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# Computing a minimum-dilation spanning tree is NP-hard ☆

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**Abstract**

In a geometric network $G = (S, E)$, the graph distance between two vertices $u, v \in S$ is the length of the shortest path in $G$ connecting $u$ to $v$. The *dilation* of $G$ is the maximum factor by which the graph distance of a pair of vertices differs from their Euclidean distance. We show that given a set $S$ of $n$ points with integer coordinates in the plane and a rational dilation $\delta > 1$, it is NP-hard to determine whether a spanning tree of $S$ with dilation at most $\delta$ exists.


*Keywords:* Spanner; Dilation; Optimization; Spanning tree; Geometric network; NP-hardness

## 1. Introduction

A *geometric network* is a weighted undirected graph whose vertices are points in $\mathbb{R}^d$, and in which each edge is a straight-line segment with weight equal to the Euclidean distance between its endpoints. Geometric networks have many applications: most naturally, many communication networks (road networks, railway networks, telephone networks) can be modelled as geometric networks.

In a geometric network $G = (S, E)$ on a set $S$ of $n$ points, the *graph distance* $d_G(u, v)$ of $u, v \in S$ is the length of a shortest path from $u$ to $v$ in $G$. Some applications require a geometric network for a given set $S$ of points that includes a relatively short path between every two points in $S$. More precisely, we consider the factor by which the graph distance $d_G(u, v)$ differs from the Euclidean distance $|uv|$. This factor is called the *dilation* $\Delta$ of the pair $(u, v)$ in $G$, and is formally expressed as:

$$\Delta_G(u, v) := \frac{d_G(u, v)}{|uv|}.$$

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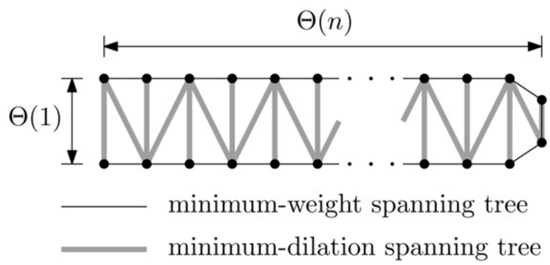


Fig. 1. An example of a minimum-weight spanning tree with bad dilation.

The dilation or stretch factor $\Delta(G)$ of a graph is the maximum dilation over all vertex pairs:

$$\Delta(G) := \max_{\substack{u,v \in S \\ u \neq v}} \Delta_G(u, v) = \max_{\substack{u,v \in S \\ u \neq v}} \frac{d_G(u, v)}{|uv|}.$$

A network $G$ is called a $t$-spanner if $\Delta(G) \leqslant t$.

An obvious 1-spanner is the complete graph. It has optimal dilation and is easy to compute, but for many applications its high cost is unacceptable. Therefore one usually seeks to construct networks that do not only have small dilation, but also have properties such as a low number of edges, a low total edge weight or a low maximum vertex degree. Such networks find applications in, for example, robotics, network topology design, broadcasting, design of parallel machines and distributed systems, and metric space searching. Therefore there has also been considerable interest from a theoretical perspective [3,12].

In this paper we focus on spanners that have small dilation and few edges. Several algorithms have been published to compute a $(1 + \varepsilon)$-spanner with $O(n)$ edges for any given set of $n$ points $S$ [10,11,13] and any $\varepsilon > 0$. Farshi and Gudmundsson did an experimental study of such algorithms [4].

Although the number of edges in the spanners from these algorithms is linear in $n$, it can still be rather large due to the hidden constants in the O-notation that depend on $\varepsilon$ and the dimension $d$. Therefore there has also been attention to the problem with the priorities reversed: given a certain number of edges, how small a dilation can we realize? Das and Heffernan [2] showed how to compute in $O(n \log n)$ time, for any constant $\varepsilon' > 0$, a spanner with at most $(1 + \varepsilon')n$ edges, maximum degree three, and constant dilation in the sense that it only depends on $\varepsilon'$ and $d$. The smallest possible number of edges for a spanner for an $n$-point set $S$ is $n - 1$, since any geometric network with finite dilation must at least connect the $n$ points of $S$, and must therefore contain a spanning tree. Eppstein [3] observed that the minimum-weight spanning tree of $S$ achieves dilation $n - 1$, and that one cannot do better than dilation $\Omega(n)$ for the vertices of a regular $n$-gon, so in a sense the minimum spanning tree is optimal. This insight was generalized by Aronov et al. [1], who showed how to compute in $O(n \log n)$ time, for any constant $k \geqslant 0$, a spanner with $n - 1 + k$ edges and dilation $O(n/(k + 1))$, and proved that this dilation is optimal in the worst case.

The minimum-weight spanning tree of a set $S$ of $n$ points always has dilation $O(n)$. In the worst case this is asymptotically optimal, since there are sets $S$ such that any spanning tree on $S$ has dilation $\Omega(n)$. For a given set of points, however, it may be possible to achieve a much smaller dilation. In Fig. 1 we show an example where the minimum-weight spanning tree has dilation $\Theta(n)$ while dilation $\Theta(1)$ is possible.

A natural question arises: Given a set $S$ of $n$ points in $\mathbb{R}^d$, what is the spanning tree of $S$ of minimum dilation? Eppstein posed the following questions:

> Is it possible to construct the exact minimum-dilation geometric spanning tree, or an approximation to it, in polynomial time? Does the minimum-dilation spanning tree have any edge crossings?

The second question was recently answered by Klein and Kutz [8], who gave a set of seven points whose minimum-dilation spanning tree has edge crossings. We give here the smallest possible example, a set of five points whose minimum-dilation spanning tree has edge crossings, and we show that sets of at most four points always admit a minimum-dilation spanning tree without edge crossings.

As for Eppstein's first question, only partial progress has been made so far. The analogous problem for weighted planar (but not geometric) graphs was shown to be NP-hard by Fekete and Kremer [5]. Gudmundsson and Smid [7] found by reduction from 3SAT that, given a geometric graph $G$, a dilation $\delta$ and a number $k \geqslant n-1$, it is NP-hard to decide whether $G$ contains a $\delta$-spanner with at most $k$ edges. Klein and Kutz [8] show that given a set of $n$ points $S$ in the plane, a dilation $\delta$ and a number $k \geqslant n-1$, it is NP-hard to decide whether there is a plane $\delta$-spanner with at most $k$ edges. Giannopoulos et al. [6] show that finding the minimum-dilation spanning *tour* of $S$ is $NP$-hard. The proofs by Gudmundsson and Smid and by Klein and Kutz are based on instances of the problem with $k > n-1$, and so Eppstein's original question whether a spanning *tree* with dilation at most $\delta$ can be found in polynomial time remained open.

We show that this problem is in fact NP-hard as well. More precisely, we show the following: Given a set $S$ of $n$ points with integer coordinates in the plane and a rational dilation $\delta > 1$, it is NP-hard to decide whether a spanning tree of $S$ with dilation at most $\delta$ exists—regardless if edge crossings are allowed or not. (The input size for the problem instance is the total bit complexity of all point coordinates and the rational representation of $\delta$.) Thus the problems studied by Gudmundsson and Smid and by Klein and Kutz remain NP-hard even if the number of edges $k$ is restricted to $n-1$.

Our NP-hardness proof[1] is a reduction from PARTITION:

PARTITION
Given a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of $n$ positive integers, is there a partition of $\{1, \ldots, n\}$ into subsets $A$ and $A'$ such that $A \cap A' = \emptyset$, $A \cup A' = \{1, \ldots, n\}$ and $\sum_{i \in A} \alpha_i = \sum_{i \in A'} \alpha_i$?

We first show that a sequence of $n$ positive integers can be transformed to $8n+8$ points in the plane, such that a partition exists if and only if there exists a geometric spanning tree $\mathcal{T}$ on $S$ with $\Delta(\mathcal{T}) \leqslant 3/2$. Conceptually, our construction is quite simple, the difficulty being to ensure that no unwanted solutions or interactions can arise.

To prove NP-hardness of the problem, we have to formulate it in a form suitable for a Turing-machine or an equivalent model; the formulation above with integer coordinates and rational $\delta$ seems most natural. Our construction does not quite fit this form yet: we construct some points as the intersection of circles. We solve this problem by showing that if the coordinates of these points are *approximated* by rational points with precision polynomial in the input size, the construction still goes through. We can then simply rescale all numbers to achieve integer point coordinates.

Eppstein's last question "or an approximation to it" remains wide open. We are not aware of any result showing how to approximate the minimum-dilation spanning tree with approximation factor $o(n)$. The only known result in this direction is by Knauer and Mulzer [9], who describe an algorithm that computes a triangulation whose dilation is within a factor of $1 + O(1/\sqrt{n})$ of the optimum. (It is not known how to compute the minimum-dilation triangulation of even a convex polygon.)

## 2. Minimum-dilation spanning trees with edge crossings

Suppose we are given a set $S$ of points. Klein and Kutz [8] have an example where $|S| = 7$ and the minimum-dilation spanning tree of $S$ has edge crossings. Below we give an example with $|S| = 5$, and prove that there is no smaller set $S$ that does not have a crossing-free minimum-dilation spanning tree.

For $u, v \in S$, we call $uv$ a $\delta$-critical edge if for every point $w \in S \setminus \{u, v\}$ we have

$$\delta \cdot |uv| < |uw| + |wv|.$$

Clearly, any spanning tree $\mathcal{T}$ of $S$ that does not include all $\delta$-critical edges has dilation $\Delta(\mathcal{T}) > \delta$.

