmatrix}, \tag{4.8}$$

where

$$S_p = P_W^t S P_W, \qquad B_p = P^t B P_W.$$

Since S is symmetric and positive definite (s.p.d.) on $\widetilde{\widetilde{W}}$, the equation for $\boldsymbol{\lambda}$ follows by eliminating $\mathbf{w}$ in (4.8):

$$B_p S_p^{-1} B_p^t \boldsymbol{\lambda} = B_p S_p^{-1} \mathbf{d},$$

which is the same as (3.7). Therefore we have the identity

$$B_p S_p^{-1} B_p^t = F_{DP}. \tag{4.9}$$

For $\boldsymbol{\lambda} \in \widetilde{M}$, we consider

$$\max_{\mathbf{w} \in \widetilde{\widetilde{W}} \setminus \{0\}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle}. \tag{4.10}$$

Since S is s.p.d. on $\widetilde{\widetilde{W}}$, the maximum in (4.10) occurs when $S_p \mathbf{w} = B_p^t \boldsymbol{\lambda}$, and it gives

$$\max_{\mathbf{w} \in \widetilde{\widetilde{W}} \setminus \{0\}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle} = \langle B_p S_p^{-1} B_p^t \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle. \tag{4.11}$$

From (4.9) and (4.11), the required identity follows. ∎

Remark 4.3. From (4.9), we can see that F_{DP} is s.p.d. on $\widetilde{M}$.

Remark 4.4. From Lemma 4.4, $E(\mathbf{w}_n) \in \widetilde{\widetilde{W}}$ for $\mathbf{w}_n \in \widetilde{W}_n$, and (3.10), we obtain the following lower bound of the FETI-DP operator:

$$\begin{aligned}\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle &= \max_{\mathbf{w}\in\widetilde{\widetilde{W}}\setminus\{0\}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda}\rangle^2}{\langle S\mathbf{w}, \mathbf{w}\rangle} \\ &\geq \max_{\mathbf{w}_n\in\widetilde{W}_n\setminus\{0\}} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda}\rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n)\rangle} \\ &= \langle \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle.\end{aligned}$$

The following lemma can be found in Bramble and Pasciak [5].

LEMMA 4.5. *For $\mathbf{w}_i \in W_i$, we have*

$$C_1\beta^2\langle S_i\mathbf{w}_i, \mathbf{w}_i\rangle \leq |\mathbf{w}_i|^2_{1/2,\partial\Omega_i} \leq C_2\langle S_i\mathbf{w}_i, \mathbf{w}_i\rangle,$$

where β is the inf-sup constant of the finite element space associated with the subdomain Ω_i and the constants C_1 and C_2 are independent of h_i and H_i.

Since we have chosen the inf-sup stable $P_1(h) - P_0(2h)$ finite element space, the constant β is independent of h_i and H_i.

We also have the following result, which is derived in [17, Lemma 5.1].

LEMMA 4.6. *For $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_N) \in W$, we have*

$$\|\mathbf{w}_i - \mathbf{w}_j\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C \max_{l\in\{i,j\}} \left\{\left(1 + \log\frac{H_l}{h_l}\right)^2\right\} \left(|\mathbf{w}_i|^2_{1/2,\partial\Omega_i} + |\mathbf{w}_j|^2_{1/2,\partial\Omega_j}\right).$$

DEFINITION 4.7. *We define the mortar projection $\pi_{ij} : [H^{1/2}_{00}(\Gamma_{ij})]^2 \to W_{ij}$ by*

$$\int_{\Gamma_{ij}} (\mathbf{v} - \pi_{ij}\mathbf{v}) \cdot \boldsymbol{\lambda}_{ij}\, ds \;= 0 \;\; \forall \boldsymbol{\lambda}_{ij} \in M_{ij}.$$

From Lemma 2.2 in [1], π_{ij} is continuous on $H^{1/2}_{00}(\Gamma_{ij})$; i.e., there exists a constant C such that

$$\|\pi_{ij}\mathbf{v}\|_{H^{1/2}_{00}(\Gamma_{ij})} \leq C\|\mathbf{v}\|_{H^{1/2}_{00}(\Gamma_{ij})} \qquad \forall \mathbf{v} \in [H^{1/2}_{00}(\Gamma_{ij})]^2. \tag{4.12}$$

We now provide an upper bound of the FETI-DP operator.

