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Aperture-Angle and Hausdorff-Approximation of Convex Figures

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Abstract The *aperture angle* $\alpha(x, Q)$ of a point $x \notin Q$ in the plane with respect to a convex polygon Q is the angle of the smallest cone with apex x that contains Q. The *aperture angle approximation error* of a compact convex set C in the plane with respect to an inscribed convex polygon $Q \subset C$ is the minimum aperture angle of any $x \in C \setminus Q$ with respect to Q. We show that for any compact convex set C in the plane and any $k > 2$, there is an inscribed convex k-gon $Q \subset C$ with aperture angle approximation error $(1 - \frac{2}{k+1})\pi$. This bound is optimal, and settles a conjecture by Fekete from the early 1990s.

The same proof technique can be used to prove a conjecture by Brass: If a polygon P admits no approximation by a sub-k-gon (the convex hull of k vertices of P) with Hausdorff distance σ, but all subpolygons of P (the convex hull of some vertices of P) admit such an approximation, then P is a $(k+1)$-gon. This implies the follow-

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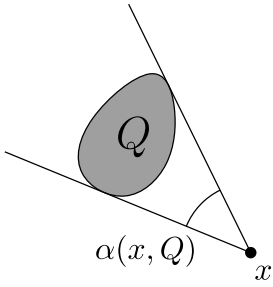
ing result: For any $k > 2$ and any convex polygon P of perimeter at most 1 there is a sub-k-gon Q of P such that the Hausdorff-distance of P and Q is at most $\frac{1}{k+1} \sin \frac{\pi}{k+1}$.

Keywords Hausdorff approximation · Aperture angle · Convex figure · Subpolygon

1 Introduction



Let Q be a compact set in the plane, and let x be a point outside Q. The *aperture angle* $\alpha(x, Q)$ of x with respect to Q is the angle of the smallest cone with apex x that contains Q (that is, its boundary consists of two rays emanating from x tangent to Q). The aperture angle plays a role in various applications related to sensing, and has been studied in a number of papers. Bose et al. [1], for instance, consider two disjoint convex polygons P and Q in the plane, and give algorithms to compute the maximum aperture angle and the minimum aperture angle with respect to Q when x is allowed to vary in P. Hoffmann et al. [7] introduce the *angle hull* of a connected region Q inside a simple polygon P, consisting of all those points $x \in P$ with $\alpha(x, Q)$ at least a given angle. They give bounds on the length of the boundary of the angle hull, and apply this to the problem of exploring an unknown environment. Cheong and van Oostrum [5] give bounds on the complexity of the angle hull of a convex polygon in a polygonal environment, and apply this to the problem of motion planning under directional uncertainty.

We consider the problem of placing k sensors or transmitters in a compact convex room C. A point in C is covered perfectly if it lies inside the convex hull $Q \subset C$ of the sensors. However, if C has more than k vertices or even a smooth boundary, we must have $Q \subsetneq C$, and it is not possible to achieve this for all points of C. For points $x \in C \setminus Q$, we would like to maximize the coverage by the sensors, and measure this using the aperture angle $\alpha(x, Q)$. For given C and $Q \subset C$, let us denote the worst coverage as

$$\alpha(C, Q) := \min_{x \in C} \alpha(x, Q),$$

where we set $\alpha(x, Q) = \pi$ for $x \in Q$. Since C is compact and $\alpha(x, Q)$ is continuous, the minimum is indeed attained in C and this is well-defined. We are looking for the best placement of k sensors, so we seek to maximize $\alpha(C, Q)$ over all convex k-gons inscribed in C. Let us denote this quantity as $\alpha(C, k)$, defined formally as

$$\alpha(C, k) := \max_{Q \in \mathcal{C}_k(C)} \alpha(C, Q),$$

where $\mathcal{C}_k(C)$ is the family of convex k-gons inscribed to C. In other words, we study the approximation of convex sets by inscribed k-gons with respect to the "aperture-angle distance." This distance measure is attractive as it is naturally scale-invariant, without needing to be normalized by some global property of C (such as its perimeter or area). We are now interested in the following question: Given k, what aperture

angle can we *guarantee* for any possible compact convex C? In other words, we ask for the following quantity

$$\alpha(k) := \inf_{C\in\mathcal{C}} \alpha(C,k) = \inf_{C\in\mathcal{C}} \max_{Q\in\mathcal{C}_k(C)} \min_{x\in C} \alpha(x,Q),$$

where $\mathcal{C}$ is the family of compact convex figures in the plane. This question was first asked by Fekete in 1990, and circulated at several open problem sessions in the early 1990s [6]. An upper bound for $\alpha(k)$ is given by the regular convex $(k+1)$-gon P_{k+1}. Since any k-gon Q inscribed in P_{k+1} must "miss" a vertex of P_{k+1}, we have $\alpha(P_{k+1},Q) \le (1-2/(k+1))\pi$ (the interior angle at each vertex of P_{k+1}). For a lower bound, we can walk around the boundary of a given C and place a vertex of Q whenever the tangent direction has changed by $2\pi/k$. It is easy to see that this achieves $\alpha(C,Q) \ge (1-2/k)\pi$, and so we have $\alpha(k) \ge (1-2/k)\pi$.

A discrepancy between the two bounds remained, and Fekete conjectured that the upper bound is correct:

Conjecture 1 *For any $k \ge 2$, the smallest value of $\alpha(k)$ is achieved by the regular $(k+1)$-gon, and we therefore have*

$$\alpha(k) = \left(1 - \frac{2}{k+1}\right)\pi.$$

Fekete already showed that his conjecture holds for $k=2$ and $k=3$, and experiments in Jenkner's Master thesis [8] indicate that it should hold for general k. The problem was also published by Brass and Lassak [3], and it appears again as Problem 5 in Sect. 11.3 of Brass et al.'s encyclopedic collection of research problems in discrete geometry [4], with a short proof of the case $k \in \{2,3\}$.

If P is a *convex polygon*, then it is known that $\alpha(P,k)$ can be attained by an inscribed *subpolygon*. Here, a *subpolygon* of P is the convex hull of a subset of P's vertices. This provides for an interesting similarity between Conjecture 1 and the following conjecture by Brass on Hausdorff approximation by subpolygons:

Conjecture 2 *Let $\mathcal{P}$ be a family of convex polygons in $\mathbb{R}^2$ that is closed under taking subpolygons. If $\mathcal{P}$ has an element that is hardest to approximate by its k-vertex subpolygons with respect to the Hausdorff metric, then one can also find a $(k+1)$-gon in $\mathcal{P}$ with this property.*

Conjecture 2 appears as Conjecture 5 in Sect. 11.5 of Brass et al. [4]. It was first suggested by Brass in 2000 [2].