Fig. 2 shows a set of five points $S = \{a, b, c, d, e\}$. The reader may verify that the edges $ab$, $bc$, and $cd$ are 8/7-critical. To complete the spanning tree, it remains to add either $ae$, $be$, $ce$, or $de$ to $\mathcal{T}$. Adding $ae$ would make $d_{\mathcal{T}}(b, e)$ longer than $(8/7)|be|$, while choosing $ce$ or $de$ would make $d_{\mathcal{T}}(a, e)$ longer than $(8/7)|ae|$. On the other hand, including $be$ results in $d_{\mathcal{T}}(a, e) = (8/7)|ae|$ and $\Delta(\mathcal{T}) = 8/7$. The minimum-dilation spanning tree of $S$ thus consists of the edges $ab$, $bc$, $cd$ and $be$, where $cd$ and $be$ intersect.

[1] Note that we cannot claim NP-completeness of the problem, as it is not known how to do the necessary distance computations involving sums of square roots in polynomial time.

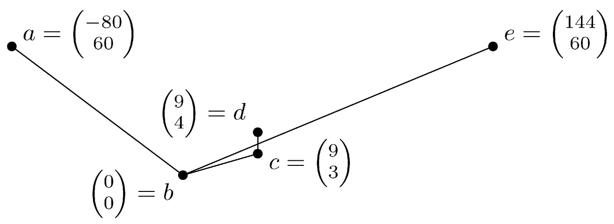


Fig. 2. A set of five points whose minimum-dilation spanning tree has dilation 8/7 and has an edge crossing (not to scale).

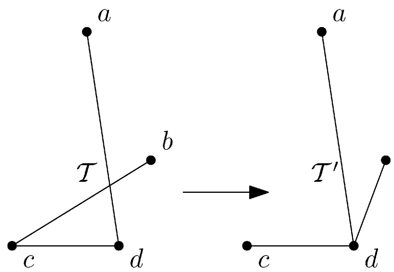


Fig. 3. Any minimum-dilation spanning tree on four points that has an edge crossing can be transformed into a minimum-dilation spanning tree without any edge crossing.

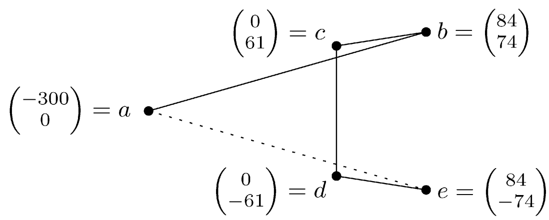


Fig. 4. A set of points whose minimum-dilation spanning path and minimum-dilation spanning tour have dilation 73/37 and have edge crossings (not to scale).

**Theorem 1.** *For $n \geqslant 5$, there are sets of $n$ points in the plane that do not have a minimum-dilation spanning tree without edge crossings. For $n < 5$, every set of $n$ points in $\mathbb{R}^d$ has a minimum-dilation spanning tree without edge crossings.*

**Proof.** For $n = 5$, an example is given in Fig. 2. The example can easily be extended with additional points.

For $n < 5$, observe that intersections between possible edges are possible only if $n = 4$ and the points are co-planar and in convex position. Suppose $\mathcal{T}$ is a minimum-dilation spanning tree with an edge crossing on four such points $a, b, c, d$. Without loss of generality, assume $ad$ and $bc$ are the intersecting edges, $cd$ is the third edge, and $b$ lies closer to $d$ than to $c$ (see Fig. 3). We now create another spanning tree $\mathcal{T}'$ by taking $\mathcal{T}$ and replacing edge $bc$ by edge $bd$. This increases only $d_{\mathcal{T}}(b, c)$. Hence we get:

$$\Delta(\mathcal{T}') = \max\left\{\Delta(\mathcal{T}), \frac{d_{\mathcal{T}'}(b,c)}{|bc|}\right\} \leqslant \max\left\{\Delta(\mathcal{T}), \frac{d_{\mathcal{T}}(b,d)}{|bd|}\right\} = \Delta(\mathcal{T}).$$

So $\mathcal{T}'$ is a minimum-dilation spanning tree of $a, b, c$ and $d$ without edge crossings. □

Aronov et al. [1] already observed that minimum-dilation spanning *paths* may have edge crossings. Fig. 4 shows an example. To get a spanning path of dilation at most 73/37, we need to include edges $bc$, $cd$ and $de$, because these are all 73/37-critical. To complete the spanning path, we need to include $ab$ (or its symmetric counterpart $ae$), which indeed yields a spanning path of dilation 73/37 (where $d_{\mathcal{T}}(b, e) = (73/37)|be|$), and $ab$ intersects $cd$. The unique minimum-dilation spanning tour of the same set of points is $ab$, $bc$, $cd$, $de$, $ea$ and also has edge crossings.

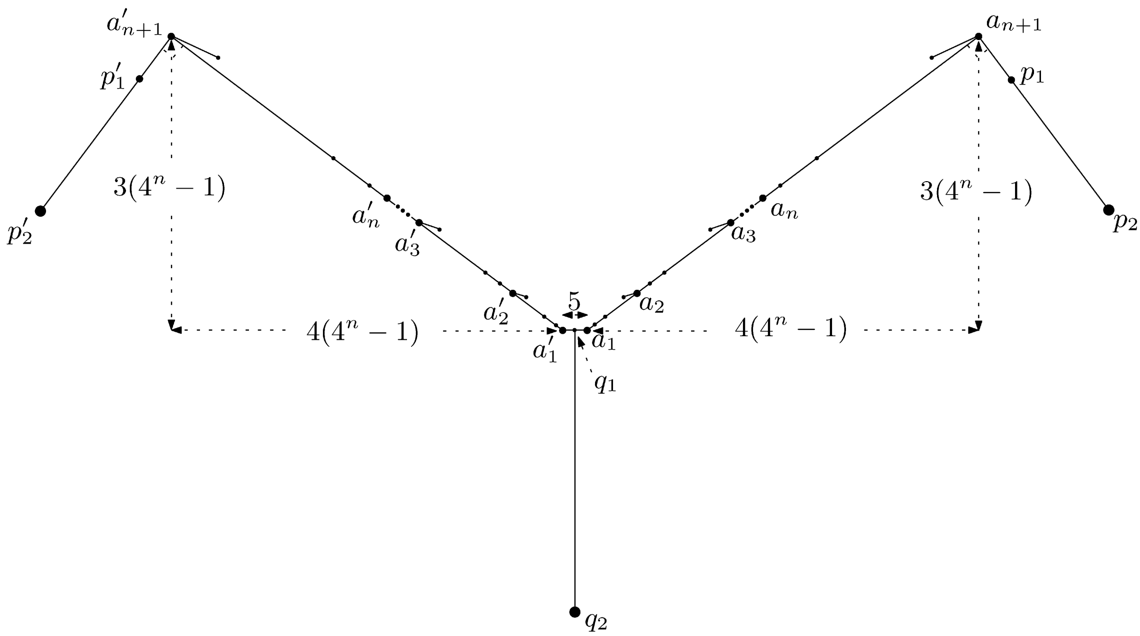


Fig. 5. Construction of $S$.

## 3. Computing a minimum-dilation tree is NP-hard

For a set $S$ of points in the plane, let us define $\Delta(S) := \min_{\mathcal{T}} \Delta(\mathcal{T})$, where the minimum is taken over all spanning trees $\mathcal{T}$ of $S$.

Our NP-hardness proof is a reduction from PARTITION. The basic idea is simple: Given an instance of PARTITION, that is, a sequence of $n$ positive integers, we construct a set $S$ of $8n + 8$ points in the plane such that $\Delta(S) \leqslant 3/2$ if and only if the partition problem has a solution.

In Section 3.1 we show how to construct this set $S$. In Section 3.2 we then show that if no partition exists, then $\Delta(S) > 3/2$. In Section 3.3 we show that if a partition exists, then $\Delta(S) \leqslant 3/2$, and there is a spanning tree with dilation $3/2$ on $S$ that does not have any edge crossings. Finally we show in Section 3.4 that the entire construction can be done in such a way that the points of $S$ have integer coordinates with total bit complexity polynomial in the bit complexity of the PARTITION instance. Together, we prove the following:

**Theorem 2.** *Given a set $S$ of points with integer coordinates in the plane and two positive integers $P$ and $Q$, it is NP-hard to decide whether a geometric spanning tree of $S$ with dilation at most $P/Q$ exists. The problem remains NP-hard if the spanning tree is restricted not to have edge crossings.*

### 3.1. Construction of S

We are given an instance of PARTITION, that is, a sequence $(\dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\alpha}_n)$ of $n$ positive integers. We define $\dot{\sigma} := \sum_{i=1}^{n} \dot{\alpha}_i$, and define the scaled quantities $\alpha_i = \dot{\alpha}_i/(10\dot{\sigma})$ and $\sigma := \sum_{i=1}^{n} \alpha_i$. By construction, we have $\sigma = 1/10$.

Fig. 5 shows the general structure of our construction of $S$. It is symmetric around the $y$-axis, and so we only need to describe the right half of the construction.

