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THE CLASSIFICATION OF REAL PROJECTIVE STRUCTURES ON COMPACT SURFACES

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ABSTRACT. Real projective structures ($\mathbb{R}P^2$ -structures) on compact surfaces are classified. The space of projective equivalence classes of real projective structures on a closed orientable surface of genus $g > 1$ is a countable disjoint union of open cells of dimension $16g - 16$. A key idea is Choi's admissible decomposition of a real projective structure into convex subsurfaces along closed geodesics. The deformation space of convex structures forms a connected component in the moduli space of representations of the fundamental group in $\mathbf{PGL}(3, \mathbb{R})$, establishing a conjecture of Hitchin.

1. REAL PROJECTIVE STRUCTURES

Projective differential geometry began in the early twentieth and late nineteenth century as an attempt to apply infinitesimal methods on manifolds to concepts from projective geometry. Most of the work, culminating in the 1930's, concentrated on local questions. Global questions became more prominent with Chern's work on the Gauss-Bonnet theorem and characteristic classes. Thurston's work [43] in the late 1970's on geometrization of 3-manifolds underscored the importance of geometric structures in low-dimensional topology, renewing interest in global projective differential geometry. In this note we summarize some recent advances in two-dimensional projective differential geometry. Although many of these ideas can be expressed in terms of affine connections and projective connections, we prefer

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the setting of geometric structures defined by local coordinate systems modelled on projective space, as initiated by Ehresmann [15] (compare also [43] and [25]).

1.1. Generalities on geometric structures. Let \mathbb{RP}^2 be the real projective plane and $\mathbf{PGL}(3, \mathbb{R})$ the group of projective transformations $\mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ and M a compact surface. A *real projective structure* (\mathbb{RP}^2 -structure) on M is a maximal atlas of coordinate charts locally modeled on \mathbb{RP}^2 with coordinate changes lying in $\mathbf{PGL}(3, \mathbb{R})$. An \mathbb{RP}^2 -manifold is a manifold together with an \mathbb{RP}^2 -structure. If M is an \mathbb{RP}^2 -manifold, then a *geodesic* on M is a curve which in local coordinates maps to a projective line in \mathbb{RP}^2 .

An \mathbb{RP}^2 -manifold M can be *developed* into \mathbb{RP}^2 as follows. The coordinate atlas globalizes to define a local diffeomorphism of the universal covering $\tilde{M} \rightarrow \mathbb{RP}^2$, called the *developing map*. The developing map restricts to projective charts on coordinate patches in \tilde{M} . The deck transformations of \tilde{M} define automorphisms of \tilde{M} . The resulting *holonomy homomorphism* $\pi_1(M) \rightarrow \mathbf{PGL}(3, \mathbb{R})$ corresponds to the coordinate changes in the atlas for the \mathbb{RP}^2 -structure. The pair consisting of the developing map and the holonomy homomorphism is unique up to the $\mathbf{PGL}(3, \mathbb{R})$ -action by composition and conjugation respectively.

1.2. Convexity. The most important \mathbb{RP}^2 -structures are the *convex structures*. An \mathbb{RP}^2 -manifold M is *convex* if its universal covering surface is equivalent to a convex domain Ω in an affine patch of \mathbb{RP}^2 . In that case the fundamental group $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is represented as a discrete group of projective transformations acting properly and freely on Ω . Equivalently an \mathbb{RP}^2 -manifold is convex if its developing map is a diffeomorphism with convex image. In that case the holonomy homomorphism is an isomorphism of $\pi_1(M)$ onto a discrete subgroup of $\mathbf{PGL}(3, \mathbb{R})$ which acts properly on this convex set.

