Continuous Variable Structure Controller for Robot Manipulators using Sliding Mode Observer

Kang-Bark Park and Ju-Jang Lee

Department of Electrical Engineering
Korea Advanced Institute of Science and Technology
373–1 Kusong-dong, Yusong-gu, Taejon 305–701 KOREA

Abstract

In this paper, a continuous variable structure controller for robot robot mainpulator is proposed. The proposed method guarantee that the tracking error converges to zero maintaining the smoothness of the actual control signal. In order to estimate the acceleration data, a sliding mode observer is used.

1 Introduction

The basic theory of a switching control system was proposed in the Soviet Union in 1950's [1] and thereafter the term "variable structure system (VSS)" and "sliding mode control (SMC)" have been used for such systems [2].

Even though the VSS has the invariance property against system uncertainties and external disturbances, it has the drawback so called "*chattering phenomena*". Since the VSS uses switching functions to get an invariance property under the existence of system uncertainties and disturbances, it is inevitable for the control input to show the chattering phenomena.

In order to overcome the chattering problem, lots of techniques have been used [6]-[7].

Almost all of the researchers has used continuation techniques. The continuation techniques are focused on the smoothing of switching function. The switching function has been replaced by the saturation function and/or a sigmoid function.

However, all of these methods suffer from the same difficulty, that is, there is no quantitative rule for assigning boundary layer thickness of their algorithms. Thus, the corresponding effects of the chattering alleviation may not be guaranteed. In addition, these approaches can not guarantee the error convergence to zero, i.e., only the boundedness of the error within some predetermined boundary layer thickness can be guaranteed.

Thus, in this paper, a smooth variable structure controller that guarantees the error convergence to zero is proposed for robot manipulators. The main concept is to consider the combination of a low pass filter (LPF) and the robot as a *virtual* plant, and design a virtual variable structure controller u for this virtual plant as shown in Fig. 1. In this figure, τ is a *real* control signal and u means a virtual control signal. The left block represents a virtual controller, the middle block denotes a Low Pass Filter(LPF), and the right block is a robot manipulator. The virtual control signal u is designed for the virtual plant (LPF + Plant). Then, although the virtual control signal u shows chattering phenomenon due to the switching function, the *real* control signal τ shows a smooth curve because it is an output of a LPF whose input is u as shown in Fig. 2.

Therefore, the proposed control scheme guarantees the smoothness of the *real* control signal without sacrificing tracking accuracy. That is, although the proposed controller does not use the saturation function, the control signal τ shows a smooth curve.

The control scheme mentioned above, however, has to use the *acceleration* data which is not easy to obtain in usual case. In this paper, thus, the smooth variable structure controller using sliding mode observer is proposed to estimate the acceleration of the joint of the robot manipulators. The closed-loop system is locally exponentially stable and region of attraction is also analyzed.

2 Virtual Plant Dynamics

The dynamic equation for an n-link robot manipulator is linearly parameterizable as follows:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Y(q,\dot{q},\ddot{q})\theta = \tau \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \Re^n$ are the joint angular displacements, velocities, and accelerations, respectively. $M(q)$ is the $n \times n$ positive definite inertia matrix, $C(q,\dot{q})$ is the $n \times n$ matrix corresponding to Coriolis and centrifugal factors, $G(q)$ is the $n \times 1$ vector of gravitational torques, τ is the $n \times 1$ input torque vector, θ is a constant p-dimensional vector of inertia parameters and Y is an $n \times p$ matrix of known functions of the generalized coordinates and their higher derivatives. It can be assumed that the parameter vector θ is uncertain and there exists $\theta_0 \in \Re^p$ and $\rho \in \Re_+$, both known, such that $\|\tilde{\theta}\| := \|\theta - \theta_0\| \leq \rho$ [4].

From the LPF's input-output relation, the following equation can be obtained. $\dot{\tau} + \Lambda\tau = u$ where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and $\lambda_i > 0, i = 1, 2, \ldots, n$. Thus, for the virtual control signal u, the dynamic equation of the virtual plant can be described by

$$M\dddot{q} + \dot{M}\ddot{q} + \Lambda M\ddot{q} + C\ddot{q} + \dot{C}\dot{q} + \Lambda C\dot{q} + \dot{G} + \Lambda G = u \quad (2)$$

where u is the $n \times 1$ virtual control input vector.