LEMMA 4.8. *For $\boldsymbol{\lambda} \in \widetilde{M}$, we have*

$$\max_{\mathbf{w}\in\widetilde{\widetilde{W}}\setminus\{0\}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda}\rangle^2}{\langle S\mathbf{w}, \mathbf{w}\rangle} \leq C\frac{1}{\beta^2} \max_{i=1,\ldots,N} \left\{\left(1 + \log\frac{H_i}{h_i}\right)^2\right\} \langle \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle.$$

Proof. From the definition of π_{ij}, we have

(4.13)

$$\langle B\mathbf{w}, \boldsymbol{\lambda}\rangle = \sum_{i=1}^{N}\sum_{j\in m_i}\int_{\Gamma_{ij}} (\mathbf{w}_i - \mathbf{w}_j) \cdot \boldsymbol{\lambda}_{ij}\, ds = \sum_{i=1}^{N}\sum_{j\in m_i}\int_{\Gamma_{ij}} \pi_{ij}(\mathbf{w}_i - \mathbf{w}_j) \cdot \boldsymbol{\lambda}_{ij}\, ds.$$

Let $\mathbf{z}_n \in W_n$ be $\mathbf{z}_n|_{\Gamma_{ij}} = \pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)$. Since $\binom{1}{0}, \binom{0}{1} \in M_{ij}$ and $\mathbf{w} \in \widetilde{\widetilde{W}}$, we have

$$\int_{\Gamma_{ij}} \mathbf{z}_n \, ds = \int_{\Gamma_{ij}} (\mathbf{w}_i - \mathbf{w}_j) \, ds = 0, \tag{4.14}$$

so that $\mathbf{z}_n \in \widetilde{W}_n$. From (3.10), the definition of $\widehat{F}_{DP}$, (4.13) can be bounded by

$$\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2 = \langle BE(\mathbf{z}_n), \boldsymbol{\lambda} \rangle^2 \le \langle \widehat{F}_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \langle SE(\mathbf{z}_n), E(\mathbf{z}_n) \rangle, \tag{4.15}$$

where $E(\mathbf{z}_n) = (\mathbf{z}_1, \dots, \mathbf{z}_N)$ is the zero extension of $\mathbf{z}_n$. From Lemma 4.5, (2.5), (4.12), and Lemma 4.6, we obtain

$$\begin{aligned}
\langle SE(\mathbf{z}_n), E(\mathbf{z}_n) \rangle &= \sum_{i=1}^N \langle S_i \mathbf{z}_i, \mathbf{z}_i \rangle \\
&\le C \frac{1}{\beta^2} \sum_{i=1}^N |\mathbf{z}_i|^2_{1/2, \partial\Omega_i} \\
&\le C \frac{1}{\beta^2} \sum_{i=1}^N \sum_{j \in m_i} \|\mathbf{w}_i - \mathbf{w}_j\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \\
&\le C \frac{1}{\beta^2} \max_{i=1,\dots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \sum_{i=1}^N |\mathbf{w}_i|^2_{1/2, \partial\Omega_i} \\
&\le C \frac{1}{\beta^2} \max_{i=1,\dots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \langle S\mathbf{w}, \mathbf{w} \rangle.
\end{aligned} \tag{4.16}$$

The estimates (4.15) and (4.16) give the required bound. □

From Remark 4.4 and Lemmas 4.4 and 4.8, we obtain the condition number bound of our FETI-DP algorithm.

THEOREM 4.9. *The FETI-DP algorithm with the Neumann–Dirichlet preconditioner* (3.10) *has the condition number bound*

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \le C \frac{1}{\beta^2} \max_{i=1,\dots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\}.$$

5. Numerical results. In this section, we provide numerical tests for the proposed FETI-DP algorithm. The following Stokes problem is considered:

$$\begin{aligned}
-\triangle \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
-\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega,
\end{aligned} \tag{5.1}$$

where Ω is a unit rectangle in $\mathbb{R}^2$ and $\mathbf{f}$ is given from the exact solution

$$\mathbf{u} = \begin{pmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{pmatrix} \quad \text{and} \quad p = x^2 - y^2.$$

We consider only the uniform partition of Ω. The notation $N = 4 \times 4$ means that Ω is partitioned into 4×4 square subdomains. With this partition, we triangulate

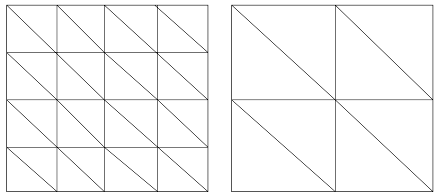

FIG. 1. *Triangulations for velocity (left) and for pressure (right) when* $n = 5$.