We create $3n + 1$ points lying on the line with slope $3/4$ through the point $(5/2, 0)$:

$$a_i = \begin{pmatrix} 5/2 \\ 0 \end{pmatrix} + (4^{i-1} - 1)\begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \forall 1 \leqslant i \leqslant n+1,$$

$$b_i = a_i + \frac{4^{i-1}}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \forall 1 \leqslant i \leqslant n,$$

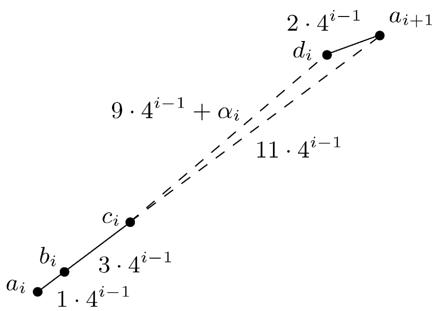


Fig. 6. The construction between $a_i$ and $a_{i+1}$.

$$c_i = b_i + \frac{3 \cdot 4^{i-1}}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \forall 1 \leqslant i \leqslant n.$$

The distances between these points are as follows:

$$\begin{aligned}
|a_i a_{i+1}| &= 15 \cdot 4^{i-1}, \\
|a_i b_i| &= 1 \cdot 4^{i-1}, \\
|b_i c_i| &= 3 \cdot 4^{i-1}, \\
|c_i a_{i+1}| &= 11 \cdot 4^{i-1}, \\
|a_1 a_{n+1}| &= 5(4^n - 1).
\end{aligned}$$

So far, we have not made any use of the quantities $\alpha_i$. They appear in the definition of the $n$ points $d_i$, for $1 \leqslant i \leqslant n$. These points lie slightly above the line $a_1 a_{n+1}$, and are defined by the two equations:

$$\begin{aligned}
|d_i a_{i+1}| &= 2 \cdot 4^{i-1}, \\
|c_i d_i| &= 9 \cdot 4^{i-1} + \alpha_i.
\end{aligned}$$

Fig. 6 shows the interval between $a_i$ and $a_{i+1}$. Since $|c_i a_{i+1}| = 11 \cdot 4^{i-1}$ and $0 < \alpha_i \leqslant 1/10$, it is clear that $d_i$ exists. We add two more points at the far end:

$$\begin{aligned}
p_1 &= a_{n+1} + \left(\frac{4^n}{9} - \frac{179}{1800}\right) \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \\
p_2 &= a_{n+1} + 4\left(\frac{4^n}{9} - \frac{179}{1800}\right) \begin{pmatrix} 3 \\ -4 \end{pmatrix}.
\end{aligned}$$

Both points lie on the line through $a_{n+1}$ with slope $-4/3$, and so $\angle a_1 a_{n+1} p_2$ is a right angle. We have

$$\begin{aligned}
|a_{n+1} p_1| &= \frac{5}{9} 4^n - \frac{179}{360}, \\
|a_{n+1} p_2| &= 4|a_{n+1} p_1| = \frac{5}{9} 4^{n+1} - \frac{179}{90}.
\end{aligned}$$

We denote the mirror images under reflection in the $y$-axis of the $4n+3$ points $a_i, b_i, c_i, d_i, p_i$ constructed so far as $a_i', b_i', c_i', d_i', p_i'$. Our point set $S$ consists of $8n+8$ points, namely the original points, their mirror images, and two more points on the $y$-axis:

$$\begin{aligned}
q_1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
q_2 &= \begin{pmatrix} 0 \\ -\frac{25}{9} 4^n + \frac{11}{18} \end{pmatrix}.
\end{aligned}$$

We have

$$p_2 - q_2 = \left(\frac{4^{n+1}}{3} - \frac{101}{150}\right) \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

so $q_2p_2$ is parallel to $a_1a_{n+1}$, and

$$|q_2p_2| = |q_2p_2'| = \frac{5}{3}\cdot 4^{n+1} - \frac{101}{30}.$$

We now prove some basic properties of the constructed point set $S$.

**Lemma 1.** *We have* $\cos\angle c_ia_{i+1}d_i > 1 - 4^{1-i}/22 \geqslant 21/22$, *and the* $y$*-coordinate of* $d_i$ *is strictly smaller than the* $y$*-coordinate of* $a_{i+1}$, *for* $1 \leqslant i \leqslant n$.

**Proof.** Since $\alpha_i \leqslant \frac{1}{10}$, the cosine theorem gives

$$\begin{aligned}\cos\angle c_ia_{i+1}d_i &= \frac{|c_ia_{i+1}|^2 + |d_ia_{i+1}|^2 - |c_id_i|^2}{2\cdot|c_ia_{i+1}|\cdot|d_ia_{i+1}|}\\ &\geqslant \frac{11^2 + 2^2 - (9 + 4^{1-i}/10)^2}{2\cdot 11\cdot 2}\\ &> 1 - 4^{1-i}/22 \geqslant 21/22 > 4/5,\end{aligned}$$

and so $\angle c_ia_{i+1}d_i$ is smaller than the angle of $a_1a_{n+1}$ with the horizontal. $\square$

**Corollary 1.** *The cosine of the angle of segment* $c_id_i$ *with the horizontal is more than* $(\frac{4}{5}\cdot 11 - 2)/(9 + 4^{1-i}\alpha_i) \geqslant 68/91$.

For $u, v \in S$, we call $uv$ a *critical edge* if for every $w \in S\setminus\{u, v\}$ we have

$$\frac{8}{5}|uv| < |uw| + |wv|.$$

As we observed in the previous section, any spanning tree $\mathcal{T}$ on $S$ that does not include a critical edge $uv$ must have dilation $\Delta(\mathcal{T}) > 8/5$. Let us call the point $w \in S\setminus\{u, v\}$ minimizing the sum $|uw| + |wv|$ the *nearest neighbor* of $uv$.

**Lemma 2.** *The following edges are all critical*: $q_1a_1$, $a_{n+1}p_1$, $p_1p_2$, $a_ib_i$, $b_ic_i$, $d_ia_{i+1}$ (*where* $1 \leqslant i \leqslant n$), *and their mirror images.*

**Proof.** The nearest neighbor of $q_1a_1$ is $b_1$. The edge $q_1a_1$ is critical since

$$|q_1b_1| + |b_1a_1| = \sqrt{3.3^2 + 0.6^2} + 1 > \frac{8}{5}\cdot\frac{5}{2}.$$

The nearest neighbor of $a_{n+1}p_1$ is $d_n$. Since $\angle d_na_{n+1}p_1$ is obtuse, we have

$$|a_{n+1}d_n| + |d_np_1| \geqslant 2\cdot 4^{n-1} + \frac{5}{9}4^n - \frac{179}{360} = \frac{19}{18}4^n - \frac{179}{360} > \frac{8}{5}\left(\frac{5}{9}4^n - \frac{179}{360}\right) = \frac{8}{5}|a_{n+1}p_1|,$$

and so $a_{n+1}p_1$ is critical.

The nearest neighbor of $p_1p_2$ is $a_{n+1}$. The edge is critical since

$$|p_1a_{n+1}| + |a_{n+1}p_2| = \frac{5}{3}|p_1p_2| > \frac{8}{5}|p_1p_2|.$$

The edge $a_1b_1$ is critical since its nearest neighbor is $q_1$ and

$$|a_1q_1| + |q_1b_1| > 5 > \frac{8}{5}|a_1b_1|.$$

For $2 \leqslant i \leqslant n$, the nearest neighbor of $a_ib_i$ is $d_{i-1}$. By Lemma 1, the $y$-coordinate of $d_{i-1}$ is strictly smaller than the $y$-coordinate of $a_i$, so $\cos\angle d_{i-1}a_ib_i < 0$, which bounds $|d_{i-1}b_i|^2 > |d_{i-1}a_i|^2 + |a_ib_i|^2 = (\frac{\sqrt{5}}{2}\cdot 4^{i-1})^2$. We get

$$|a_id_{i-1}| + |d_{i-1}b_i| > 2\cdot 4^{i-2} + \frac{\sqrt{5}}{2}\cdot 4^{i-1} > \frac{8}{5}|a_ib_i|,$$

and so $a_i b_i$ is critical.

The nearest neighbor of $b_i c_i$ is $a_i$, and the edge is critical since

$$|b_i a_i| + |a_i c_i| = 5 \cdot 4^{i-1} > \frac{8}{5}|b_i c_i|.$$

For $1 \leqslant i \leqslant n-1$, the nearest neighbor of $d_i a_{i+1}$ is $b_{i+1}$, and the edge is critical since

$$|d_i b_{i+1}| + |b_{i+1} a_{i+1}| > 2|b_{i+1} a_{i+1}| = 2 \cdot 4^i > \frac{8}{5}|d_i a_{i+1}|.$$

Finally, the nearest neighbor of $d_n a_{n+1}$ is $p_1$, and

$$|d_n p_1| + |p_1 a_{n+1}| > 2|p_1 a_{n+1}| = 2\left(\frac{5}{9}4^n - \frac{179}{360}\right) > \frac{8}{5}\left(2 \cdot 4^{n-1}\right) = \frac{8}{5}|d_n a_{n+1}|$$

implies that $d_n a_{n+1}$ is critical. $\square$

The enumeration in Lemma 2 is exhaustive: these are all the critical edges. However, to form the connection between $c_i$ and $d_i$, only two choices are possible—this is the choice at the heart of our NP-hardness argument.