Introducing the state vectors $\overline{x}_1 = q$, $\overline{x}_2 = \dot{q}$, $\overline{x}_3 = \ddot{q}$, the virtual plant (2) can be rewritten in the following state-space representation:

$$\begin{aligned} \dot{\overline{x}}_1 &= \overline{x}_2 \\ \dot{\overline{x}}_2 &= \overline{x}_3 \\ \dot{\overline{x}}_3 &= \beta(\overline{x}_1, \overline{x}_2, \overline{x}_3) + M(\overline{x}_1)^{-1}u \end{aligned} \quad (3)$$

where

$$\beta(\overline{x}_1, \overline{x}_2, \overline{x}_3) = -M(\overline{x}_1)^{-1}\left\{\left(\dot{M} + \Lambda M + C\right)\overline{x}_3 + \left(\dot{C} + \Lambda C\right)\overline{x}_2 + \dot{G} + \Lambda G\right\} \quad (4)$$

which defines a locally observable representation in the sense of the lie-algebra, as defined by Hermann and Krener [9].

Then, introducing the tracking error state vectors $x_1 = q - q_d$, $x_2 = \dot{q} - \dot{q}_d$, $x_3 = \ddot{q} - \ddot{q}_d$ where $q_d \in \Re^n$ represents the desired trajectory vector, the similar representation can be obtained for the tracking error space as the following:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \beta(x_1 + q_d, x_2 + \dot{q}_d, x_3 + \ddot{q}_d) + M^{-1}u - \dddot{q}_d \end{aligned} \quad (5)$$

where $q_d^i \in C^3[0, \infty)$.

It is obvious that the above representation can be rewritten as the following:

$$\dot{x} = Ax + B(y + \beta_1) \quad (6)$$

where $x^T = [x_1^T, x_2^T, x_3^T]$, $y = M_0^{-1}u$, $\beta_1 = \beta - \dddot{q}_d + (M^{-1}M_0 - I_n)y$, and

$$A = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ I_n \end{bmatrix}$$

and $I_n \in \Re^{n\times n}$ is an identity matrix. The following assumption is assumed to be valid.

Assumption 1 *There exists a known function $\overline{\beta} > 0$ and a posotive known constant $c_0 > 0$ such that*

$$\begin{aligned} \|\beta - \hat{\beta}\| &\le \overline{\beta} & (7) \\ \|M^{-1}M_0 - I_n\| &\le c_0 < 1 & (8) \end{aligned}$$

for all their arguments where $\hat{\beta}$ is the estimate of β.

3 Design of Continuous VSS using Full States

Consider the following function augmented sliding surface $s \in \Re^n$ so that $s(0) = 0$ is assured.

$$s = x_3 + \Lambda_1 x_2 + \Lambda_2 x_1 - Ne^{-\Lambda_3 t} \quad (9)$$

where $\Lambda_i = diag(\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in})$, $\lambda_{ij} > 0$, $i = 1, 2, 3$, and $j = 1, 2, \ldots, n$, $N = x_3(0) + \Lambda_1 x_2(0) + \Lambda_2 x_1(0)$. Then, the control law can be obtained from the sliding mode existence condition $s^T\dot{s} \le 0$, that is,

$$\dot{s} = y + \beta_1 + \Lambda_1 x_3 + \Lambda_2 x_2 + N\Lambda_3 e^{-\Lambda_3 t}.$$

Therefore, the control law can be given by

$$y = \dddot{q}_d - \Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t} - \hat{\beta} - \Psi s - \rho \cdot sgn(s)$$

where ρ is a scalar positive bounding function satisfying the following inequality:

$$\|\beta - \hat{\beta} - (M^{-1}M_0 - I_n)\,y\| \le \rho \quad (10)$$

From the Assumption 1, ρ can be obtained by the following equation:

$$\rho = c_0\left\|\dddot{q}_d - \hat{\beta} - \Psi s - \Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t}\right\| + c_0\rho + \overline{\beta}.$$

Therefore, ρ can be defined as the following:

$$\rho = \frac{c_0}{1 - c_0}\left\|\dddot{q}_d - \hat{\beta} - \Psi s - \Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t}\right\| + \frac{\overline{\beta}}{1 - c_0}.$$

Then, for the virtual control law $u = M_0 y$, the following theorem can be stated.