In this paper, we prove both Conjecture 1 and 2. Our proof relies on a combinatorial analysis of worst-approximable polygons in Sect. 3. Here, a polygon P is *worst-approximable* if every proper subpolygon R of P admits a better approximation by k-vertex subpolygons than P does. Our analysis in Sect. 3 makes no use of the geometry of the problem, and applies to any approximation measure that is *monotone* in the sense that "growing" $Q \subset P$ cannot increase the approximation error (a formal definition of monotonicity is in Sect. 2).

In Sect. 4 we bring back in the geometry of the aperture angle problem, and prove that a convex polygon that is worst-approximable under aperture-angle approximation is in fact a $(k+1)$-gon. This implies a positive answer to Conjecture 1 for the case of polygons. The case of arbitrary compact convex sets is then proven using a limit argument.

Similarly, we study the geometry of the Hausdorff problem in Sect. 5 and prove that any polygon that is worst-approximable under Hausdorff approximation is again a $(k+1)$-gon.

As an application of this result, we finally consider the family $\mathcal{P}$ of convex polygons with unit perimeter. We show that the $(k+1)$-gon P in $\mathcal{P}$ that admits the worst approximation by a k-vertex subpolygon is the regular $(k+1)$-gon. Together with our result on Conjecture 2, this implies that for every convex polygon P with unit perimeter and every $k>2$ there is a k-vertex subpolygon Q of P such that the Hausdorff distance between P and Q is at most $\frac{1}{k+1}\sin\frac{\pi}{k+1}$. This result is the "subpolygon version" of a result by Popov, who had proven that any convex figure C of perimeter one admits an inscribed k-gon Q with Hausdorff-distance at most $\frac{1}{2k}\sin\frac{\pi}{k}$ [10]. Popov's result is not known to be tight. Popov conjectured that the regular $(k+1)$-gon is the worst case [9, 10]. Our result shows that the equivalent statement is true for approximation by subpolygons.

2 Preliminaries

Let P be a convex n-gon and let V be the set of its vertices. For three vertices $p,u,q\in V$, we say that u *lies between* p *and* q and write $p\preceq u\preceq q$ if a counterclockwise traversal of P starting at p encounters u before q (and u is allowed to coincide with p or q). If we do not allow u to coincide with p, we write $p\prec u\preceq q$, see Fig. 1a.

For any subset $V'\subset V$, the convex hull of V' is a *subpolygon* of P. A subpolygon Q is *proper* if $Q\neq P$. We are interested in subpolygons Q of P that closely approximate P. Let $\varphi(P,Q)\geq 0$ denote the *approximation error* of P with respect to a subpolygon Q of P. We will consider two different error functions: the aperture-angle and the Hausdorff-distance (to be defined below). Let $\mathcal{C}_k(P)$ denote the set of convex subpolygons of P with at most k vertices and let $\varphi_k(P)$ denote the smallest error that can be achieved by a polygon in $\mathcal{C}_k(P)$, that is,

$$\varphi_k(P) := \min_{Q\in\mathcal{C}_k(P)} \varphi(P,Q).$$

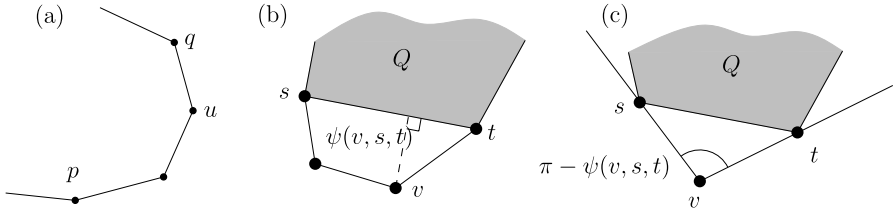


Fig. 1 **a** u lies between p and q, that is, $p\prec u\prec q$. **b** Illustrating the Hausdorff distance, and **c** the aperture-angle

Clearly $\varphi_k(P) = 0$ if P has at most k vertices. We require that $\varphi(P, Q)$ can be expressed as

$$\varphi(P, Q) = \max_{v \in V} \varphi(v, Q),$$

where $\varphi(v, Q)$ is zero if v is a vertex of Q, and is otherwise of the form

$$\varphi(v, Q) = \psi(v, s, t),$$

where s and t are the two consecutive vertices of Q with $s \prec v \prec t$. Furthermore, we require the function ψ to be *monotone*, that is, if $s \preceq s' \prec v \prec t' \preceq t$, then $\psi(v, s', t') \leq \psi(v, s, t)$.

If we set $\psi(v, s, t)$ to be the distance of v from the segment st, then $\varphi(P, Q)$ is the Hausdorff-distance between P and Q, as shown in Fig. 1b.

If we set $\psi(v, s, t) = \pi - \angle svt$, then $\pi - \varphi(P, Q)$ is the smallest aperture angle of any vertex of P with respect to Q, see Fig. 1c. Since it is easy to see that $\alpha(x, Q)$ is minimized within P at the vertices of P, this implies $\varphi(P, Q) = \pi - \alpha(P, Q)$. Note that we use the complement of the angle since we want to minimize the error, but maximize the angle.

3 Properties of Worst-Approximable Polygons

Let us call a convex polygon P *worst-approximable* if for every proper subpolygon Q of P we have $\varphi_k(Q) < \varphi_k(P)$. In this section we study the nature of worst-approximable polygons. Our arguments are purely combinatorial, using only the monotonicity of ψ.

We start by introducing some basic concepts. A pair $(p, q) \in V^2$ is called a *diagonal*. For a given approximation error $\sigma > 0$, a diagonal (p, q) is called *feasible* if for every $v \in V$ with $p \prec v \prec q$ we have $\psi(v, p, q) \leq \sigma$. By monotonicity of ψ, if (p, q) is feasible, then so is any (p', q') with $p \preceq p' \prec q' \preceq q$. A feasible diagonal (p, q) is called a *chord* if it is the feasible diagonal starting at p of largest combinatorial length, where the *combinatorial length* of a diagonal (p, q) is the *number of edges* on the polygon boundary from p to q (in counter-clockwise direction). In other words, (p, q) is a chord if q is the last vertex on P (starting from p) such that (p, q) is feasible.

The *chord graph* $\mathcal{G}$ is a directed graph with vertex set V, such that (p, q) is an edge of $\mathcal{G}$ if (p, q) is a chord. Clearly, every vertex of $\mathcal{G}$ has out-degree one.