TABLE 1

CG iterations (condition number) when n (the number of nodes) increases with a fixed $N = 4\times 4$ *(the number of subdomains):* $\widehat{F}_{DP}^{-1}F_{DP}$ *(with preconditioner),* F_{DP} *(without preconditioner).*

n	Matching		Nonmatching	
	F_{DP}	$\widehat{F}_{DP}^{-1}F_{DP}$	F_{DP}	$\widehat{F}_{DP}^{-1}F_{DP}$
5	12(5.23)	9(2.62)	16(8.35)	12(3.75)
9	24(2.50e+1)	13(4.39)	50(1.15e+2)	15(5.79)
17	37(6.68e+1)	15(5.94)	86(5.01e+2)	17(7.93)
33	45(1.45e+2)	17(7.75)	119(1.31e+3)	20(9.88)
65	58(2.69e+2)	19(9.85)	153(3.29e+3)	22(1.20e+1)

TABLE 2

CG iterations (condition number) when N (the number of subdomains) increases with $n = 5$ *and 9:* $\widehat{F}_{DP}^{-1}F_{DP}$ *(with preconditioner),* F_{DP} *(without preconditioner).*

n	N	Matching		Nonmatching	
		F_{DP}	$\widehat{F}_{DP}^{-1}F_{DP}$	F_{DP}	$\widehat{F}_{DP}^{-1}F_{DP}$
5	4×4	12(5.23)	9(2.62)	16(8.35)	12(3.75)
	8×8	12(5.42)	9(2.62)	16(9.18)	12(3.68)
	16×16	10(5.54)	9(2.55)	16(9.57)	11(3.42)
	32×32	10(5.61)	9(2.53)	16(10.88)	12(3.78)
9	4×4	24(2.50e+1)	13(4.39)	50(1.15e+2)	15(5.79)
	8×8	25(2.60e+1)	13(4.35)	53(1.19e+2)	15(6.21)
	16×16	24(2.62e+1)	12(4.27)	57(1.34e+2)	16(6.27)
	32×32	23(2.62e+1)	12(4.27)	56(1.25e+2)	16(6.24)

each subdomain in the following manner. Let $n = 4k + 1$ with k an integer. For matching triangulations, we take the same uniform triangulation in each subdomain with $(n-1)/2+1$ nodes, including end points on both horizontal and vertical edges. We denote it by $\Omega_i^{2h_i}$, a triangulation for the pressure. A triangulation for the velocity, $\Omega_i^{h_i}$, is then given by dividing each triangle in $\Omega_i^{2h_i}$ into four triangles, as in Figure 1. For nonmatching grids, we take randomly $(n-1)/2+1$ nodes on each horizontal and vertical edge and then generate a nonuniform structured triangulation $\Omega_i^{2h_i}$ from the nodes. A triangulation $\Omega_i^{h_i}$ is obtained from $\Omega_i^{2h_i}$, as before.

We solved the FETI-DP operator with and without preconditioner, varying N and n on both matching and nonmatching triangulations. The CG (conjugate gradient) iteration is terminated when the relative residual reduces by 10^{-6}.

In Tables 1 and 2, the number of CG iterations and the corresponding condition number are shown, varying N and n. Table 1 shows the numbers when $(n-1)$ increases two-fold with fixed $N = 4\times 4$. The preconditioner performs well, and the condition numbers seem to exhibit $\log^2$-growth as n increases. The preconditioner is much more efficient for the nonmatching case. Table 2 shows the performance of the

TABLE 3

Errors (factors) on matching grids: $\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$ *(broken* H^1*-norm error for velocity),* $\|\mathbf{u}-\mathbf{u}^h\|_0$ *(*L^2*-norm error for velocity),* $\|p-p^h\|_0$ *(*L^2*-norm error for pressure).*

$N = 4\times 4$	$n = 5$	$n = 9$	$\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|p-p^h\|_0$
n	N	N			
5	4×4		3.37e-1	3.75e-3	1.07e-1
9	8×8	4×4	1.72e-1 (0.510)	1.02e-3 (0.272)	5.99e-2 (0.559)
17	16×16	8×8	8.64e-2 (0.502)	2.64e-4 (0.258)	3.08e-2 (0.514)
33	32×32	16×16	4.32e-2 (0.500)	6.65e-5 (0.258)	1.55e-2 (0.503)
65		32×32	2.16e-2 (0.500)	1.66e-5 (0.249)	7.79e-3 (0.502)