Theorem 1 *Applying the following virtual control input (11) to the virtual plant (6), the closed-loop system is globally exponentially stable.*

$$u = M_0\left(\dddot{q}_d - \Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t} - \hat{\beta} - \Psi s - (\rho + \eta) \cdot sgn(s)\right) \quad (11)$$

where $\eta = [\eta_1, \eta_2, \ldots, \eta_n]^T$, $\eta_i > 0$, $i = 1, 2, \ldots, n$.

Proof For the Lyapunov function candidate $V = \frac{1}{2}s^T s$, its time derivative can be derived as the following:

$$\begin{aligned} \dot{V} &= s^T\dot{s} = s^T\left(\dot{x}_3 + \Lambda_1 x_3 + \Lambda_2 x_2 + N\Lambda_3 e^{-\Lambda_3 t}\right) \\ &= s^T\Big((M^{-1}M_0 - I_n)(\dddot{q}_d - \Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t} - \hat{\beta} - (\rho + \eta) \cdot sgn(s)) \\ &\quad + \beta - \hat{\beta} - \Psi s + (\rho + \eta) \cdot sgn(s)\Big) \\ &\le -\Psi s^T s - \sum_{i=1}^{n}\{\eta_i\,|s_i|\}. \end{aligned}$$

Thus, it is easy to know that V is a positive definite function of s and $\dot{V}$ is a negative definite function of s. Therefore, V is a Lyapunov function. Furthermore, since $s(0) = 0$, $V(0) = 0$ and hence $s = 0$ $\forall t \geq 0$. Hence, it is obvious that the tracking error exponentially converges to zero. ■

Corollary 1 *The system is in the sliding mode all the time, that is, $s \equiv 0 \ \forall \ t \geq 0$.*

Remark 1 *Since the virtual control input u contains switching function $sgn(\cdot)$, u produces a high frequency switching signal. However, as shown in Fig. 1, the real control signal τ does not contain high frequency components because it is made by low pass filtering u. Therefore, from the result of the above theorem, using the real control signal with no chattering, the tracking error exponentially converges to zero maintaining the robustness against parameter uncertainties all the time.*

4 Design of Sliding Mode Observer

Based on the general scheme for the sliding mode observer [8], the following state observer can be designed:

$$\begin{aligned} \dot{\hat{x}}_1 &= K_1\tilde{x}_1 + \hat{x}_2 + \Gamma_1 sgn(\tilde{x}_1) \\ \dot{\hat{x}}_2 &= K_2\tilde{x}_2 + \hat{x}_3 + \Gamma_2 sgn(\tilde{x}_2) \\ \dot{\hat{x}}_3 &= K_3\tilde{x}_2 + \Gamma_3 sgn(\tilde{x}_2) \end{aligned} \tag{12}$$

where $\hat{x}_1, \hat{x}_2$, and $\hat{x}_3$ are the estimated values of x_1, x_2, and x_3, respectively, $\tilde{(\cdot)} = (\cdot) - \hat{(\cdot)}$, $K_1, K_2, K_3, \Gamma_1, \Gamma_2$, and Γ_3 are $n \times n$ positive definite matrices. M_0 is the "nominal" inertia matrix. In the simplest case, it will be only a diagonal constant matrix, e.g., $M_0 = m_0 I$ where $m_0 > 0$. The nonlinear function $\hat{\beta}$ is introduced to cope with the influence of β in (4) and to account for differences between the tracking errors and the observation errors.