Let $k > 2$ and $\sigma > 0$ be fixed, and assume that P is a convex n-gon with $\varphi_k(P) > \sigma$, but such that for every proper subpolygon R of P we have $\varphi_k(R) \leq \sigma$. As before, let V be the set of P's vertices, and let $\mathcal{G}$ be the chord graph of P for the approximation error σ. For $u \in V$, let P_u be the convex hull of $V \setminus \{u\}$, see Fig. 2a. By assumption, P_u has a k-gon approximation Q_u with error $\leq \sigma$. Without loss of generality, we can assume that Q_u consists of $k - 1$ chords and of an edge st, where $s \prec u \prec t$, as illustrated in Fig. 2b. For all $x \in V \setminus \{u\}$ with $s \prec x \prec t$ we have $\psi(x, s, t) \leq \sigma$, but $\psi(u, s, t) > \sigma$.

We call st the *base* of Q_u, and note that Q_u is completely determined by t, since the other $k - 1$ vertices can be found by following $k - 1$ arcs from t in the chord

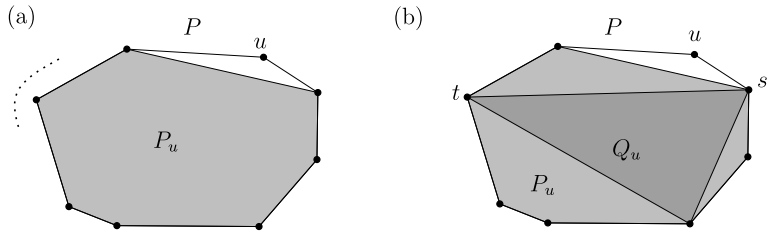


Fig. 2 **a** Illustrating P and P_u, and **b** Q_u with base st

graph. Let $s : V \mapsto V$ and $t : V \mapsto V$ be functions mapping u to the clockwise (s) and counter-clockwise (t) endpoints of the base of Q_u.

Lemma 1 *The function t is a bijection.*

Proof Assume that there exist two subpolygons Q_u and Q_v, $u \neq v$, of P that both have base st. Since st is the base of Q_u and $s \prec v \prec t$, we have $\psi(u, s, t) > \sigma$ and $\psi(v, s, t) \leq \sigma$. This, however, contradicts $\psi(v, s, t) > \sigma$ due to st being the base of Q_v. Thus, the base of each Q_u is distinct. Since Q_u is completely determined by $t(u)$, the function t is an injection, and therefore a bijection. □

For a base st, let the *witness* $w(st)$ of st be the unique vertex with $s \prec w(st) \prec t$ with $\psi(w(st), s, t) > \sigma$. The witness of the base of Q_u is of course u. There are thus n bases in total, and their witnesses are all distinct.

Lemma 2 *It is impossible for two bases st and $s't'$ to be nested, that is, to realize the order*

$$s \preceq s' \prec t' \preceq t.$$

Proof Assume the opposite, that is, $s \preceq s' \prec t' \preceq t$. Since $w(s't') \neq w(st)$ we have $\psi(w(s't'), s, t) \leq \sigma$. By monotonicity of ψ this implies $\psi(w(s't'), s', t') \leq \sigma$, a contradiction. □

Next we study the nature of the chord graph and prove that it is surprisingly symmetric. We denote the vertices V of P as $v_0, v_1, \ldots, v_{n-1}$ in counter-clockwise order. Throughout the paper, arithmetic on indices of v will be modulo n.

Lemma 3 *Every vertex in the chord graph has in-degree and out-degree one. There is an integer $m > 1$ such that every chord is of the form (v_i, v_{i+m}).*

Proof From the definition of chord graphs it immediately follows that the out-degree of each vertex is one. Since the number of arcs in the chord graph is n, it suffices to show that no vertex has in-degree greater than one. Assume that q is a vertex with in-degree at least two, that is, there are chords (p, q) and (p', q), with $p \prec p' \prec q$. Let $u = t^{-1}(p)$ and $u' = t^{-1}(p')$, and so $t(u) = p$ and $t(u') = p'$. Since we can find $s(u)$ from $t(u)$ by following $k - 1$ arcs in the chord graph, $t(u) = t(u')$ implies

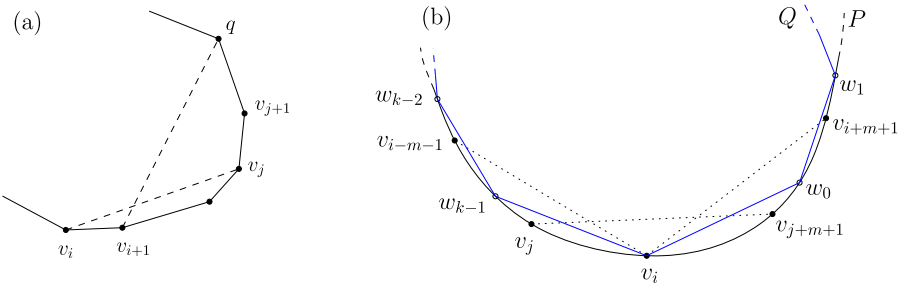


Fig. 3 **a** Illustrating the proof of Lemma 3, and **b** the proof of Lemma 5

$s(u) = s(u')$. Now we have $s(u') \preceq s(u) \preceq t(u) \prec t(u')$, a contradiction to Lemma 2. It follows that no vertex of the chord graph has in-degree greater than one.

Next we consider two chords (v_i, v_j) and (v_{i+1}, q), as shown in Fig. 3a. If $q \neq v_{j+1}$ then there has to be a chord (p, v_{j+1}) with $v_i \prec p \prec v_{i+1} \prec v_j \prec v_{j+1} \prec q$, a contradiction. This implies that two consecutive chords must have the same combinatorial length, and so all chords do. □

We will use m to denote the combinatorial length of all chords. For every $0 \leq i < n$, (v_i, v_{i+m}) is the chord starting at v_i, and (v_{i-m}, v_i) is the chord ending at v_i.

Recall that every Q_u has $k-1$ chords and a base. Since the combinatorial length of every chord is m by Lemma 3, it follows immediately that every base must have length $n-(k-1)m$:

Corollary 1 *Every base is of the form* $(v_i, v_{i+n-(k-1)m})$.

It turns out that we can prove even stronger properties about the bases.

Lemma 4 *All bases have combinatorial length* $m+1$, *and so* $n = km+1$.