**Lemma 3.** *If $\mathcal{T}$ is a spanning tree on $S$ with $\Delta(\mathcal{T}) \leqslant 8/5$, then it contains exactly one of the edges $c_i d_i$ and $c_i a_{i+1}$, and exactly one of the edges $c'_i d'_i$ and $c'_i a'_{i+1}$, for each $1 \leqslant i \leqslant n$.*

**Proof.** Consider points $c_i d_i$, for some $1 \leqslant i \leqslant n$. If $\mathcal{T}$ contains neither $c_i d_i$ nor $c_i a_{i+1}$, then the shortest path from $c_i$ to $d_i$ in $\mathcal{T}$ must make use of a point $w \in S \setminus \{c_i, d_i, a_{i+1}\}$, and its length is at least $|c_i w| + |w d_i|$. The point $w$ minimizing this expression is $b_i$, but since

$$|d_i b_i| + |b_i c_i| > (11 + 3 - 2)4^{i-1} + 3 \cdot 4^{i-1} > \frac{8}{5}|c_i d_i|,$$

this is not good enough. It follows that $\mathcal{T}$ must contain at least one of the edges $c_i d_i$ or $c_i a_{i+1}$. Since by Lemma 2 it also contains $d_i a_{i+1}$, it cannot contain both edges. $\square$

*3.2. If there is no partition, then $\Delta(S) > 3/2$*

In fact, we will prove a slightly stronger claim: If there is no solution to the PARTITION instance, then $\Delta(S) > 3/2 + \xi$, where $\xi := 1/(4^{n+4}\dot{\sigma})$. Throughout this section, we will assume that a spanning tree $\mathcal{T}$ on $S$ exists with $\Delta(\mathcal{T}) \leqslant 3/2 + \xi$. We define

$$A := \left\{i \in \{1, \dots, n\} \mid \mathcal{T} \text{ contains } c_i d_i\right\},$$
$$A' := \left\{i \in \{1, \dots, n\} \mid \mathcal{T} \text{ contains } c'_i d'_i\right\},$$

and our aim is to show that $A$, $A'$ are a solution to the PARTITION instance.

We set $\sigma_A = \sum_{i \in A} \alpha_i$ and $\sigma_{A'} = \sum_{i \in A'} \alpha_i$. We need to show that $\sigma_A = \sigma_{A'}$, that $A \cap A' = \emptyset$, and that $A \cup A' = \{1, \dots, n\}$.

**Lemma 4.** *We have $A \cup A' = \{1, \dots, n\}$.*

**Proof.** Let us assume that for some $1 \leqslant i \leqslant n$, neither $c_i d_i$ nor $c'_i d'_i$ is in $\mathcal{T}$. We consider the dilation of the pair $d'_i d_i$. The shortest path from $d_i$ to $d'_i$ in $\mathcal{T}$ must go through both $a_{i+1}$ and $a'_{i+1}$ by Lemma 2, and so its length is at least

$$\begin{aligned} d_{\mathcal{T}}(d'_i, d_i) &\geqslant 2|d_i a_{i+1}| + 2|a_1 a_{i+1}| + |a_1 a'_1| \\ &= 4 \cdot 4^{i-1} + 10\left(4^i - 1\right) + 5 \\ &= 11 \cdot 4^i - 5. \end{aligned}$$

On the other hand, $|d'_i d_i| = |a'_{i+1} a_{i+1}| - 2\ell$, where $\ell$ is the length of the projection of $d_i a_{i+1}$ on the $x$-axis. By Lemma 1, we have $\ell > \frac{4}{5}|d_i a_{i+1}|$, and so

$$\begin{aligned}|d_i'd_i| &= |a_{i+1}'a_{i+1}| - 2\ell\\ &< |a_{i+1}'a_{i+1}| - \frac{8}{5}|d_i a_{i+1}|\\ &= 8\big(4^i - 1\big) + 5 - \frac{16}{5}4^{i-1}\\ &= \frac{36}{5}\cdot 4^i - 3,\end{aligned}$$

and so

$$\begin{aligned}d_{\mathcal{T}}(d_i', d_i)/|d_i'd_i| &> \big(11\cdot 4^i - 5\big)/\left(\frac{36}{5}\cdot 4^i - 3\right)\\ &> 3/2 + 1/4^4 \geqslant 3/2 + \xi.\end{aligned}$$

This is a contradiction, so no such $i$ can exist, and the lemma follows. □

**Lemma 5.** *We have $A \cap A' = \emptyset$ and $\sigma_A = \sigma_{A'} = 1/20$. Also, $\mathcal{T}$ contains the edge $q_1q_2$.*

**Proof.** The spanning tree $\mathcal{T}$ must contain the $6n + 6$ critical edges enumerated in Lemma 2 since $8/5 > 3/2 + \xi$. By Lemma 3, it must also contain $n$ edges connecting each $c_i$ to either $d_i$ or $a_{i+1}$, and by symmetry also $n$ edges connecting each $c_i'$ to either $d_i'$ or $a_{i+1}'$. Since $S$ consists of $8n + 8$ points, $\mathcal{T}$ has $8n + 7$ edges, and so there is only one edge unaccounted for. This edge must connect $q_2$ to some point $q \in S \setminus \{q_2\}$. We note that $|q_2q| \geqslant |q_2q_1| = \frac{25}{9}4^n - \frac{11}{18}$ (see Fig. 5).

Since $\Delta(\mathcal{T}) \leqslant 3/2 + \xi$, we have

$$\begin{aligned}d_{\mathcal{T}}(p_2', q_2) + d_{\mathcal{T}}(q_2, p_2) &\leqslant \frac{3}{2}\big(|p_2'q_2| + |q_2p_2|\big) + \xi\big(|p_2'q_2| + |q_2p_2|\big)\\ &< 5\cdot 4^{n+1} - \frac{101}{10} + \frac{1}{10\dot{\sigma}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}d_{\mathcal{T}}(p_2', q_2) + d_{\mathcal{T}}(q_2, p_2) &= d_{\mathcal{T}}(p_2', q) + |qq_2| + |q_2q| + d_{\mathcal{T}}(q, p_2)\\ &\geqslant d_{\mathcal{T}}(p_2', p_2) + 2|qq_2|\\ &\geqslant d_{\mathcal{T}}(p_2', p_2) + \frac{50}{9}4^n - \frac{11}{9}.\end{aligned}$$

Now,

$$\begin{aligned}d_{\mathcal{T}}(p_2', p_2) &= |p_2'a_{n+1}'| + d_{\mathcal{T}}(a_{n+1}', a_1') + |a_1'a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1}p_2|\\ &= \frac{10}{9}4^{n+1} + \frac{46}{45} + d_{\mathcal{T}}(a_1', a_{n+1}') + d_{\mathcal{T}}(a_1, a_{n+1}).\end{aligned}$$

What is $d_{\mathcal{T}}(a_1, a_{n+1})$? Since the shortest path from $a_1$ to $a_{n+1}$ in $\mathcal{T}$ must go through all $a_i$, we can express it as

$$d_{\mathcal{T}}(a_1, a_{n+1}) = \sum_{i=1}^{n} d_{\mathcal{T}}(a_i, a_{i+1}).$$

We now observe that $d_{\mathcal{T}}(a_i, a_{i+1}) = |a_ia_{i+1}|$ if $\mathcal{T}$ contains $c_ia_{i+1}$, that is if $i \notin A$, and $d_{\mathcal{T}}(a_i, a_{i+1}) = |a_ia_{i+1}| + \alpha_i$ if $i \in A$. This implies

$$d_{\mathcal{T}}(a_1, a_{n+1}) = |a_1a_{n+1}| + \sum_{i\in A}\alpha_i = 5\big(4^n - 1\big) + \sigma_A,$$

and similarly we have $d_{\mathcal{T}}(a_1', a_{n+1}') = 5(4^n - 1) + \sigma_{A'}$.

This gives

$$d_{\mathcal{T}}(p_2', p_2) = \frac{130}{9}4^n - \frac{404}{45} + \sigma_A + \sigma_{A'}.$$

Putting everything together we get

$$\begin{aligned} 5 \cdot 4^{n+1} - \frac{101}{10} + \frac{1}{10\dot{\sigma}} &> d_{\mathcal{T}}(p_2', q_2) + d_{\mathcal{T}}(q_2, p_2) \\ &\geqslant d_{\mathcal{T}}(p_2', p_2) + \frac{50}{9}4^n - \frac{11}{9} \\ &= 5 \cdot 4^{n+1} - \frac{102}{10} + \sigma_A + \sigma_{A'}, \end{aligned}$$

which implies $\sigma_A + \sigma_{A'} < 1/10 + 1/(10\dot{\sigma})$.

If there is an $i \in A \cap A'$, then Lemma 4 implies $\sigma_A + \sigma_{A'} \geqslant 1/10 + \alpha_i = 1/10 + \dot{\alpha}_i/(10\dot{\sigma})$. Since $\dot{\alpha}_i$ is a positive integer, this is a contradiction, and so $A \cap A' = \emptyset$ and $\sigma_A + \sigma_{A'} = 1/10$.

We now show that only $q = q_1$ is possible. We use again

$$\begin{aligned} d_{\mathcal{T}}(q_2, p_2) &\leqslant \frac{3}{2}|q_2 p_2| + \xi |q_2 p_2| \\ &< 10 \cdot 4^n - \frac{101}{20} + \frac{1}{20\dot{\sigma}} \\ &\leqslant 10 \cdot 4^n - \frac{100}{20}. \end{aligned}$$

If $q$ is on the left side of the $y$-axis, then the path from $q_2$ to $p_2$ in $\mathcal{T}$ passes through $a_1'$, and we have

$$\begin{aligned} d_{\mathcal{T}}(q_2, p_2) &\geqslant |q_2 q| + |a_1' q_1| + d_{\mathcal{T}}(q_1, p_2) \\ &\geqslant |q_2 q_1| + 2|q_1 a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2| \\ &\geqslant 10 \cdot 4^n - \frac{52}{20}, \end{aligned}$$

a contradiction. Similarly, $q$ cannot be on the right side of the $y$-axis, and the only remaining possibility is $q = q_1$.