By subtracting the observer equations (12) from the virtual plant equations (5), the following equations can be obtained

$$\dot{\tilde{x}}_1 = -K_1\tilde{x}_1 + \tilde{x}_2 - \Gamma_1 sgn(\tilde{x}_1) \tag{13}$$

$$\dot{\tilde{x}}_2 = -K_2\tilde{x}_2 + \tilde{x}_3 - \Gamma_2 sgn(\tilde{x}_2) \tag{14}$$

$$\dot{\tilde{x}}_3 = -K_3\tilde{x}_2 - \Gamma_3 sgn(\tilde{x}_2) + y + \beta_1. \tag{15}$$

Since the angular position and velocity can be obtained from the encoder and the tachometers, respectively, it is assumed that x_1 and x_2 are measureable as described in the following assumption.

Assumption 2 *The angular position(x_1) and the angular velocity(x_2) are measureable.*

For the simplicity's sake consider Γ_1, Γ_2 to be the diagonal matricies, i.e., $\Gamma_1 = \gamma_1 I$, $\Gamma_2 = \gamma_2 I$. Then the following lemma can be obtained.

Lemma 1 *The region $\{(x_1, x_2) | \tilde{x}_1 = \tilde{x}_2 = 0\}$ is invariant as long as $|\tilde{x}_3^i| < \gamma_2$.*

Proof Consider the following Lyapunov function candidate $V_1 = \frac{1}{2}\tilde{x}_1^T\tilde{x}_1$. Then, from the equation (13), it is obvious that $\dot{V}_1 \leq -\tilde{x}_1^T K_1 \tilde{x}_1$ as long as $|\tilde{x}_2^i| < \gamma_1$. Since it is assumed in Assumption 2 that x_1 and x_2 are measureable, $|\tilde{x}_2^i| < \gamma_1$ can be guaranteed. Therefore, $\tilde{x}_1 \equiv 0$.

Similarly, by taking the other Lyapunov function candidate $V_2 = \frac{1}{2}\tilde{x}_2^T\tilde{x}_2$, from the equation (14), it is also shown that $\dot{V}_2 \leq -\tilde{x}_2^T K_2 \tilde{x}_2$ as long as $|\tilde{x}_3^i| < \gamma_2$. Therefore, $\tilde{x}_2 \equiv 0$ if $|\tilde{x}_3^i| < \gamma_2$. ■

The generalized notion of the solution of a differential equation with discontinuities on the right-hand side such as (13)–(15) is given by [3]; the dynamics on the switching surface is an average of the dynamics on each side of the discontinuity surface. The region within which the switching surface is invariant is called "sliding patch" [8]. For the sliding patch, the following lemma can be obtained.

Lemma 2 *On the region of $\mathcal{R} = \{(x_1, x_2, x_3) | \tilde{x}_1 = \tilde{x}_2 = 0, |\tilde{x}_3^i| < \gamma_2\}$, which is called the sliding patch, $x_3 = \hat{x}_3 + \Gamma_2 sgn(\tilde{x}_2)$, and the error dynamics of $[\tilde{x}_1^T, \tilde{x}_2^T, \tilde{x}_3^T]^T$ is given by the following reduced-order dynamics:*

$$\dot{\tilde{x}}_3 = -\Gamma\tilde{x}_3 + y + \beta_1 \tag{16}$$

where $\Gamma = \Gamma_3\Gamma_2^{-1} = \gamma I$ with $\gamma > 0$.

Proof Since $\tilde{x}_1(t) \equiv \tilde{x}_2(t) \equiv 0$, it is clear from the equation (14) that $\tilde{x}_3 = \Gamma_2 sgn(\tilde{x}_2)$. Thus, $x_3 = \hat{x}_3 + \Gamma_2 sgn(\tilde{x}_2)$. In addition, it is also obvious that $sgn(\tilde{x}_2) = \Gamma_2^{-1}\tilde{x}_3$. Therefore, from the equation (15), the following reduced-order dynamics can be obtained:

$$\dot{\tilde{x}}_3 = -\Gamma\tilde{x}_3 + y + \beta_1 \tag{17}$$

where $\Gamma = \Gamma_3\Gamma_2^{-1} = \gamma I$ with $\gamma > 0$. ■

Remark 2 *From the above Lemma, it can be known that the acceleration data can be estimated as the following in the sliding patch $\mathcal{R}$:*

$$x_3 = \hat{x}_3 + \Gamma_2 sgn(\tilde{x}_2) \tag{18}$$

5 Stability Analysis for the Closed-Loop System

It will be analyzed in this section that the stability of the closed loop system with the variable structure controller (11) using the estimated state through the state estimator (12) in place of the true state. Since x_1 and x_2 are assumed to be measureable in Assumption 2, they can be used directly. The acceleration, however, is not accessible directly. From Remark 2, it was proved that the acceleration x_3 is given by (18) in the sliding patch $\mathcal{R}$.

Instead of the vector x, therefore, $\left[x_1^T, x_2^T, (\hat{x}_3 + \Gamma_2 sgn(\tilde{x}_2))^T\right]^T$ is used in the controller (11). Let us define the new state vector $\overline{z} = \left[x_1^T, x_2^T, \varsigma_3^T\right]^T$ where $\varsigma_3 = \hat{x}_3 + \Gamma_2 sgn(\tilde{x}_2)$, and the modified sliding surface

$$\overline{s} = \varsigma_3 + \Lambda_1 x_2 + \Lambda_2 x_1 - \overline{N}e^{-\Lambda_3 t} \tag{19}$$

where $\overline{N} = \varsigma_3(0) + \Lambda_1 x_2(0) + \Lambda_2 x_1(0)$. Then, the control law using the full states (11), can be modified by

$$\begin{aligned} u = M_0(&\ddot{q}_d - \Lambda_1\varsigma_3 - \Lambda_2 x_2 - \overline{N}\Lambda_3 e^{-\Lambda_3 t} \\ &-\hat{\beta}(x_1+q_d, x_2+\dot{q}_d, \varsigma_3+\ddot{q}_d) - \Psi\overline{s} - (\overline{\rho}+\eta)\cdot sgn(\overline{s})) \end{aligned} \tag{20}$$

where $\overline{\rho}$ is defined as

$$\begin{aligned} \overline{\rho} = &\frac{c_0}{1-c_0}\left\|\ddot{q}_d - \hat{\beta} - \Psi\overline{s} - \Lambda_1\varsigma_3 - \Lambda_2 x_2 - \overline{N}\Lambda_3 e^{-\Lambda_3 t}\right\| \\ &+\frac{1}{1-c_0}\overline{\beta}(x_1 + q_d, x_2 + \dot{q}_d, \varsigma_3 + \ddot{q}_d). \end{aligned} \tag{21}$$

Using the above control law (20), the following theorem can be derived.

Theorem 2 *The closed-loop system with the plant (6) and the proposed control law (20) using the observer (12) is exponentially stable provided that the state is in the sliding patch $\mathcal{R}$.*

Proof From Remark 2, in the sliding patch $\mathcal{R}$, $x = \overline{z}$, $s = \overline{s}$, and the control law (20) is identical with (11). Therefore, in the sliding patch $\mathcal{R}$, the closed-loop system dynamics of (6) and (12) is described as:

$$\dot{z} = \overline{A}z + \overline{B}(y + \beta_1) \tag{22}$$

where $z = [x_1^T, x_2^T, x_3^T, \tilde{x}_3^T]^T \in \Re^{4n}$

$$\overline{A} = \begin{bmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma \end{bmatrix} \qquad \overline{B} = \begin{bmatrix} 0 \\ 0 \\ I_n \\ I_n \end{bmatrix}. \tag{23}$$

In the sliding patch $\mathcal{R}$, it is obvious that the sliding mode existence condition $s^T\dot{s} < 0$ can be guaranteed when the control law (20) is used. In addition, since it is also clear that $s(0) = 0$ is guaranteed in the sliding patch $\mathcal{R}$, the system is in the sliding mode all the time, that is, $s \equiv 0 \ \ \forall\ t \geq 0$ as Corollary 1. Therefore, one of the resulting dynamics can be obtained from $s = 0$ as the following:

$$s = x_3 + \Lambda_1 x_2 + \Lambda_2 x_1 - Ne^{-\Lambda_3 t} = 0. \tag{24}$$

Obviously, the above resultant dynamics (24) is exponentially stable.