Proof By Corollary 1 all bases have combinatorial length $b := n-(k-1)m$. Assume that $b \neq m+1$. Since a base must have larger combinatorial length than a chord, it suffices to consider the case when $b > m+1$. Consider two consecutive bases (v_0, v_b) and (v_1, v_{b+1}). Consider the diagonal (v_1, v_b). Since $b > m+1$, this diagonal is not feasible, and so there is a vertex $w \in V$ with $v_1 \prec w \prec v_b$ with $\psi(w, v_1, v_b) > \sigma$.

By monotonicity, this implies $\psi(w, v_0, v_b) > \sigma$, and $\psi(w, v_1, v_{b+1}) > \sigma$. However, $w(v_0 v_b)$ is the only vertex in the range $v_0 \prec x \prec v_b$ with $\psi(x, v_0, v_b) > \sigma$, and so $w = w(v_0 v_b)$. Similarly, $w(v_1 v_{b+1})$ is the only vertex in the range $v_1 \prec x \prec v_{b+1}$ with $\psi(x, v_1, v_{b+1}) > \sigma$, and so $w = w(v_1 v_{b+1})$. It follows that $w(v_0 v_b) = w(v_1 v_{b+1})$, a contradiction since all witnesses are distinct. □

Lemma 5 *Let* $v_i \in V$, *and consider the two witnesses* $w(v_{i-m-1} v_i)$ *and* $w(v_i v_{i+m+1})$. *If* $n > k+1$ *then at least one of the two witnesses lies in the range* $s(v_i) \prec x \prec t(v_i)$.

Proof Set $v_j := s(v_i)$ and note that $v_{i-m-1} \prec v_j \prec v_i \prec v_{j+m+1} \prec v_{i+m+1}$, as shown in Fig. 3b. Consider the k witnesses $w_a := w(v_{i+am}v_{i+am+m+1})$, where $0 \le a \le k-1$. Note that $w_0 = w(v_i v_{i+m+1})$, and $w_{k-1} = w(v_{i-m-1}v_i)$ since $v_{i-m-1} = v_{i+(k-1)m}$.

The proof is done by contradiction, thus assume $v_{i-m-1} \prec w_{k-1} \preceq v_j$ and $v_{j+m+1} \preceq w_0 \prec v_{i+m+1}$. Now consider the subpolygon R with the $k+1$ vertices $\{v_i, w_0, w_1, \ldots, w_{k-1}\}$. We will show that $\varphi_k(R) > \sigma$, which is a contradiction to the definition of P.

First, we observe that $v_{i+am} \prec w_a \preceq v_{i+(a+1)m}$ and that $w_{k-1} \prec v_i \prec w_0$. This implies that the vertices of R are $v_i, w_0, w_1, \ldots, w_{k-1}$ in this order. It remains to verify the approximation error.

From the definition of a witness we have $\psi(w_a, v_{i+am}, v_{i+am+m+1}) > \sigma$ and $w_{a-1} \preceq v_{i+am} \prec w_a \prec v_{i+am+m+1} \preceq w_{a+1}$, for $1 \le a \le k-2$. Putting these two observations together implies that $\psi(w_a, w_{a-1}, w_{a+1}) > \sigma$, since ψ is a monotone function.

Similarly, since $\psi(w_0, v_i, v_{i+m+1}) > \sigma$ and $w_0 \prec v_{i+m+1} \preceq w_1$, we have $\psi(w_0, v_i, w_1) > \sigma$.

Since $\psi(w_{k-1}, v_{i-m-1}, v_i) > \sigma$ and $w_{k-2} \preceq v_{i-m-1} \prec w_{k-1}$, we also have $\psi(w_{k-1}, w_{k-2}, v_i) > \sigma$.

Finally, since $v_i = w(v_j v_{j+m+1})$, we have $\psi(v_i, v_j, v_{j+m+1}) > \sigma$. Since $w_{k-1} \preceq v_j \prec v_i \prec v_{j+m+1} \preceq w_0$, we also have $\psi(v_i, w_{k-1}, w_0) > \sigma$.

Hence, for any subpolygon Q of R with k vertices, the vertex v of R not in Q has approximation error $\varphi(v, Q) > \sigma$, implying $\varphi(R, Q) > \sigma$. □

4 Aperture-Angle Approximation

In order to proceed with our arguments, we need to get back into the geometry of the problem. In this section we consider the case of aperture angle approximation, that is, our error function is $\psi(v, s, t) = \pi - \angle svt$ as illustrated in Fig. 1c.

For two points p, q in the plane, let $H^+(p, q)$ be the half-plane to the right of the oriented line from p to q. For any $0 < \sigma < \pi$ we define

$$D_\sigma(p, q) := \{x \in H^+(p, q) \mid \angle pxq \ge \pi - \sigma\}.$$

The region $D_\sigma(p, q)$ is the intersection of a disk containing p and q on its boundary with $H^+(p, q)$, see Fig. 4a. Its boundary consists of a circular arc with endpoints p and q and of the segment pq. For fixed σ, the regions $D_\sigma(p, q)$ for any pair p, q are affinely similar—that is, one can be obtained from the other by a rigid motion and a scaling—and, in particular, the radius of the circular arc is proportional to the distance $d(p, q)$. If $0 < \sigma < \pi/2$, then the center of the circular arc lies to the left of the oriented line pq, and so the circular arc spans less than a semi-circle. We observe that $\psi(v, s, t) \le \sigma$ if and only if $v \in D_\sigma(s, t)$ (note that $s \preceq v \preceq t$ implies that $v \in H^+(s, t)$).

We will need a simple geometric fact, which can be proven using elementary calculations, see Fig. 4b.

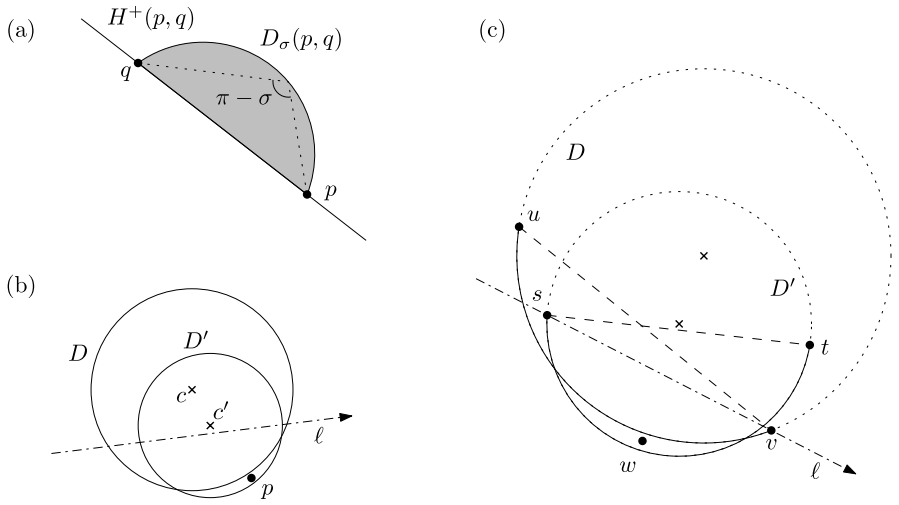


Fig. 4 **a** The region $D_\sigma(p,q)$. **b** The radius of D' is smaller than that of D. **c** D and D' fulfill the assumptions of Lemma 6