The basic results on convex \mathbb{RP}^2 -structures on a closed surface S are due to Kuiper, Benzécri [2],[3], with subsequent work by Koszul [33],[34], Kuiper [36], Vey [44],[45], and Kobayashi [30],[31],[32]. It follows from this work that if M is a convex \mathbb{RP}^2 -manifold with $\chi(M) < 0$, then the universal covering space of M is a strictly convex domain $\Omega \subset \mathbb{RP}^2$; the boundary $\partial\Omega$ is a C^1 curve which is either a conic (in which case the convex \mathbb{RP}^2 -structure on M arises from a hyperbolic structure on M) or is nowhere $C^{1+\epsilon}$ for some $\epsilon > 0$. The first example of such a convex \mathbb{RP}^2 -manifold whose universal covering has non-smooth boundary is due to Kac-Vinberg [29] (see also [19]).

1.3. Gluing. When S has boundary, we assume that the boundary is represented by closed geodesics each having a geodesically convex collar neighborhood and whose holonomy has distinct positive eigenvalues. We call such a boundary component *principal*.

One obtains new \mathbb{RP}^2 -manifolds from old ones by *gluing* structures on surfaces along principal boundary components. Suppose M_0 is a (possibly disconnected) compact \mathbb{RP}^2 -manifold. Suppose that

$$b_1, b_2 \subset \partial M_0$$

are boundary components with collar neighborhoods $b_i \subset N(b_i) \subset M_0$ ($i = 1, 2$).

Let $f : N(b_1) \rightarrow N(b_2)$ be a projective isomorphism. Let M be the manifold obtained by identifying $N(b_1)$ with $N(b_2)$ by f .

Lemma 1. *Then M inherits an \mathbb{RP}^2 -structure such that the quotient map $M \rightarrow M_0$ is an \mathbb{RP}^2 -map.*

This lemma follows immediately from the definitions in terms of local coordinates. When M_0 consists of convex surfaces, then in many important cases, the glued surface M is convex (compare Goldman [23]):

Lemma 2. *Suppose that each component of M_0 has negative Euler characteristic. If M_0 is convex with all boundary components principal, then M is convex.*

1.4. Example: Convex annuli and π -annuli. Here is a simple example of a convex \mathbb{RP}^2 -structure with principal boundary. Let $\Delta \subset \mathbb{RP}^2$ be a triangle bounded by three lines in general position; in an appropriate system of homogeneous coordinates, Δ consists of all $[x, y, z] \in \mathbb{RP}^2$ such that

$$x > 0, \quad y > 0, \quad z > 0.$$

The stabilizer of Δ in $\mathbf{PGL}(3, \mathbb{R})$ corresponds to diagonal matrices with positive entries. Suppose that $T \in \mathbf{PGL}(3, \mathbb{R})$ is represented by a matrix

$$\tilde{T} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

where $a > b > c > 0$. Then T has an attracting fixed point at the point p_1 corresponding to the first coordinate line, a fixed point of saddle type at the point p_2 corresponding to the second coordinate line, and a repelling fixed point at the point p_3 corresponding to the third coordinate line. Two fixed points p_i, p_j span T -invariant lines, denoted l_{ij} , corresponding to the \tilde{T} -invariant planes in \mathbb{R}^3 . For $1 \leq i < j \leq 3$, let s_{ij} denote the component of $l_{ij} - \{p_i, p_j\}$ meeting $\bar{\Delta}$ and s'_{ij} the other component of $l_{ij} - \{p_i, p_j\}$. Thus \mathbb{RP}^2 decomposes as the disjoint union of:

- four T -invariant open triangular regions (one of which is Δ);
- six T -invariant open line segments $s_{12}, s_{13}, s_{23}, s'_{12}, s'_{13}, s'_{23}$;
- three fixed points p_1, p_2, p_3 .

(Compare Figure 1.)

The cyclic group $\langle T \rangle$ generated by T is discrete and acts properly and freely on Δ , with quotient space an open annulus. There are two natural compactifications of $\Delta/\langle T \rangle$:

$$A_1 = \Delta_1/\langle T \rangle, \quad A_3 = \Delta_3/\langle T \rangle$$

where

$$\Delta_1 = \Delta \cup s_{12} \cup s_{13}$$

and

$$\Delta_3 = \Delta \cup s_{13} \cup s_{23}.$$

Both A_1 and A_3 are convex \mathbb{RP}^2 -manifolds with principal boundary, whose interiors are projectively isomorphic. However, unless $ac = b^2$, the projective isomorphism between the interiors does not extend to one between A_1 and A_3 .