For the simplicity of the analysis, let us assume that $\Lambda_1 = 3\Lambda$, $\Lambda_2 = 3\Lambda^2$, $\Lambda_3 = \Lambda^3$ so that the poles of the resultant dynamics are placed on $-\Lambda$.

The other dynamics can be obtained from the last equation of (22) which is obtained from (16). Since the condition $s \equiv 0$ implies $\dot{s} = 0$ as well as $s = 0$, the following equations can be obtained:

$$\dot{s} = \dot{x}_3 + \Lambda_1 x_3 + \Lambda_2 x_2 + N\Lambda_3 e^{-\Lambda_3 t} = 0.$$

This implies the following equation:

$$y + \beta_1 = -\Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t}. \tag{25}$$

Using the above equations (24) and (25), the last equation of (22) which is obtained from (16) can be rewritten as follows:

$$\begin{aligned} \dot{\tilde{x}}_3 &= -\Gamma\tilde{x}_3 + y + \beta_1 \\ &= -\Gamma\tilde{x}_3 - \Lambda_1 x_3 - \Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t} \\ &= -\Gamma\tilde{x}_3 + \Lambda_1\left(\Lambda_1 x_2 + \Lambda_2 x_1 - Ne^{-\Lambda_3 t}\right) \\ &\quad -\Lambda_2 x_2 - N\Lambda_3 e^{-\Lambda_3 t} \\ &= -\Gamma\tilde{x}_3 + \left(\Lambda_1^2 - \Lambda_2\right)x_2 + \Lambda_1\Lambda_2 x_1 \\ &\quad -\left(\Lambda_1 N - N\Lambda_3\right)e^{-\Lambda_3 t}. \end{aligned} \tag{26}$$

Thus, from the equations (24) and (26), it is obvious that $q \rightarrow q_d$ and $\hat{x}_3 \rightarrow x_3$ exponentially as $t \rightarrow \infty$. ■

Since the exponential stability is guaranteed in the sliding patch $\mathcal{R}$, the region of attraction has to be determined. For the simplicity of the analysis, it is assumed that $s = \hat{\Lambda}x = x_3 + 2\Lambda x_2 + \Lambda^2 x_1$ where $\hat{\Lambda} = \left[\Lambda^2 \ \ 2\Lambda \ \ 1\right] \in \Re^{n\times 3n}$, $\Lambda = \lambda I_n$, and $\Psi = \psi I_n$. The following lemma will be used.

Lemma 3 *There exists a positive definite matrix $Q = Q^T \in \Re^{4n\times 4n}$ such that $\overline{B}^T P = \overline{\Lambda}$ where $\overline{\Lambda} = \left[\hat{\Lambda} \ \ 0_n\right] \in \Re^{n\times 4n}$, $0_n \in \Re^{n\times n}$ is the zero matrix, P is the unique positive definite matrix satisfying the Lyapunov equation $\overline{A}_0^T P + P\overline{A}_0^T + Q = 0$ where $\overline{A}_0 = \left(I_{4n} - \overline{B}\overline{\Lambda}\right)\overline{A} - \overline{B}\Psi\overline{\Lambda}$.*

Proof By Kalman-Yakubovich lemma [10], it is sufficient to show that $\overline{A}_0$ is asymptotically stable and $\overline{\Lambda}\left(sI_{4n} - \overline{A}_0\right)^{-1}\overline{B}$ is strictly positive real. After some simple manipulation, one can obtain the following result: $det\left(sI_{4n} - \overline{A}_0\right) = (s+\gamma)(s+\psi)(s+\lambda)^2$. Thus, $\overline{A}_0$ is asymptotically stable. In addition, it is also easy to show the following equation:

$$\overline{\Lambda}\left(sI_{4n} - \overline{A}_0\right)^{-1}\overline{B} = \frac{1}{s+\psi}I_n \tag{27}$$