It remains to show that $\sigma_A = \sigma_{A'} = 1/20$. If this is not the case, we can without loss of generality assume $\sigma_A > 1/20$. Since $\sum_{i \in A} \dot{\alpha}_i - \dot{\sigma}/2 > 0$ is an integer, we have $\sum_{i \in A} \dot{\alpha}_i - \dot{\sigma}/2 \geqslant 1$, and so $\sigma_A \geqslant 1/20 + 1/(10\dot{\sigma})$. On the other hand, we have

$$\begin{aligned} 10 \cdot 4^n - \frac{101}{20} + \frac{1}{20\dot{\sigma}} &> d_{\mathcal{T}}(q_2, p_2) \\ &= |q_2 q_1| + |q_1 a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2| \\ &= 10 \cdot 4^n - \frac{102}{20} + \sigma_A, \end{aligned}$$

and so $\sigma_A < 1/20 + 1/(20\dot{\sigma})$, a contradiction. □

### *3.3. If a set partition exists, then $\Delta(S) \leqslant 3/2$*

Let us call a tree $\mathcal{T}$ on $S$ a *standard tree* if it consists of the critical edges, the edge $q_1 q_2$, and for each $1 \leqslant i \leqslant n$ either $c_i d_i$ or $c_i a_{i+1}$ and either $c_i' d_i'$ or $c_i' a_{i+1}'$. In the following lemmas we will show that *any* standard tree has dilation less than 3/2 for nearly all pairs of points in $S$, excluding only the pairs $(d_i', d_i)$ (for $1 \leqslant i \leqslant n$), $(q_2, p_2)$, and $(q_2, p_2')$. These remaining pairs are where the existence of a solution to the PARTITION instance is critical.

Let $\mathcal{T}$ be an arbitrary standard tree. Let $H$ be the set of points of $S$ to the right of the $y$-axis, except $p_1$ and $p_2$, and symmetrically, let $H'$ be the set of points of $S$ to the left of the $y$-axis, except $p_1'$ and $p_2'$:

$$\begin{aligned} H &:= \{a_i, b_j, c_j, d_j \mid 1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n\}, \\ H' &:= \{a_i', b_j', c_j', d_j' \mid 1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n\}. \end{aligned}$$

Below, in Lemmas 6 and 7, we first prove that the dilation on paths within $H \cup \{q_1\}$ is less than 3/2. By symmetry, these lemmas also apply to paths within $H' \cup \{q_1\}$. Next, in Lemmas 8, 9, and 10, we analyze the dilation on paths between $H$ and $H'$, except paths from $d_i'$ to $d_i$ (for $1 \leqslant i \leqslant n$) such that $\mathcal{T}$ contains neither $c_i d_i$ nor $c_i' d_i$. Lemma 11 deals with paths from $\{p'_2, p'_1, p_1, p_2\}$ to $\{p'_2, p'_1\} \cup H' \cup \{q_1\} \cup H \cup \{p_1, p_2\} = S \setminus \{q_2\}$. It remains to consider the dilation on pairs that involve $\{q_2\}$: Lemma 12 treats this case, except for the pairs $(q_2, p_2)$ and $(q_2, p'_2)$. We then show in Lemma 13 that if a solution to the PARTITION instance exists, we can get dilation at most $3/2$ also on $(q_2, p_2)$, $(q_2, p'_2)$ and on all pairs $(d'_i, d_i)$ (for $1 \leqslant i \leqslant n$). Thus we prove that if a solution to the PARTITION instance exists, $\Delta(S) \leqslant 3/2$.

For a point $w$, denote by $w^{\downarrow}$ the orthogonal projection of $w$ on the line through $a_1$ and $a_{n+1}$. Let $P_{\mathcal{T}}(u, v)$ be the path from $u$ to $v$ in $\mathcal{T}$. The edges and vertices of the path may depend on the choice of $\mathcal{T}$: for example, $d_i$ lies on $P_{\mathcal{T}}(a_1, a_{n+1})$ if and only if $\mathcal{T}$ contains $c_i d_i$.

We first concentrate on the dilation between points $a_i, b_i, c_i$ and $d_i$ in one half of the tree.

**Lemma 6.** *Let $w \in H$. For any pair of points $u, v \in P_{\mathcal{T}}(a_{n+1}, w)$ (not necessarily vertices) we have $d_{\mathcal{T}}(u, v) < (22/21)|u^{\downarrow} v^{\downarrow}|$ and $\Delta_{\mathcal{T}}(u, v) < 22/21 < 3/2$.*

**Proof.** By Lemma 1 the cosine of the angle between any segment of the path $P_{\mathcal{T}}(a_1, a_{n+1})$ and the line $a_1 a_{n+1}$ is more than 21/22; hence the path is monotone in its projection on the line $a_1 a_{n+1}$ and each segment has length at most 22/21 times the length of its projection. Since $|u^{\downarrow} v^{\downarrow}| \leqslant |uv|$, it follows that $\Delta_{\mathcal{T}}(u, v) < 22/21$. □

Note that the lemma applies to any pair of points $u, v \in P_{\mathcal{T}}(a_{n+1}, w)$, not only to *vertices* of $P_{\mathcal{T}}(a_{n+1}, w)$. This will be convenient later when we bound the dilation of a path by cutting it into pieces and bounding the dilation for each piece separately. The endpoints of these pieces may lie in the interior of edges.

**Lemma 7.** *For any pair of vertices $u, v \in H \cup \{q_1\}$, we have $\Delta_{\mathcal{T}}(u, v) < 3/2$.*

**Proof.** We first deal with the case of $u, v \in H$. Without loss of generality, let $u$ lie above and to the right of $v$. If $u$ lies on the path $P_{\mathcal{T}}(v, a_{n+1})$, the lemma follows from Lemma 6. Otherwise, $u = d_i$ for some $1 \leqslant i \leqslant n$, and $\mathcal{T}$ does not contain the edge $c_i d_i$. Furthermore, since $v$ lies below and to the left of $u$, the vertex $v$ must be one of the points $\{q_1, a_1, b_1, c_1, d_1, a_2, \ldots, a_i, b_i, c_i\}$. Of these points, $c_i$ is closest to $d_i$ in the projection on the line through $a_1$ and $a_{n+1}$. Thus $|d_i^{\downarrow} c_i| = |d_i^{\downarrow} c_i^{\downarrow}| \leqslant |d_i^{\downarrow} v^{\downarrow}|$. Now we have:

$$\begin{aligned}
\frac{d_{\mathcal{T}}(d_i, v)}{|d_i v|} &\leqslant \frac{d_{\mathcal{T}}(d_i, d_i^{\downarrow}) + d_{\mathcal{T}}(d_i^{\downarrow}, v)}{|d_i^{\downarrow} v^{\downarrow}|} \\
&\leqslant \frac{d_{\mathcal{T}}(d_i, d_i^{\downarrow})}{|d_i^{\downarrow} c_i|} + \frac{d_{\mathcal{T}}(d_i^{\downarrow}, v)}{|d_i^{\downarrow} v^{\downarrow}|} \\
&< \frac{4}{9} + \frac{22}{21} = \frac{94}{63} < \frac{3}{2}.
\end{aligned}$$

This concludes the proof for the case of $u, v \in H$. Now suppose $v = q_1$. If $u = a_1, b_1, c_1$ or $d_1$, it can easily be verified that $\Delta_{\mathcal{T}}(u, q_1) < 3/2$ (regardless whether the path $P_{\mathcal{T}}(d_1, q_1)$ passes through $a_2$). If $u$ is any other point in $H$, then the path $P_{\mathcal{T}}(u, q_1)$ passes through $a_2$. Now observe:

$$d_{\mathcal{T}}(a_2, q_1) \leqslant \frac{35}{2} + \alpha_1 < \frac{22}{21} \cdot 17 = \frac{22}{21} |a_2, q_1^{\downarrow}|.$$

Hence we can apply the same arguments as for $u, v \in H$ to bound the dilation $\Delta_{\mathcal{T}}(u, q_1)$. □

In the following three lemmas we turn our attention to pairs of points in opposite halves of the tree (still excluding $p_1, p_2, p'_1, p'_2$ and $q_2$).

**Lemma 8.** *For any pair of points (not necessarily vertices) $u, v \in P_{\mathcal{T}}(a_{n+1}, a'_{n+1})$, we have $\Delta_{\mathcal{T}}(u, v) < 91/68 < 3/2$.*

**Proof.** By Corollary 1, the cosine of the angle of any segment of $P_{\mathcal{T}}(a_{n+1}, a'_{n+1})$ and the $x$-axis is more than 68/91. Hence the path is $x$-monotone and its dilation is less than $91/68$. □

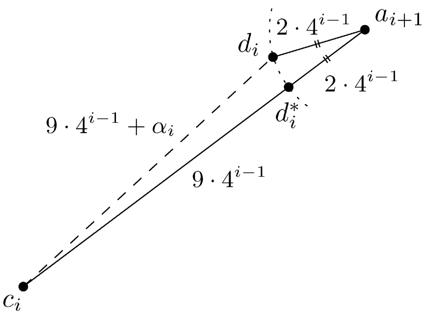


Fig. 7. The point $d_i^*$.