Projectively isomorphic to A_1 is the annulus

$$A'_1 = \Delta'_1/\langle T \rangle$$

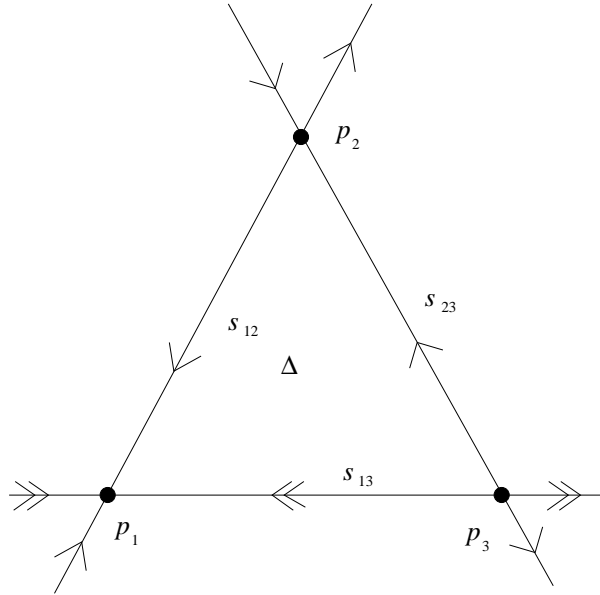


FIGURE 1. Dynamics of a projective transformation

where

$$\Delta'_1 = \Delta' \cup s'_{12} \cup s_{13}$$

and Δ' consists of all $[x, y, z]$ with $x > 0, z > 0, y < 0$ and s'_{12} is the component of $l_{12} - \{p_1, p_2\}$ disjoint from $\bar{\Delta}$. Then the result of gluing A_1 to A'_1 is an annulus A with $\mathbb{R}\mathbb{P}^2$ -structure obtained as a quotient of

$$H = s_{12} \cup \Delta \cup s_{13} \cup \Delta' \cup s'_{12}$$

by $\langle T \rangle$. In inhomogeneous coordinates

$$Y = y/x, \quad Z = z/x$$

on the affine plane $\mathbb{R}\mathbb{P}^2 - l_{23} \approx \mathbb{R}^2$ the interior of H is the half-plane defined by $Z > 0$ and the boundary of H is the two components of the Z -axis. In contrast to Lemma 2, the annulus A is not convex. Since the interior of the universal covering of A spans π radians, such an annulus is called a π -annulus in [6]. Figure 2 depicts schematically how the two convex annuli are glued to form a π -annulus.

These manifolds can be embedded in closed $\mathbb{R}\mathbb{P}^2$ -manifolds. For example, A_1 and A are both subsurfaces of the *Hopf torus*

$$M = (\mathbb{R}\mathbb{P}^2 - (\{p_1\} \cup l_{23})) / \langle T \rangle.$$

In the above coordinates (Y, Z) on the affine plane $\mathbb{R}\mathbb{P}^2 - l_{23}$, the origin is p_1 and T is the linear contraction

$$\begin{bmatrix} Y \\ Z \end{bmatrix} \mapsto \begin{bmatrix} (b/a) Y \\ (c/a) Z \end{bmatrix}.$$

The cyclic group $\langle T \rangle$ acts properly and freely on the complement with quotient M . A fundamental domain for this action is depicted in Figure 3.