Lemma 6 *Consider two disks D and D', whose centers lie to the left of an oriented line ℓ. If $D' \cap \ell$ is contained in $D \cap \ell$, but there is a point $p \in D' \setminus D$ to the right of ℓ, then the radius of D' is smaller than the radius of D.*

Lemma 7 *If P is a convex n-gon that is worst-approximable with respect to the aperture angle and $k > 2$ then $n = k + 1$.*

Proof Let P be a worst-approximable convex n-gon with respect to the aperture angle, and assume that the statement of the lemma is false, that is, $n > k+1$. Let $\sigma := \max_R \varphi_k(R)$, where the maximum is taken over all proper subpolygons R of P. Since P is worst-approximable, we have $\varphi_k(P) > \sigma$, and since for every proper subpolygon R of P we have $\varphi_k(R) \le \sigma$, the results of Sect. 3 apply. Since $\alpha(3) = \pi/2$ [4], we have $\sigma < \varphi_k(P) \le \pi/2$.

Let now v be a vertex of V that maximizes the Euclidean distance $d(s(v), t(v))$. According to Lemma 5 there is an incident base, say uv, such that $u \prec s(v) \prec w(uv) \prec v$ (the other case being symmetric). We let $w := w(uv)$, $s := s(v)$, $t := t(v)$, and consider the sequence of five vertices $u \prec s \prec w \prec v \prec t$.

Let D be the disk supporting $D_\sigma(u, v)$, and let D' be the disk supporting $D_\sigma(s, t)$, see Fig. 4c. By our observation above, the only vertex in the range $u \preceq x \preceq v$ not in D is w, and the only vertex in the range $s \preceq x \preceq t$ not in D' is v. In particular, we have $s \in D$, $w \notin D$, $w \in D'$, and $v \notin D'$, see Fig. 4c.

Let ℓ be the directed line from s to v. Since $\sigma < \pi/2$, the center of D lies to the left of the oriented line uv, and therefore to the left of ℓ. Similarly, the center of D' lies to the left of st, and therefore to the left of ℓ. Since $s \in D$ and $v \notin D'$, we have $D' \cap \ell \subset D \cap \ell$. Furthermore, since $s \prec w \prec v$, the point w lies to the right of ℓ, and in $D' \setminus D$.

By Lemma 6 the radius of D' is therefore less than the radius of D. Since $D_\sigma(u, v)$ and $D_\sigma(s, t)$ are affinely similar, this implies that $d(s, t) < d(u, v)$. This, however, is a contradiction to our choice of v, and our assumption $n > k + 1$ is false. □

A limit argument now gives the following theorem:

Theorem 1 *For any compact convex set $C \subset \mathbb{R}^2$ and any integer $k > 2$ there is a convex k-gon Q contained in C such that $\alpha(C, Q) \geq \alpha_k$, where $\alpha_k = (1 - 2/(k + 1))\pi$. This bound is best possible.*

Proof We start by proving the theorem for the special case when C is a convex polygon. Among all subpolygons R of C with $\varphi_k(R) \geq \varphi_k(C)$, let P be one with the minimal number of vertices. This implies that for every proper subpolygon R of P we have $\varphi_k(R) < \varphi_k(C) \leq \varphi_k(P)$, and so P is worst-approximable. By Lemma 7, P is a $(k + 1)$-gon. It follows that P has at least one interior angle that is at least α_k. Choosing Q to be the convex hull of the remaining k vertices gives $\varphi(P, Q) \leq \pi - \alpha_k$, and so $\varphi_k(C) \leq \varphi_k(P) \leq \pi - \alpha_k$, proving the theorem.

Next we consider a general compact convex figure C in the plane. We choose a sequence P_i of convex polygons inscribed within C that converges to C with respect to the Hausdorff-distance. For each P_i there is a subpolygon $Q_i \subset P_i$ with k vertices and $\alpha(P_i, Q_i) \geq \alpha_k$.

We interpret the k vertices of Q_i as a point $q_i \in \mathbb{R}^{2k}$. Since $Q_i \subset C$, this sequence is bounded, and so the Bolzano–Weierstrass theorem guarantees the existence of a subsequence that converges to a point $q \in \mathbb{R}^{2k}$. We interpret q again as a k-vertex polygon Q. It is easy to see that Q is a convex polygon with at most k vertices.

It remains to show that $\alpha(C, Q) \geq \alpha_k$. Let $p \in C \setminus Q$. There is a sequence of points $p_i \in P_i$ with $\lim_{i\to\infty} p_i = p$. Since $\alpha(P_i, Q_i) \geq \alpha_k$, that implies that there are vertices x_i, y_i of Q_i such that $\angle x_i p_i y_i \geq \alpha_k$. We consider the sequence (x_i, y_i) in $\mathbb{R}^4$ and apply again the Bolzano–Weierstrass theorem. We pass to a subsequence where $x := \lim_{i\to\infty} x_i$ and $y := \lim_{i\to\infty} y_i$ exist. The points x and y are necessarily vertices of Q. The angle $\angle xpy$ is a continuous function in (x, y, p) as long as the three points remain distinct. Since $p \notin Q$, this implies $\angle xpy \geq \alpha_k$, and the theorem follows.

The regular $(k + 1)$-gon shows that the bound is indeed best possible. □

Theorem 1 implies that $\alpha(k) = \alpha_k$, positively answering Conjecture 1 (the case $k = 2$ was already known to be true).

5 Hausdorff-Approximation

In this section we consider the case of Hausdorff approximation, and our error function ψ is the distance between v and the segment st, as illustrated in Fig. 1b. We continue with the analysis of worst-approximable polygons of Sect. 3: $k > 2$ and $\sigma > 0$ are fixed, and we consider a convex n-gon P with $\varphi_k(P) > \sigma$, but such that for every proper subpolygon R of P we have $\varphi_k(R) \leq \sigma$.