To facilitate the analysis of the dilation of pairs that involve a point $d_i$ or $d'_i$, we introduce an auxiliary point $d_i^*$ on $a_i a_{i+1}$:

$$d_i^* = c_i + \frac{9 \cdot 4^{i-1}}{5}\begin{pmatrix}4\\3\end{pmatrix} = a_{i+1} - \frac{2 \cdot 4^{i-1}}{5}\begin{pmatrix}4\\3\end{pmatrix},$$

and we similarly define $d_i^{*\prime}$ on $a'_i a'_{i+1}$. We have $|d_i^* a_{i+1}| = |d_i a_{i+1}| = 2 \cdot 4^{i-1}$, see Fig. 7. Since by Lemma 1 we have $\cos \angle c_i a_{i+1} d_i > 1 - 4^{1-i}/22$, we can use the cosine theorem to bound $|d_i d_i^*|$:

$$\begin{aligned}|d_i d_i^*|^2 &= |d_i a_{i+1}|^2 + |d_i^* a_{i+1}|^2 - 2|d_i a_{i+1}||d_i^* a_{i+1}| \cos \angle c_i a_{i+1} d_i\\ &= 2\bigl(2 \cdot 4^{i-1}\bigr)^2 (1 - \cos \angle c_i a_{i+1} d_i)\\ &< 2\bigl(2 \cdot 4^{i-1}\bigr)^2 \frac{1}{22 \cdot 4^{i-1}} = \frac{4^i}{11}\end{aligned}$$

and so

$$|d_i d_i^*| < \sqrt{4^i/11}. \tag{1}$$

**Lemma 9.** *For any pair of points $(d'_i, u)$, where $1 \leqslant i \leqslant n$ and $u$ is a point (not necessarily a vertex) on $P_{\mathcal{T}}(a_{n+1}, a_1)$, we have $\Delta_{\mathcal{T}}(d'_i, u) < 3/2$.*

**Proof.** If $d'_i$ lies on the path $P_{\mathcal{T}}(a'_1, a'_{n+1})$, the lemma follows from Lemma 8. Otherwise, $\mathcal{T}$ contains $c'_i a'_{i+1}$ (and not $c'_i d'_i$).

The ratio $|uu^{\downarrow}|/|a_1 u^{\downarrow}|$ is maximized for $u = d_j$, for some $j$, so with Eq. (1) we get:

$$\frac{|uu^{\downarrow}|}{|a_1 u^{\downarrow}|} \leqslant \max_j \frac{|d_j d_j^{\downarrow}|}{|a_1 d_j^{\downarrow}|} < \max_j \frac{|d_j d_j^*|}{|a_1 d_j^*|} < \max_j \frac{2^j/\sqrt{11}}{5(\frac{9}{10}4^j - 1)} < \frac{1}{20}.$$

We set $m' = \frac{9}{10}4^i - 1$ and $m = |a_1 u^{\downarrow}|/5$, and have

$$\begin{aligned}d_i^{*\prime} &= a'_1 + m'\begin{pmatrix}-4\\3\end{pmatrix},\\ u^{\downarrow} &= a_1 + m\begin{pmatrix}4\\3\end{pmatrix}\end{aligned}$$

and thus:

$$\begin{aligned}|d'_i d_i^{*\prime}| &\leqslant \frac{1}{20} \cdot 5m' = \frac{1}{4}m',\\ |uu^{\downarrow}| &\leqslant \frac{1}{20} \cdot 5m = \frac{1}{4}m.\end{aligned}$$

We can now bound $d_{\mathcal{T}}(u, d_i')$:

$$\begin{aligned}
d_{\mathcal{T}}(u, d_i') &\leqslant d_{\mathcal{T}}(u, a_1) + |a_1 a_1'| + d_{\mathcal{T}}(a_1', d_i^{*\prime}) + d_{\mathcal{T}}(d_i^{*\prime}, d_i') \\
&\leqslant (|u^{\downarrow} a_1| + \sigma) + 5 + (|a_1' d_i^{*\prime}| + \sigma) + 2|a_{i+1}' d_i'| \\
&= 5m + \frac{1}{10} + 5 + 5m' + \frac{1}{10} + 4^i \\
&= 5m + 5m' + \frac{10}{9}(m' + 1) + \frac{26}{5} \\
&= 5m + \frac{55}{9}m' + \frac{284}{45}.
\end{aligned}$$

On the other hand,

$$|ud_i'| \geqslant |u^{\downarrow} d_i^{*\prime}| - |u^{\downarrow} u| - |d_i' d_i^{*\prime}| > \left|u^{\downarrow} d_i^{*\prime}\right| - \frac{1}{4}(m + m').$$

To prove the lemma we need to show that $d_{\mathcal{T}}(u, d_i') \leqslant \frac{3}{2}|ud_i'|$. This follows from:

$$10m + \frac{110}{9}m' + \frac{568}{45} \leqslant 3\left|u^{\downarrow} d_i^{*\prime}\right| - \frac{3}{4}(m + m'),$$

which follows from:

$$\begin{aligned}
&\left(\left(10 + \frac{3}{4}\right)m + \left(\frac{110}{9} + \frac{3}{4}\right)m' + \frac{568}{45}\right)^2 \\
&\quad < 116m^2 + 169m'^2 + 279mm' + 360m + 360m' + 225 \\
&\quad = 225m^2 + 225m'^2 + 126mm' + 360m + 360m' + 225 - \left(109m^2 + 56m'^2 - 153mm'\right) \\
&\quad < 225m^2 + 225m'^2 + 126mm' + 360m + 360m' + 225 \\
&\quad = 9(4m + 5 + 4m')^2 + 9(3m - 3m')^2 \\
&\quad = \left(3\left|u^{\downarrow} d_i^{*\prime}\right|\right)^2,
\end{aligned}$$

completing the proof. □

Note that the above lemma applies symmetrically to pairs of points $(d_i, u)$ where $1 \leqslant i \leqslant n$ and $u$ is a point (not necessarily a vertex) on $P_{\mathcal{T}}(a_{n+1}', a_1')$.

**Lemma 10.** *For any pair of vertices $d_i', d_j$ with $1 \leqslant i, j \leqslant n$ and $i \neq j$, we have $\Delta_{\mathcal{T}}(d_i', d_j) < 3/2$.*

**Proof.** If $d_i'$ is on the path from $a_{n+1}'$ to $a_1'$, or if $d_j$ is on the path from $a_{n+1}$ to $a_1$, the lemma follows from Lemma 9.

Otherwise, $\mathcal{T}$ contains $c_i' a_{i+1}'$ (not $c_i' d_i'$) and $c_j' a_{j+1}'$ (not $c_j d_j$). Without loss of generality, assume that $i < j$. We set $m' = \frac{9}{10}4^i - 1$ and $m = \frac{9}{10}4^j - 1$, and have:

$$m + 1 = 4^{j-i}(m' + 1). \tag{2}$$

By Eq. (1) we have:

$$\begin{aligned}
|d_i' d_i^{*\prime}| &< \sqrt{\frac{4^i}{11}} = \sqrt{\frac{10}{99}(m' + 1)} < \frac{1}{3}\sqrt{m' + 1}, \\
|d_j d_j^*| &< \sqrt{\frac{4^j}{11}} = \sqrt{\frac{10}{99}(m + 1)} < \frac{1}{3}\sqrt{m + 1}.
\end{aligned}$$

We now bound $d_{\mathcal{T}}(d_i', d_j)$:

$$
\begin{aligned}
d_{\mathcal{T}}(d_i', d_j) &\leqslant |d_i' a_{i+1}'| + \left(|a_{i+1}' a_1'| + \frac{1}{10}\right) + |a_1' a_1| + \left(|a_1 a_{j+1}| + \frac{1}{10}\right) + |a_{j+1} d_j| \\
&= \frac{5}{9}(m'+1) + \left(\frac{50}{9}m' + \frac{5}{9} + \frac{1}{10}\right) + 5 + \left(\frac{50}{9}m + \frac{5}{9} + \frac{1}{10}\right) + \frac{5}{9}(m+1) \\
&= \frac{55}{9}(m'+m) + \frac{334}{45}.
\end{aligned}
$$

With Eq. (2) we now get:

$$
d_{\mathcal{T}}(d_i', d_j) < \frac{55}{9}(4^{j-i}+1)(m'+1).
$$

On the other hand

$$
\begin{aligned}
|d_i' d_j| &\geqslant |d_i^{*\prime} d_j^*| - |d_i' d_i^{*\prime}| - |d_j d_j^*| \\
&> \sqrt{(4m'+5+4m)^2 + (3m-3m')^2} - \frac{1}{3}\sqrt{m'+1} - \frac{1}{3}\sqrt{m+1}.
\end{aligned}
$$

For bounding $d_{\mathcal{T}}(d_i', d_j)/|d_i' d_j|$ we now consider two cases: $j = i+1$, and $j > i+1$. We first consider the case $j = i+1$. By Eq. (2) we now have $m = 4m'+3$, and thus:

$$
\begin{aligned}
&0 < 60.75(m'+1)^2 - 41\sqrt{m'+1}\,(m'+1) - 121(m'+1) + 9 \quad \left(\because m'+1 = \frac{9}{10}\cdot 4^i \geqslant 3.6\right) \\
&\quad = (400 + 81 - 420.25)(m'+1)^2 - 41\sqrt{m'+1}\,(m'+1) - (120+1)(m'+1) + 9 \\
&400(m'+1)^2 - 120(m'+1) + 9 + 81(m'+1)^2 > 420.25(m'+1)^2 + 41\sqrt{m'+1}\,(m'+1) + \left(\sqrt{m'+1}\right)^2, \\
&\sqrt{\{20(m'+1)-3\}^2 + \{9(m'+1)\}^2} > 20.5(m'+1)^2 + \sqrt{m'+1}, \\
&|d_i' d_j| > \sqrt{(20m'+17)^2 + (9m'+9)^2} - \sqrt{m'+1} \\
&\qquad > 20.5(m'+1).
\end{aligned}
$$