TABLE 4

Errors (factors) on nonmatching grids when n increases with a fixed $N = 4\times 4$*:* $\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$ *(broken* H^1*-norm error for velocity),* $\|\mathbf{u}-\mathbf{u}^h\|_0$ *(*L^2*-norm error for velocity),* $\|p-p^h\|_0$ *(*L^2*-norm error for pressure).*

n	$\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|p-p^h\|_0$
5	3.41e-1	3.79e-3	1.05e-1
9	1.78e-1 (0.521)	1.10e-3 (0.290)	6.08e-2 (0.579)
17	8.95e-2 (0.502)	2.85e-4 (0.259)	3.16e-2 (0.517)
33	4.48e-2 (0.500)	7.21e-5 (0.252)	1.58e-2 (0.500)
65	2.24e-2 (0.500)	1.81e-5 (0.251)	7.93e-3 (0.501)

TABLE 5

Errors (factors) on nonmatching grids when N increases with $n = 5$ *and* 9*:* $\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$ *(broken* H^1*-norm error for velocity),* $\|\mathbf{u}-\mathbf{u}^h\|_0$ *(*L^2*-norm error for velocity),* $\|p-p^h\|_0$ *(*L^2*-norm error for pressure).*

n	N	$\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|p-p^h\|_0$
5	4×4	1.78e-1	1.10e-3	6.08e-2
	8×8	8.95e-2 (0.502)	2.94e-4 (0.269)	3.28e-2 (0.539)
	16×16	4.49e-2 (0.501)	7.33e-5 (0.249)	1.63e-2 (0.496)
	32×32	2.25e-2 (0.501)	1.84e-5 (0.251)	8.18e-3 (0.501)
9	4×4	3.37e-1	3.75e-4	1.07e-1
	8×8	1.72e-1 (0.510)	1.02e-3 (0.272)	5.99e-2 (0.559)
	16×16	8.64e-2 (0.502)	2.64e-4 (0.258)	3.08e-2 (0.514)
	33×32	4.32e-2 (0.500)	6.65e-5 (0.258)	1.55e-2 (0.503)

preconditioner when N increases with $n = 5$ and 9. The CG iteration becomes stable as N increases.

We have further observed the convergent behaviors of the approximated solution. The H^1 and L^2-errors for velocity and pressure are examined. The errors and reduction factors are shown in Table 3 for various N and n on matching grids. Three cases are considered: when $n-1$ increases two-fold with a fixed $N = 4\times 4$ and when N increases two-fold in both horizontal and vertical edges of Ω with a fixed $n = 5$ or 9. For all cases, we can see that the H^1-error for velocity and the L^2-error for pressure reduce by half, and the L^2-error for velocity reduces by quarter. These factors are optimal for the $P_1(h) - P_0(2h)$ finite element space.

Tables 4–5 show errors and reduction factors for the nonmatching case. In Table 4, we observe the optimal convergence of errors as $(n-1)$ increases two-fold with a fixed $N = 4\times 4$. When $n = 5$ and 9, as N increases, the errors also show the optimal convergence in Table 5.

As mentioned in section 2, if the inf-sup constant of the space $V\times P$ is independent

TABLE 6
Inf-sup constant β_0 when N increases with $n = 5$ and $n = 9$.

N	$n = 5$		$n = 9$	
	Nonmatching	Matching	Nonmatching	Matching
4×4	0.5780	0.5785	0.5921	0.5924
8×8	0.5293	0.5294	0.5352	0.5353
16×16	0.5008	0.5010	0.5041	0.5042
32×32	0.4827	0.4828	0.4854	0.4848

TABLE 7
Inf-sup constant β_0 when n increases with a fixed $N = 4 \times 4$.

n	Nonmatching	Matching
5	0.5780	0.5785
9	0.5921	0.5294
17	0.5966	0.5967
33	0.5973	0.5979
65	0.5983	0.5983

of N and n, then the optimality of approximation can be shown for the space $V \times P$. Let β^* and β be the inf-sup constants for the space $V \times P$ and the $P_1(h) - P_0(2h)$ finite elements, respectively, and β_0 be the inf-sup constant for the space $V \times \Pi_0$; see (2.9) and (2.2) for the definition. The constant β^* depends on β and β_0 from the trick conceived by Boland and Nicolaides [4]. Hence, if the constant β_0 is independent of n and N, then the same holds for β^*. We showed that β_0 is independent of n and N numerically for various N and n. The results are given in Tables 6 and 7 for both matching and nonmatching cases. We observe that β_0 becomes stable as N or n increases.