We need a small geometric result similar to Lemma 2:

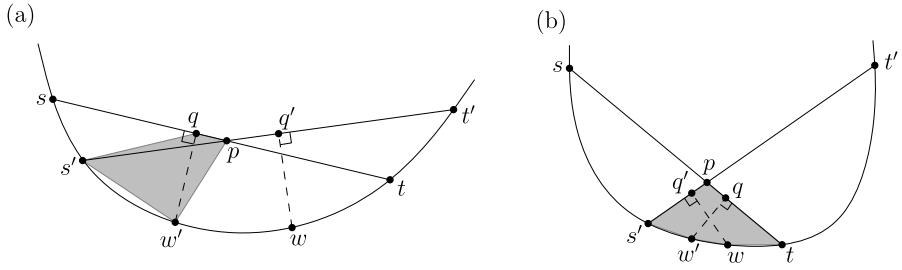


Fig. 5 Proof of Lemma 8

Lemma 8 *With respect to the Hausdorff-approximation, it is impossible for two bases st and $s't'$ to realize the order $s \prec s' \prec w(s't') \prec w(st) \prec t \prec t'$.*

Proof Suppose that there exist two bases st and $s't'$ such that $s \prec s' \prec w(s't') \prec w(st) \prec t \prec t'$. Let $w := w(st)$, let $w' := w(s't')$, let q be the point on the segment st minimizing the distance $d(w', q)$, and let q' be the point on $s't'$ minimizing $d(w, q')$.

The bases st and $s't'$ must intersect in a point p lying in P. We first assume that q lies on the segment sp, see Fig. 5a. Since $ss'w't$ is a convex quadrilateral, so is $pqs'w'$. Its diagonals $s'p$ and $w'q$ intersect, implying $d(w', s't') < d(w', q) \leq \sigma$, a contradiction to $w' = w(s't')$. It follows that q must lie on the segment pt, and by symmetry q' lies on $s'p$.

Since $ps'w'wt$ is a convex pentagon (the intersection of the two subpolygons $ss'w'wt$ and $s'w'wtt'$ of P), q lies on its edge pt, and q' lies on its edge $s'p$, the chain $q'w'wq$ is a convex quadrilateral, see Fig. 5b. The sum of the lengths of its diagonals $q'w$ and qw' must be larger than the sum of the lengths of the opposite sides $q'w'$ and qw. This, however, is a contradiction to $d(w, q') \leq \sigma < d(w, q)$ and $d(w', q) \leq \sigma < d(w', q')$, and the lemma follows. □

Lemma 9 *There is an integer $0 < r \leq m$ such that for every v_i we have $s(v_i) = v_{i-r}$.*

Proof Let r be the smallest integer > 0 such that there is a vertex v_i with $s(v_i) = v_{i-r}$. We will show that then $s(v_{i+1}) = v_{i-r+1}$, and by induction this implies the lemma. Assume the opposite, that is, $s(v_{i+1}) \neq v_{i-r+1}$. By definition of r, we cannot have $v_{i-r+1} \prec s(v_{i+1}) \prec v_{i+1}$, and therefore $s(v_{i+1}) \prec s(v_i) \prec v_i \prec v_{i+1} \prec t(v_{i+1}) \prec t(v_i)$, which is a contradiction to Lemma 8. □

Note that the above lemma also implies $t(v_i) = v_{i-r+m+1}$. From now on, let r be as in Lemma 9. We may assume $r \leq m/2$, otherwise we can work with the mirror image of P.

The rest of the proof is similar in spirit to Lemma 7, but is technically more complicated.

Lemma 10 *We have $3r > m + 1$.*

Proof Assume $3r \leq m+1$, and let v_i be a vertex minimizing the Euclidean distance $d(v_i, v_{i-r})$. We concentrate on the vertices

$$s_1 = v_{i-2r}, \qquad s_2 = v_{i-r}, \qquad s_3 = v_i, \qquad s_4 = v_{i+r},$$
$$t_1 = v_{i-2r+m+1}, \qquad t_2 = v_{i-r+m+1}, \qquad t_3 = v_{i+m+1}.$$

Since $3r \leq m+1$, we have $s_1 \prec s_2 \prec s_3 \prec s_4 \preceq t_1 \prec t_2$. We have $s_2 = w(s_1t_1)$, $s_3 = w(s_2t_2)$, and $s_4 = w(s_3t_3)$. By our choice of v_i, we have $d(s_2, s_3) \leq d(s_1, s_2)$ and $d(s_2, s_3) \leq d(s_3, s_4)$. We will show that this is impossible, implying that the assumption $3r \leq m+1$ is false.

By rotating P, we can assume that the line s_1t_1 is horizontal. Let S_1 be the horizontal strip of width σ bounded from above by s_1t_1, and let S_2 be the strip of width σ to the right of the oriented line from s_2 to t_2. The intersection $R = S_1 \cap S_2$ is a rhombus. Its top-left corner p_1 is the intersection point of the segments s_1t_1 and s_2t_2. We denote the remaining corners counter-clockwise as p_2, p_3, and p_4, see Fig. 6. Since $s_1 \prec s_3 \prec s_4 \preceq t_1$, we have $s_3, s_4, t_1 \in S_1$. Since we also have $s_4, t_1 \in S_2$, it follows that $s_4, t_1 \in R$.

We now first show that s_4 must lie strictly right of s_3. If t_1 lies right of s_3, this follows from the fact that s_4 lies strictly to the right of the oriented line s_3t_1 and to the left of the oriented line s_1s_3. If t_1 lies left of s_3, then $d(s_3, s_1t_1) \leq \sigma$ implies $d(s_3, t_1) \leq \sigma$. If s_4 lies not strictly right of s_3, then the angle $\angle s_3s_4t_1$ is right or obtuse, and so $d(s_3, s_4) \leq d(s_3, t_1) \leq \sigma$, a contradiction to $s_4 = w(s_3t_3)$.

We will now first consider the case that the angle $\angle s_2p_1t_1$ is at least 90°, so R has obtuse angles at p_1 and p_3 and acute angles at p_2 and p_4, see Fig. 6a. Let z be the point on the segment p_1p_2 at distance σ from p_1. Since $s_2 = w(s_1t_1)$, s_2 must lie below z, and so the segment zp_1 is a subset of s_2t_2. Since s_3 lies in S_1 but $d(s_3, s_2t_2) > \sigma$, s_3 must lie strictly to the right of p_3. Since $s_4 \in R$ and strictly right of s_3, this implies that s_3 lies strictly left of p_4. Since s_3 lies strictly in between the vertical lines through p_3 and p_4, and s_4 lies in between the vertical lines through s_3 and p_4, we have $d(s_3, s_4) < d(p_3, p_4)$.