Hence:

$$
\frac{d_{\mathcal{T}}(d_i', d_j)}{|d_i' d_j|} < \frac{\frac{55}{9}(4^{j-i}+1)}{20.5} < \frac{275/9}{20.5} < \frac{3}{2}.
$$

It remains to consider the case where $j > i+1$. By Eq. (2) we have

$$
m - m' = (4^{j-i}-1)(m'+1).
$$

Thus we get:

$$
\begin{aligned}
|d_i' d_j| &> \sqrt{(4m+5+4m')^2 + (3m-3m')^2} - \frac{1}{3}\sqrt{m+1} - \frac{1}{3}\sqrt{m'+1} \\
&> \sqrt{16(m+m')^2 + 9(m-m')^2} - \frac{1}{3}\sqrt{m'+1} - \frac{1}{3}\sqrt{m+1} \\
&> 5(m-m') - \frac{1}{3}(m'+1+m+1) \\
&= 5(4^{j-i}-1)(m'+1) - \frac{1}{3}(4^{j-i}+1)(m'+1) \\
&= \frac{14}{3}(4^{j-i}+1)(m'+1) - 10(m'+1).
\end{aligned}
$$

Hence:

$$
\frac{d_{\mathcal{T}}(d_i', d_j)}{|d_i' d_j|} < \frac{\frac{55}{9}(4^{j-i}+1)}{\frac{14}{3}(4^{j-i}+1) - 10} = \frac{55/3}{14 - 30/(4^{j-i}+1)} \leqslant \frac{55/3}{208/17} = \frac{935}{624} < \frac{3}{2}. \qquad \square
$$

We now study pairs of vertices involving $p_1$, $p_2$, $p_1'$ and/or $p_2'$, but not $q_2$.

**Lemma 11.** *For any pair of vertices $u, v$ where $u \in \{p_1, p_2, p'_1, p'_2\}$ and $v \in S \setminus \{q_2\}$, we have $\Delta_{\mathcal{T}}(u, v) < 3/2$.*

**Proof.** We assume that $u \in \{p_1, p_2\}$ (the case of $u \in \{p'_1, p'_2\}$ is symmetric). We now distinguish four cases for $v$: first $v \in \{p_1, p_2, a_{n+1}\}$, second $v \in H \setminus \{a_{n+1}\}$, third $v \in H' \cup \{q_1\}$, and fourth $v \in \{p'_1, p'_2\}$.

First, if $v \in \{p_1, p_2, a_{n+1}\}$, then the connection between $u$ and $v$ is a straight line and the dilation is 1.

Second, if $v \in H \setminus \{a_{n+1}\}$, then the path from $u$ to $v$ goes through $a_{n+1}$, and $\angle u a_{n+1} v \geqslant \pi/2$. Hence the dilation for the pair $(u, v)$ is (using Lemma 6):

$$\frac{d_{\mathcal{T}}(u, v)}{|uv|} < \frac{|ua_{n+1}| + \frac{22}{21}|a_{n+1}v|}{(|ua_{n+1}| + |a_{n+1}v|)/\sqrt{2}} < \frac{22}{21}\sqrt{2} < \frac{3}{2}.$$

Third, if $v \in H' \cup \{q_1\}$, let $w$ be a point where the segment $uv$ intersects the path from $a_{n+1}$ to $a_1$ (which is a part of the path from $u$ to $v$). By the analysis of the previous case $\Delta_{\mathcal{T}}(u, w) < 3/2$, and by Lemmas 8 or 9 we have $\Delta_{\mathcal{T}}(w, v) < 3/2$; hence $\Delta_{\mathcal{T}}(u, v) < 3/2$.

Finally, if $v \in \{p'_1, p'_2\}$, let $w$ be defined as above, and let $w'$ be a point where the segment $uv$ intersects the path from $a'_1$ to $a'_{n+1}$. By the analysis of the second case $\Delta_{\mathcal{T}}(u, w) < 3/2$ and $\Delta_{\mathcal{T}}(w', v) < 3/2$, and by Lemma 8 we have $\Delta_{\mathcal{T}}(w, w') < 3/2$; hence $\Delta_{\mathcal{T}}(u, v) < 3/2$. □

It remains to consider the dilation on pairs of points that involve $q_2$. We only consider the dilation on pairs of points $(u, q_2)$ where $u \notin \{p_2, p'_2\}$: the dilation of $(p_2, q_2)$ and $(p'_2, q_2)$ depends critically on the choice of standard tree and we will defer its analysis to Lemma 13.

**Lemma 12.** *For any vertex $u \in S \setminus \{p_2, p'_2\}$ we have $\Delta_{\mathcal{T}}(u, q_2) < 3/2$.*

**Proof.** We distinguish four cases: first $u = q_1$, second $u$ is on the path from $a_1$ to $a_{n+1}$, third $u = d_i$ for some $1 \leqslant i \leqslant n$, and finally $u = p_1$ (the cases in which $u$ lies to the left of the $y$-axis are symmetric).

First, if $u = q_1$, then the connection between $u$ and $q_2$ is a straight line and the dilation is 1.

Second, if $u$ lies on the path from $a_1$ to $a_{n+1}$, let $r = (0, -15/8)$ be the intersection of $q_1q_2$ with the line through $a_1$ and $a_{n+1}$. With the sine rule we get:

$$\begin{aligned}
|u^{\downarrow}r| + |rq_2| &= \frac{\sin\angle ru^{\downarrow}q_2 + \sin\angle u^{\downarrow}q_2r}{\sin\angle q_2ru^{\downarrow}}|u^{\downarrow}q_2| \\
&\leqslant \frac{2\sin(\frac{1}{2}(\angle ru^{\downarrow}q_2 + \angle u^{\downarrow}q_2r))}{\sin\angle q_2ru^{\downarrow}}|u^{\downarrow}q_2| \\
&= \frac{2\sin(\frac{1}{2}(\pi - \angle q_2ru^{\downarrow}))}{\sin\angle q_2ru^{\downarrow}}|u^{\downarrow}q_2| \\
&= \frac{2/\sqrt{5}}{4/5}|u^{\downarrow}q_2| = \frac{\sqrt{5}}{2}|u^{\downarrow}q_2|.
\end{aligned}$$

With Lemma 6 we now get:

$$\begin{aligned}
\frac{d_{\mathcal{T}}(u, q_2)}{|uq_2|} &\leqslant \frac{d_{\mathcal{T}}(u, a_1) + d_{\mathcal{T}}(a_1, r) + |rq_2|}{|u^{\downarrow}q_2|} \\
&\leqslant \frac{\frac{22}{21}|u^{\downarrow}a_1| + (|a_1r| + \frac{5}{4}) + |rq_2|}{|u^{\downarrow}q_2|} \\
&< \frac{\frac{22}{21}|u^{\downarrow}a_1| + |a_1r| + |rq_2|}{|u^{\downarrow}q_2|} + \frac{5/4}{|q_1q_2|} \\
&< \frac{22}{21} \cdot \frac{|u^{\downarrow}r| + |rq_2|}{|u^{\downarrow}q_2|} + \frac{5/4}{\frac{25}{9}4^n - \frac{11}{18}} \\
&\leqslant \frac{11}{21}\sqrt{5} + \frac{5}{42} < \frac{3}{2}.
\end{aligned}$$

Third, if $u = d_i$, we get:

$$\begin{aligned}
\frac{d_{\mathcal{T}}(u,q_2)}{|uq_2|} &\leqslant \frac{d_{\mathcal{T}}(d_i,a_1)+d_{\mathcal{T}}(a_1,r)+|rq_2|}{|d_i^* q_2|} \\
&\leqslant \frac{(2|d_i^* a_{i+1}|+\frac{22}{21}|d_i^* a_1|)+(|a_1 r|+\frac{5}{4})+|rq_2|}{|d_i^* q_2|} \\
&= \frac{\frac{22}{21}|d_i^* a_1|+|a_1 r|+|rq_2|}{|d_i^* q_2|}+\frac{2|d_i^* a_{i+1}|+\frac{5}{4}}{|d_i^* q_2|} \\
&< \frac{22}{21}\cdot\frac{|d_i^* r|+|rq_2|}{|d_i^* q_2|}+\frac{4^i+\frac{5}{4}}{\frac{25}{9}4^n-\frac{11}{18}+\frac{27}{10}4^i-3} \\
&\leqslant \frac{11}{21}\sqrt{5}+\frac{3}{10}-\frac{\frac{5}{6}4^n-\frac{7}{3}-\frac{19}{100}4^i}{\frac{25}{9}4^n-\frac{65}{18}+\frac{27}{10}4^i} \\
&< \frac{11}{21}\sqrt{5}+\frac{3}{10}<\frac{3}{2}.
\end{aligned}$$