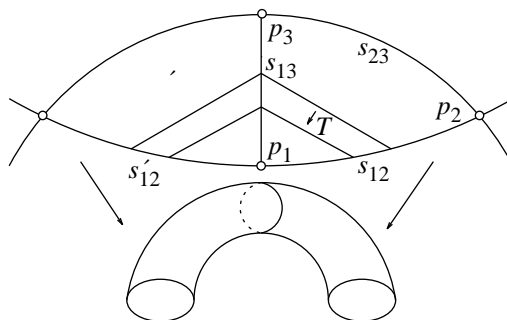


FIGURE 2. How to build a π -annulus

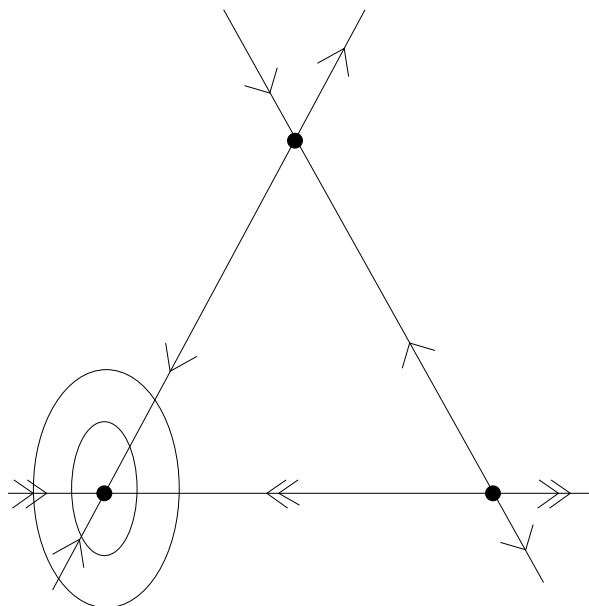


FIGURE 3. Fundamental domain of a Hopf torus

More bizarre closed $\mathbb{R}\mathbb{P}^2$ -manifolds — for example, ones whose developing maps are not covering maps onto their image — can be obtained by gluing copies of A_1 and A_3 . The first examples are due to Smillie [40] and Sullivan-Thurston [42], independently in 1976; see also [19],[21],[12],[13],[37].

1.5. Example: Hyperbolic structures on surfaces. An important class of convex $\mathbb{R}\mathbb{P}^2$ -manifolds consists of hyperbolic manifolds. Let $\Omega \subset \mathbb{R}\mathbb{P}^2$ be the interior of a conic; then the subgroup G of projective transformations of $\mathbb{R}\mathbb{P}^2$ stabilizing Ω leaves invariant a Riemannian metric g of constant negative curvature. Furthermore every isometry of g is realized by a unique projective transformation preserving Ω . For example, if Ω is the domain in $\mathbb{R}\mathbb{P}^2$ defined by

$$\Omega = \{[x_1, x_2, x_3] \mid x_1^2 + x_2^2 - x_3^2 < 0\},$$

then G corresponds to the orthogonal group $O(2, 1)$. Let M be a surface with a hyperbolic structure; composing a developing map $\tilde{M} \rightarrow H^2$ with an isometry $H^2 \rightarrow \Omega$ realizes M as a convex \mathbb{RP}^2 -surface Ω/Γ where $\Gamma \subset G$ is a discrete cocompact subgroup.

2. DEFORMATION SPACES

The classification of convex \mathbb{RP}^2 -manifolds with nonnegative Euler characteristic is due to Kuiper ([35],[36]) in the early 1950's, and \mathbb{RP}^2 -manifolds with zero Euler characteristic were classified in [19] following the classification of affine structures on closed 2-manifolds ([1],[2],[38]).

2.1. The space of all \mathbb{RP}^2 -structures. The *deformation space* $\mathbb{RP}^2(S)$ of \mathbb{RP}^2 -structures on S consists of diffeomorphisms $f : S \rightarrow M$ to an \mathbb{RP}^2 -manifold M modulo the action of the identity component $\text{Diff}(S)^0$ given by

$$\phi : f \mapsto f \circ \phi$$

where $\phi \in \text{Diff}(S)^0$. Give $\mathbb{RP}^2(S)$ the quotient topology induced from the C^∞ topology on the space of diffeomorphisms f .