which is strictly positive real. ■

Now, let us define the Lyapunov function candidate as $V(z) = z^T P z$. Differentiating the $V(z)$ with respect to time, we have

$$\begin{aligned} \dot{V} &= z^T\left(P\overline{A}_0 + \overline{A}_0^T P\right)z - 2\rho z^T P\overline{B}sgn(s) \\ &\quad +2z^T P\overline{B}\left(\eta - \beta_1\right) \\ &\leq -z^T Qz - 2\rho s^T sgn(s) + 2s^T\left(\eta - \beta_1\right) \\ &\leq -z^T Qz \ \leq\ -\lambda_{min}(Q)\left\|z\right\|^2 \ \leq\ 0 \end{aligned}$$

where $\lambda_{min}(\cdot)$ means the minimum eigenvalue of $(\cdot)$. Since the above inequality is satisfied provided that the system is in the sliding patch $\mathcal{R}$, it sould be guaranteed that z is always in the sliding patch $\mathcal{R}$. It is clear that V is bounded as $\lambda_{min}(P)\|z\|^2 \le V \le \lambda_{max}(P)\|z\|^2$ where $\lambda_{max}(\cdot)$ means the maximum eigenvalue of $(\cdot)$. Thus, a region of attraction, where z remains in the sliding patch $\mathcal{R}$ and z is asymptotically stable, is given by

$$\Omega = \left\{ z \;\middle|\; \|z\| \le \gamma_2 \sqrt{\lambda_{min}(P)/\lambda_{max}(P)} \right\}.$$

Remark 3 *Let us define $\hat{x}_1(0) = x_1(0)$ and $\hat{x}_2(0) = x_2(0)$ and consider the closed-loop system (6), (12), (16) with (20). Then once a trajectory starts in the region Ω, the overall system always shows the invariance property against parameter variations and external disturbances all the time, and the tracking error and the estimation error converges to zero exponentially. Therefore, Ω is a domain of attraction for the closed-loop system.*

6 Simulation Results

The simulation has been carried out for a two-link robot manipulator model used by Yeung and Chen [5]. The parameter values are also the same as those of Yeung and Chen.

The results are shown in Figs. 3~6 .

The tracking error of the proposed controller is presented in Fig. 3. One can easily know that the tracking error converges to zero.

Fig. 4 shows the modified sliding surface variable $\bar{s}$ defined in (19). As can be shown in this figure, $\bar{s}$ is in the sliding mode all the time even though the actual control signal τ shows a smooth curve(Fig. 6) . Therefore, the closed-loop system shows invariance property against parameter uncertainties all the time, that is, the system response always shows the same curve under the existence of parameter variations and external disturbances.

In Fig. 5, the virtual control signal u is given. Because of the switching function($sgn(\cdot)$), it chatters with the high frequency.

However, as in Fig. 6, the *real* control signal τ applied to the robot(real plant) shows the smooth curve.

7 Conclusions

In order to overcome the *chattering problem*, first order LPF and the concept of a virtual controller/plant has been used. The proposed control scheme guarantees the invariance property against parameter uncertainties maintaining the smoothness of the *real* control signal without sacrificing the tracking accuracy. That is, although the proposed controller generates a smooth control signal τ, the system is in the sliding mode all the time and the tracking error converges to zero exponentially under the existence of parameter uncertainties. The closed-loop system has been shown to be globally exponentially stable.

Since the chattering-free variable structure controller uses the acceleration data, a smooth control law using the estimated acceleration is proposed and the stability for the closed-lop system has been verified.