Finally, if $u = p_1$, we have

$$\begin{aligned}
d_{\mathcal{T}}(u,q_2) &\leqslant |q_2q_1|+|q_1a_1|+|a_1a_{n+1}|+\sigma+|a_{n+1}p_1| \\
&= \frac{25}{9}4^n-\frac{11}{18}+\frac{5}{2}+5\cdot(4^n-1)+\frac{1}{10}+\frac{5}{9}4^n-\frac{179}{360} \\
&= \frac{25}{3}4^n-\frac{421}{120} \\
&< 8.34\cdot 4^n-3.50.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|uq_2|^2 &= \left(\frac{13}{3}4^n-\frac{1079}{600}\right)^2+\left(\frac{16}{3}4^n-\frac{241}{75}\right)^2 \\
&= \frac{425}{9}16^n-\frac{1795}{36}4^n+\frac{195257}{14400} \\
&> 6.87^2\cdot 16^n-50.5632\cdot 4^n+3.68^2 \\
&= (6.87\cdot 4^n-3.68)^2 \\
&> \left(\frac{2}{3}(8.34\cdot 4^n-3.50)\right)^2,
\end{aligned}$$

and the claim follows. □

We have now completed our analysis of standard trees. It remains to show that if a solution to the PARTITION instance exists, then we can choose a standard tree with dilation $3/2$. Given $A$, $A'$ with $A \cup A' = \{1,\dots,n\}$, $A \cap A' = \emptyset$, and $\sigma_A = \sigma'_A = 1/20$, we construct a standard tree $\mathcal{T}$ as follows: If $i \in A$, then $\mathcal{T}$ contains $c_i d_i$ and $c'_i a'_{i+1}$, otherwise (that is, if $i \in A'$) $\mathcal{T}$ contains $c'_i d'_i$ and $c_i a_{i+1}$.

**Lemma 13.** *The tree $\mathcal{T}$ constructed above has dilation* $3/2$.

**Proof.** Lemmas 6–12 prove that we have $\Delta_{\mathcal{T}}(u,v) < 3/2$ for all pairs of points $u, v \in S$, except possibly for the pairs $(d'_i, d_i)$ (with $1 \leqslant i \leqslant n$), $(p_2, q_2)$, and $(p'_2, q_2)$.

By construction, for any $1 \leqslant i \leqslant n$ either $d_i$ is on the path from $a_{n+1}$ to $a_1$, or $d'_i$ is on the path from $a'_{n+1}$ to $a'_1$. Hence $\Delta_{\mathcal{T}}(d'_i, d_i) < 3/2$ by Lemma 9.

It remains to check the dilation of $(p_2, q_2)$ and $(p'_2, q_2)$. We have:

$$\begin{aligned}
d_{\mathcal{T}}(p_2, q_2) &= |p_2 a_{n+1}| + d_{\mathcal{T}}(a_{n+1}, a_1) + |a_1 q_1| + |q_1 q_2| \\
&= |p_2 a_{n+1}| + |a_{n+1} a_1| + \sigma_A + |a_1 q_1| + |q_1 q_2| \\
&= \frac{5}{9} 4^{n+1} - \frac{179}{90} + 5(4^n - 1) + \frac{1}{20} + \frac{5}{2} + \frac{25}{9} 4^n - \frac{11}{18} \\
&= \frac{5}{2} 4^{n+1} - \frac{101}{20}.
\end{aligned}$$

Since $|p_2 q_2| = \frac{5}{3} 4^{n+1} - \frac{101}{30}$, it follows that $\Delta_{\mathcal{T}}(p_2, q_2) = 3/2$. By a symmetric calculation we can show that $\Delta_{\mathcal{T}}(p_2', q_2) = 3/2$. □

*3.4. Reduction with integer coordinates*

To complete our proof of Theorem 2, we need to construct a set of points with integer coordinates. The construction in Section 3.1 does not achieve that yet, because the points $d_i$ are defined as the solution of a quadratic equation.

Instead of the points $d_i$ originally defined, we will therefore compute *approximations* $\tilde{d}_i$ with $|d_i - \tilde{d}_i| < \varepsilon$, for an $\varepsilon < 1$ to be determined more precisely later. We denote by $\tilde{S}$ the set of points obtained that way, that is, the set of points $a_i, b_i, c_i, \tilde{d}_i, p_i$ and their mirror images as well as the two points $q_i$.

In the following lemma we bound by how much the dilation of the corresponding points in $S$ and $\tilde{S}$ can differ.

**Lemma 14.** *We have* $|\Delta(S) - \Delta(\tilde{S})| < 4^{n+6} n\varepsilon$.

**Proof.** Let $u, v$ be a pair of points in $S$, let $\tilde{u}, \tilde{v}$ be the corresponding points in $\tilde{S}$, and let $\mathcal{T}$ be any spanning tree on $S$. By slight abuse of notation, we will allow $\mathcal{T}$ to also denote the corresponding tree on $\tilde{S}$. We set $X := d_{\mathcal{T}}(u, v)$, $\tilde{X} := d_{\mathcal{T}}(\tilde{u}, \tilde{v})$, $Y := |uv|$, and $\tilde{Y} := |\tilde{u}\tilde{v}|$. Since $|u\tilde{u}| < \varepsilon$ and $|v\tilde{v}| < \varepsilon$, we have $|Y - \tilde{Y}| < 4\varepsilon$. The path from $\tilde{u}$ to $\tilde{v}$ in $\mathcal{T}$ passes at most $2n$ approximated points, and so $|X - \tilde{X}| < 4n\varepsilon$.

The edges of $\mathcal{T}$ have length less than $4^{n+3}$, and so $X < (|S| - 1)4^{n+3} \leqslant 15n \cdot 4^{n+3}$. We have $Y \geqslant 1$, and $\tilde{Y} \geqslant 1$, and thus get

$$\begin{aligned}
\frac{\tilde{X}}{\tilde{Y}} - \frac{X}{Y} &= \frac{\tilde{X}Y - X\tilde{Y}}{Y\tilde{Y}} \\
&< \frac{Y(X + 4n\varepsilon) - X(Y - 4\varepsilon)}{Y\tilde{Y}} \\
&= \frac{4n\varepsilon}{\tilde{Y}} + \frac{4\varepsilon X}{Y\tilde{Y}} \\
&\leqslant 4n\varepsilon + 4\varepsilon(15n4^{n+3}) \leqslant 4^{n+6} n\varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{X}{Y} - \frac{\tilde{X}}{\tilde{Y}} &= \frac{X\tilde{Y} - \tilde{X}Y}{Y\tilde{Y}} \\
&< \frac{X(Y + 4\varepsilon) - Y(X - 4n\varepsilon)}{Y\tilde{Y}} \\
&= \frac{4\varepsilon X}{Y\tilde{Y}} + \frac{4n\varepsilon}{\tilde{Y}} \leqslant 4^{n+6} n\varepsilon.
\end{aligned}$$

Taken together this implies $|X/Y - \tilde{X}/\tilde{Y}| < 4^{n+6} n\varepsilon$, or

$$\left|\Delta_{\mathcal{T}}(u, v) - \Delta_{\mathcal{T}}(\tilde{u}, \tilde{v})\right| < 4^{n+6} n\varepsilon.$$

Since this is true for any pair $u, v$ and any spanning tree $\mathcal{T}$, the lemma follows. □

We will choose $\varepsilon < \xi/(4^{n+7} n)$, and so Lemma 14 implies that $|\Delta(S) - \Delta(\tilde{S})| < \xi/4$. We proved in the previous section that if our PARTITION instance has a solution, then $\Delta(S) \leqslant 3/2$, and therefore $\Delta(\tilde{S}) < 3/2 + \xi/4$. On the other hand, we showed in Section 3.2 that if the PARTITION instance has no solution, then $\Delta(S) \geqslant 3/2 + \xi$, and so $\Delta(\tilde{S}) > 3/2 + 3\xi/4$. It follows that by determining whether or not $\Delta(\tilde{S}) \leqslant 3/2 + \xi/2$, we can still decide the correct answer to the PARTITION instance.

Recall that the *input size* of the PARTITION instance is the total bit complexity of the $n$ integers $(\alpha_1, \dots, \alpha_n)$. Let $k$ be an integer with $k > 4n + 22 + \log n + \log \dot{\sigma}$. Clearly $k$ can be chosen to be polynomial in the input size, and we have $2^{-k} < \xi/(4^{n+7}n)$. By the above, it suffices to ensure that $|d_i - \tilde{d}_i| \leqslant 2^{-k}$, that is, it suffices to compute $\tilde{d}_i$ with $k$ bits after the binary point.

We will multiply all coordinates in our construction by $1800 \cdot 2^k$. We first observe that the coordinates of the points $p_i, q_i, a_i, b_i, c_i$ are now all integers, and by the above it suffices to approximate $d_i$ by an integer as well. Since $d_i$ is defined as the intersection of two circles with integer radii and centers with integer coordinates, an integer approximation can be computed in time polynomial in the bit complexity of the six integers involved.

The largest coordinate in our point set is less than $1800 \cdot 2^k \cdot 2 \cdot 4^{n+1}$, so all numbers can be represented with at most $2n + k + 15$ bits. This implies that the total bit complexity of our construction is polynomial in the input size of the PARTITION instance.

The threshold $3/2 + \xi/2$ can be expressed as a rational number $P/Q$, with $P = 3 \cdot 4^{n+4}\dot{\sigma} + 1$ and $Q = 2 \cdot 4^{n+4}\dot{\sigma}$. Both numbers have bit complexity polynomial in the input size as well.

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