Consider now the point s_2. We argue that it must lie strictly below p_2. Indeed, otherwise it lies inside S_1 on the segment p_1p_2. Since $d(s_2, s_1t_1) > \sigma$, this implies that

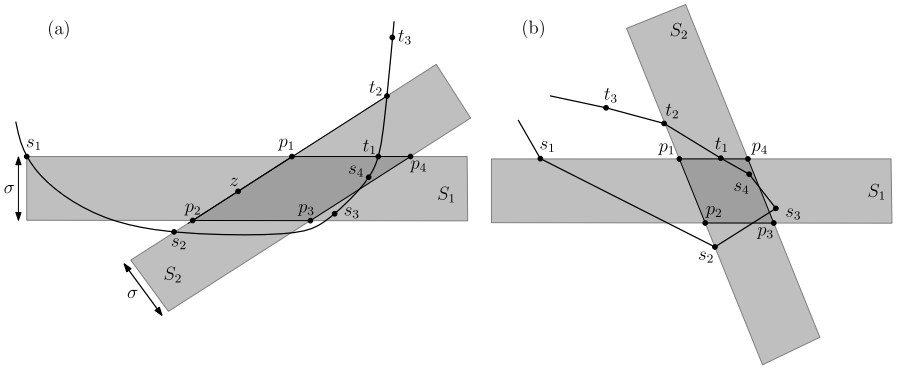


Fig. 6 Proof of Lemma 10

s_1 lies right of s_2. Then the angle $\angle s_2 s_1 p_1$ is obtuse, implying $d(s_2, s_1) \le d(s_2, p_1)$. But since s_3 lies strictly right of p_3, we have $d(s_2, s_3) > d(s_2, p_1) \ge d(s_2, s_1)$, a contradiction to our choice of $s_3 = v_i$.

Since s_2 lies strictly below p_2 on the line $p_1 p_2$ and s_3 lies in S_1 right of p_3, we have $d(s_2, s_3) > d(p_2, p_3) = d(p_3, p_4) > d(s_3, s_4)$, again a contradiction to our choice of $s_3 = v_i$.

It remains to consider the case that the angle $\angle s_2 p_1 t_1$ is less than 90°, so R has acute angles at p_1 and p_3 and obtuse angles at p_2 and p_4, see Fig. 6b. If s_3 does not lie in R, we immediately have a contradiction: Since s_4 must lie to the left of the oriented line $t_1 s_3$ and right of s_3, it cannot lie in R. If s_3 lies in R, the nearest point to s_3 on the segment $s_2 t_2$ must be s_2, as otherwise $d(s_3, s_2 t_2) \le \sigma$. This implies that s_3 lies below the line ℓ through s_2 orthogonal to $p_1 p_2$, and outside the circle T with radius σ around s_2.

Since the segments $s_2 s_4$ and $s_3 t_3$ intersect and $s_4 = w(s_3 t_3)$, we must have $d(s_2, s_4) > \sigma$. This implies that the nearest point to s_4 on $s_2 t_2$ must be different from s_2, and so s_4 must lie above the line ℓ.

Now we observe that if s_3 lies above s_2, then it lies strictly right of the intersection point of ℓ and the line $p_4 p_3$. Since s_4 lies right of s_3, this implies that s_4 cannot lie in R above ℓ, a contradiction, and so s_3 must lie below s_2. Since $s_3 \in R$, this implies $s_2 \in R$. Therefore, the point nearest to s_2 on $s_1 t_1$ must be t_1, and t_1 must lie left of s_2. This implies that the angle $\angle t_1 s_2 s_3$ is obtuse, and so $d(t_1, s_3) > d(t_1, s_2) > \sigma$. Since s_3 lies right of t_1, this implies $d(s_3, s_1 t_1) > \sigma$, a contradiction. □

Theorem 2 *If P is a convex n-gon that is worst-approximable with respect to the Hausdorff-distance and $k > 2$ then $n = k + 1$.*

Proof We assume $n > k + 1$ and set $\rho := m + 1 - r$. Since $r \le m/2$, we have $2\rho > m + 1$. Let R be the subpolygon of P formed by the vertices $v_0, v_\rho, v_{2\rho}, \ldots, v_{\ell\rho}$, where ℓ is an integer such that $(\ell + 1)\rho \le km + 1 < (\ell + 2)\rho$. Since $\rho > 1$, R is a proper subpolygon of P. We will show that $\varphi_k(R) > \sigma$, a contradiction to the assumption that P is worst-approximable.

We first show that $\ell \ge k$. We assume the contrary, that is $\ell < k$. This implies $\ell + 2 \le k + 1$. Since $km + 1 < (\ell + 2)\rho$, we have $km + 2 \le (\ell + 2)\rho$. By Lemma 10 we have $3r > m + 1$, which implies $3r \ge m + 2$, and therefore $\rho \le \frac{2}{3}m + \frac{1}{3}$. This gives

$$3km + 6 = 3(km + 2) \le 3(\ell + 2)\rho \le 3(k + 1)\left(\frac{2}{3}m + \frac{1}{3}\right) = 2km + 2m + k + 1.$$

Rearranging the terms gives $km - 2m - k + 5 \le 0$ or $(k - 2)(n - 1) \le -3$, a contradiction with $k > 2$ and $m \ge 1$. It follows that our claim holds, that is $\ell \ge k$.

It remains to prove that $\varphi_k(R) > \sigma$. Note that R has at least $k + 1$ vertices, and so a k-vertex subpolygon Q of R must exclude at least one vertex of R. We have three cases (recall that arithmetic on the indices of v is modulo n):

- $v_{i\rho}$ is excluded, for $0 < i < \ell$.
  By Lemma 9, $v_{i\rho}$ is the witness of the base $(v_{(i+1)\rho-(m+1)}, v_{(i+1)\rho})$. Since $2\rho > m + 1$, $v_{(i-1)\rho} \prec v_{(i+1)\rho-(m+1)} \prec v_{i\rho} \prec v_{(i+1)\rho}$, and so monotonicity of ψ implies that $\psi(v_{i\rho}, v_{(i-1)\rho}, v_{(i+1)\rho}) > \sigma$.

- v_0 is excluded.
  v_0 is the witness of the base $(v_{\rho-(m+1)}, v_\rho)$. Since $(\ell+1)\rho \le km+1 = n$ and $2\rho > m+1$, we have $\ell\rho \le n-\rho < n+\rho-(m+1)$, which implies $v_{\ell\rho} \prec v_{\rho-(m+1)} \prec v_0 \prec v_\rho$, and so $\psi(v_0, v_{\ell\rho}, v_\rho) > \sigma$.
- $v_{\ell\rho}$ is excluded.
  $v_{\ell\rho}$ is the witness of the base $(v_{(\ell+1)\rho-(m+1)} v_{(\ell+1)\rho})$. Since $2\rho > m+1$ and $(\ell+1)\rho \le n$, we have $v_{(\ell-1)\rho} \prec v_{(\ell+1)\rho-(m+1)} \preceq v_0$, and so $\psi(v_{\ell\rho}, v_{(\ell-1)\rho}, v_0) > \sigma$.