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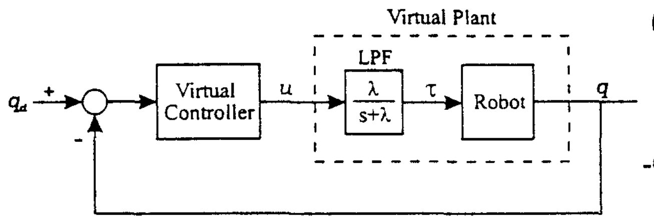

Fig. 1. Block Diagram of the Virtual System

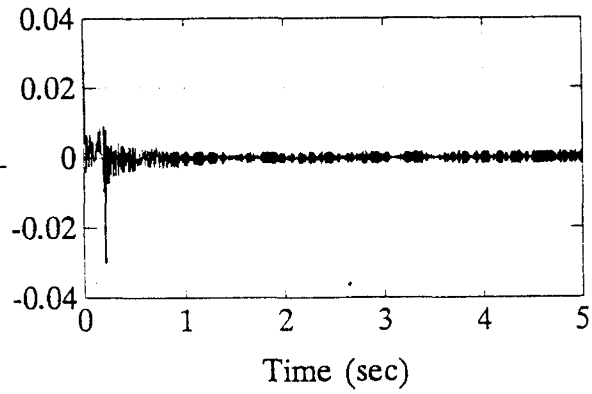

Fig. 4. Modified Sliding Surface

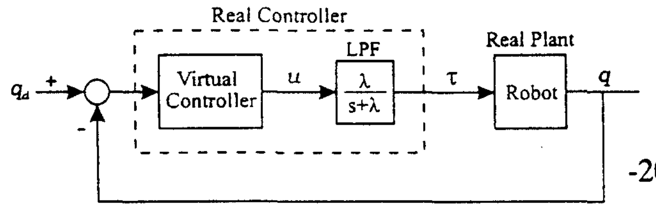

Fig. 2. Block Diagram of the Real System

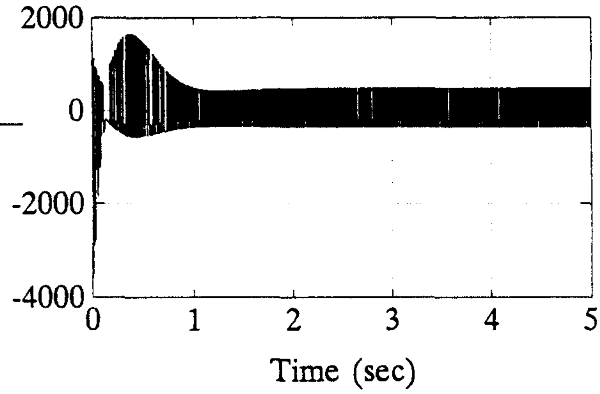

Fig. 5. Virtual Control Input u

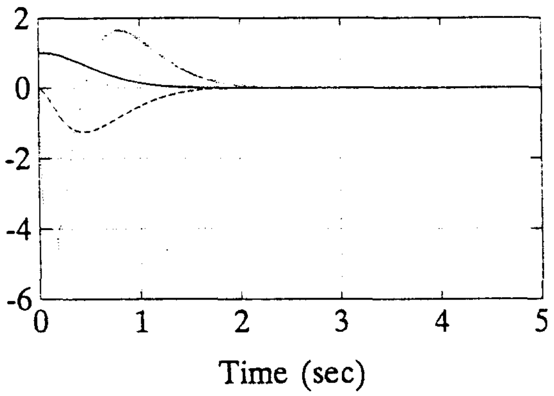

Fig. 3. Tracking Error, Velocity, and Acceleration

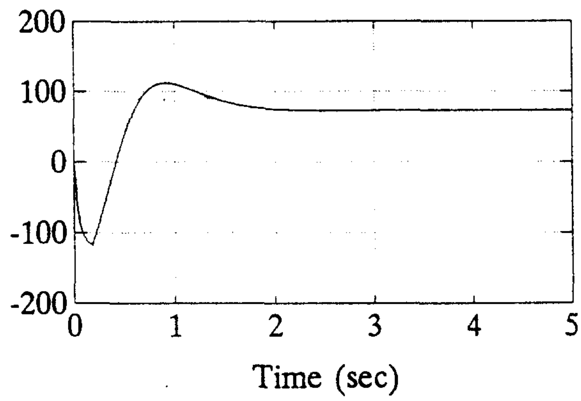

Fig. 6. Real Control Input τ