REFERENCES

[1] F. B. BELGACEM, *The mortar finite element method with Lagrange multipliers*, Numer. Math., 84 (1999), pp. 173–197.

[2] F. B. BELGACEM, *The mixed mortar finite element method for the incompressible Stokes problem: Convergence analysis*, SIAM J. Numer. Anal., 37 (2000), pp. 1085–1100.

[3] C. BERNARDI, Y. MADAY, AND A. T. PATERA, *A new nonconforming approach to domain decomposition: The mortar element method*, in Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, Vol. XI (Paris, 1989–1991), Pitman Res. Notes Math. Ser. 299, Longman Scientific and Technical, Harlow, UK, 1994, pp. 13–51.

[4] J. M. BOLAND AND R. A. NICOLAIDES, *Stability of finite elements under divergence constraints*, SIAM J. Numer. Anal., 20 (1983), pp. 722–731.

[5] J. H. BRAMBLE AND J. E. PASCIAK, *A domain decomposition technique for Stokes problems*, Appl. Numer. Math., 6 (1990), pp. 251–261.

[6] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer Ser. Comput. Math. 15, Springer-Verlag, New York, 1991.

[7] B. COCKBURN AND C.-W. SHU, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440–2463.

[8] M. DRYJA AND W. PROSKUROWSKI, *A FETI-DP method for the mortar discretization of elliptic problems with discontinuous coefficients*, in Domain Decomposition Methods in Science and Engineering, Lecture Notes Comput. Sci. Eng. 40, Springer, Berlin, 2004, pp. 345–352.

[9] M. DRYJA AND O. B. WIDLUND, *A FETI-DP method for a mortar discretization of elliptic problems*, in Recent Developments in Domain Decomposition Methods (Zürich, 2001), Lecture Notes Comput. Sci. Eng. 23, Springer, Berlin, 2002, pp. 41–52.

[10] M. DRYJA AND O. B. WIDLUND, *A generalized FETI-DP method for a mortar discretization of elliptic problems*, in Domain Decomposition Methods in Science and Engineering (Cocoyoc, Mexico, 2002), UNAM, Mexico City, 2003, pp. 27–38.

[11] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, *FETI-DP: A dual-primal unified FETI method.* I. *A faster alternative to the two-level FETI method*, Int. J. Numer. Methods Engrg., 50 (2001), pp. 1523–1544.

[12] C. Farhat and F.-X. Roux, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Int. J. Numer. Methods Engrg., 32 (1991), pp. 1205–1227.

[13] H. H. Kim and C.-O. Lee, *A preconditioner for the FETI-DP formulation with mortar methods in two dimensions*, SIAM J. Numer. Anal., 42 (2005), pp. 2159–2175.

[14] A. Klawonn, O. B. Widlund, and M. Dryja, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal., 40 (2002), pp. 159–179.

[15] J. Li, *A dual-primal FETI method for incompressible Stokes equations*, Numer. Math., 102 (2005), pp. 257–275.

[16] J. Li, *Dual-Primal FETI Methods for Stationary Stokes and Navier–Stokes Equations*, Ph.D. thesis, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, New York, 2002.

[17] J. Mandel and R. Tezaur, *On the convergence of a dual-primal substructuring method*, Numer. Math., 88 (2001), pp. 543–558.

[18] D. Stefanica, *Choosing nonmortars: Does it influence the performance of FETI-DP algorithms?*, in Domain Decomposition Methods in Science and Engineering, Lecture Notes Comput. Sci. Eng. 40, Springer, Berlin, 2004, pp. 377–384.

[19] R. Stenberg, *Analysis of mixed finite elements methods for the Stokes problem: A unified approach*, Math. Comp., 42 (1984), pp. 9–23.

[20] B. I. Wohlmuth, *A mortar finite element method using dual spaces for the Lagrange multiplier*, SIAM J. Numer. Anal., 38 (2000), pp. 989–1012.

[21] B. I. Wohlmuth, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, Lecture Notes Comput. Sci. Eng. 17, Springer-Verlag, Berlin, 2001.

[22] J. Xu and J. Zou, *Some nonoverlapping domain decomposition methods*, SIAM Rev., 40 (1998), pp. 857–914.