In all cases, $\varphi(R, Q) > \sigma$, and so $\varphi_k(R) > \sigma$. □

The following approximation result is a direct application of Theorem 2.

Theorem 3 *For any convex polygon P of perimeter at most one and any $k > 2$ there exists a subpolygon Q of P with k vertices such that $\varphi(P, Q) \le \frac{1}{k+1} \sin\frac{\pi}{k+1}$ with respect to the Hausdorff-distance. If P is a regular $(k+1)$-gon, this bound is best possible.*

Proof Let R be a subpolygon of P with the smallest number of vertices such that $\varphi_k(R) \ge \varphi_k(P)$. Then R is worst-approximable and, by Theorem 2, R is a $(k+1)$-gon. The following lemma now implies the theorem. □

Lemma 11 *Let $k > 2$ and let P be a convex $(k+1)$-gon with $\varphi_k(P) = 1$ with respect to the Hausdorff-distance. Then the perimeter of P is at least $(k+1)/\sin(\pi/(k+1))$, and this bound is tight for the regular $(k+1)$-gon.*

Proof We set $n = k+1$, and let P be a convex n-gon with $\varphi_k(P) = 1$ of minimal perimeter (the existence of such a P follows from a compactness argument). Let $v_1, \ldots, v_n$ denote the vertices of P in counter-clockwise order, and let Q_i be the subpolygon excluding the vertex v_i. We first argue that $\varphi(P, Q_i) = 1$ for all i. Indeed, if there is a vertex v_i such that $\varphi(P, Q_i) > 1$, then we can move v_i slightly along the directed line from v_i to v_{i+2}. This decreases the perimeter while keeping $\varphi_k(P) = 1$, a contradiction to the choice of P.

Let γ_i be the angle made by the oriented diagonals $v_{i-1}v_{i+1}$ and $v_i v_{i+2}$. Since the direction of the diagonal $v_{i-1}v_{i+1}$ is a tangent direction at v_i, we have

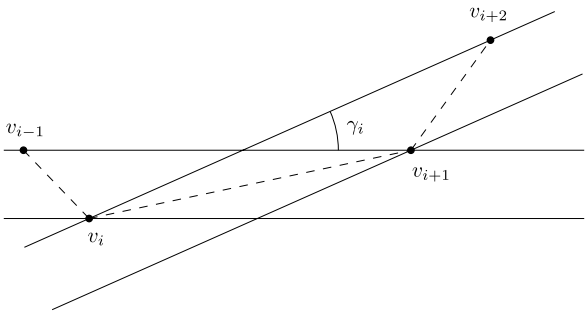
$$\sum_{i=1}^{n} \gamma_i = 2\pi.$$

The distance of v_i from the line $v_{i-1}v_{i+1}$ and the distance of v_{i+1} from the line $v_i v_{i+2}$ is one. This implies that the length of the edge $v_i v_{i+1}$ is $1/\sin(\gamma_i/2)$, see Fig. 7. We set $x_i = \gamma_i/2$, and define $f(x) = 1/\sin x$. We then have

$$\sum_{i=1}^{n} x_i = \pi,$$

$$\mathrm{peri}(R) = \sum_{i=1}^{n} f(x_i).$$

Fig. 7 Proof of Lemma 11



Since $f''(x) > 0$ for $0 < x < \pi/2$, the function $f(x)$ is convex on the interval $(0, \pi/2)$. Since $0 < x_i < \pi/2$, we can thus apply Jensen's inequality to give us

$$f\left(\frac{\sum_i x_i}{n}\right) \leq \frac{\sum_i f(x_i)}{n} = \frac{\text{peri}(R)}{n}.$$

Multiplying by n gives $\text{peri}(R) \geq nf(\pi/n) = n/\sin(\pi/n)$, completing the proof. □

6 Conclusions

We showed that worst-approximable polygons are $(k+1)$-gons for both the Hausdorff-distance and the aperture-angle distance. A large part of the argument is purely combinatorial, using only the monotonicity of the distance function ψ. To finish the argument, however, we needed to make use of some geometric properties of the two distance functions we considered; in Lemma 7 for the aperture angle, and in Lemmas 8 and 10 for the Hausdorff distance. We must ask: are we just blinded by the geometry to overlook an entirely combinatorial proof that would apply for any monotone distance function?

We gave one application of Theorem 2, but it's worth pointing out that the theorem is really far more general, and applies to many families $\mathcal{P}$ of convex polygons. In many cases the regular $(k+1)$-gon appears to be the worst case, but this is not always the case, for instance, because the family $\mathcal{P}$ does not contain it. (Consider, for instance, the family of all convex polygons with vertices on a given ellipse.)

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References

1. Bose, P., Hurtado, F., Omaña-Pulido, E., Snoeyink, J., Toussaint, G.T.: Some aperture angle optimization problems. Algorithmica **33**, 411–435 (2002)
2. Brass, P.: On the approximation of polygons by subpolygons. In: Proc. European Workshop Comput. Geom. (EuroCG), pp. 59–61 (2000)

3. Brass, P., Lassak, M.: Problems on approximation by triangles. Geombinatorics **10**, 103–115 (2001)
4. Brass, P., Moser, W., Pach, J.: Research Problems in Discrete Geometry. Springer, Berlin (2005)
5. Cheong, O., van Oostrum, R.: Reaching a polygon with directional uncertainty. Int. J. Comput. Geom. Appl. **11**, 197–214 (2001)
6. Fekete, S.: Personal communication
7. Hoffmann, F., Icking, C., Klein, R., Kriegel, K.: Moving an angle around a region. In: Algorithm Theory—SWAT'98. Lecture Notes in Computer Science, vol. 1432, pp. 71–82 (1998)
8. Jenkner, M.: Approximation konvexer Kurven. Master's thesis, Universität zu Köln (1997)
9. Popov, V.A.: Approximation of convex bodies. C. R. Acad. Bulg. Sci. **21**, 993–995 (1968) (in Russian, see Zbl 215.50601)
10. Popov, V.A.: Approximation of convex sets. B'lgar. Akad. Nauk. Otdel. Mat. Fiz. Nauk. Izv. Mat. Inst. **11**, 67–80 (1970) (in Bulgarian, with Russian and English summary)