$\mathbb{RP}^2(S)$ is closely related to the space $\text{Hom}(\pi_1(S), \mathbf{PGL}(3, \mathbb{R}))$ of homomorphisms $\pi_1(S) \rightarrow \mathbf{PGL}(3, \mathbb{R})$, which has a natural topology as the set of \mathbb{R} -points of an affine algebraic variety defined over \mathbb{Z} . The group $\mathbf{PGL}(3, \mathbb{R})$ acts on this space by conjugation. Its orbit space

$$X(S) = \text{Hom}(\pi_1(S), \mathbf{PGL}(3, \mathbb{R}))/\mathbf{PGL}(3, \mathbb{R})$$

corresponds most closely to the deformation space. However the representation variety is singular and the action of $\mathbf{PGL}(3, \mathbb{R})$ is neither proper nor free. Fortunately these pathologies can be avoided for \mathbb{RP}^2 -structures on closed surfaces of negative Euler characteristic.

Taking the holonomy homomorphism of a projective structure defines a map

$$\mathbf{hol} : \mathbb{RP}^2(S) \rightarrow X(S)$$

which is essentially a local homeomorphism (see [22] and [28] for an exposition). The following theorem is proved in [23]:

Theorem 3. *Let S be a closed surface with $\chi(S) < 0$. Then the deformation space $\mathbb{RP}^2(S)$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$.*

An alternate construction of $\mathbb{RP}^2(S)$ as a *symplectic quotient* of the space of affine connections on S is given in [24]. In this construction there are two moment maps, one corresponding to the torsion of a connection, and the other the projective curvature tensor. The corresponding transformation groups are the vector space of 1-forms on S and the diffeomorphism group, respectively.

2.2. Deformations of convex structures. The subset of $\mathbb{RP}^2(S)$ consisting of convex structures is open in $\mathbb{RP}^2(S)$ and will be denoted by $\mathfrak{P}(S)$. The restriction of \mathbf{hol} to $\mathfrak{P}(S)$ is an embedding onto an open subset of $X(S)$. (When S has boundary, we assume that the boundary is represented by closed geodesics each having a geodesically convex collar neighborhood and whose holonomy has distinct positive eigenvalues. We call such a boundary component *principal*.) The global topology of $\mathfrak{P}(S)$ was determined by Goldman in [23]:

Theorem 4. *Let S be a compact surface having n boundary components such that $\chi(S) < 0$. Then $\mathfrak{P}(S)$ is diffeomorphic to a cell of dimension $-8\chi(S)$. The map which associates to a convex $\mathbb{R}\mathbb{P}^2$ -manifold M the germ of the $\mathbb{R}\mathbb{P}^2$ -structure near ∂M is a fibration of $\mathfrak{P}(S)$ over an open $2n$ -cell. Its fiber is an open cell of dimension $-8\chi(S) - 2n$.*

Corollary 5. *Let S be a closed orientable surface of genus $g > 1$. Then the deformation space $\mathfrak{P}(S)$ of convex $\mathbb{R}\mathbb{P}^2$ -structures on S is diffeomorphic to an open cell of dimension $16(g - 1)$.*

The deformation space $\mathfrak{P}(S)$ is an analogue of the Teichmüller space $\mathfrak{T}(S)$ of S , which is classically known (Fricke-Klein [17]) to be an open cell of dimension $6(g - 1)$. As in 1.5, every hyperbolic structure on S defines a convex $\mathbb{R}\mathbb{P}^2$ -structure; thus $\mathfrak{T}(S)$ embeds in $\mathfrak{P}(S)$. The mapping class group of S acts properly discontinuously on $\mathfrak{P}(S)$ as well as on $\mathfrak{T}(S)$; indeed $\mathfrak{P}(S)$ admits an equivariant retraction onto $\mathfrak{T}(S)$. Furthermore projective duality defines a natural involution $\mathfrak{P}(S) \rightarrow \mathfrak{P}(S)$ whose stationary set equals $\mathfrak{T}(S)$. A Riemannian metric on $\mathfrak{P}(S)$ similar to the Weil-Petersson metric on $\mathfrak{T}(S)$ is constructed in Darvishzadeh-Goldman [14]. (A possibly related construction may be derived from Cheng-Yau [5].)

3. THE ADMISSIBLE DECOMPOSITION THEOREM

Recently, Choi ([6]) has proved the following “admissible decomposition theorem” (compare also [8] and [9]), which answered a question raised by Thurston and Goldman in 1977 (see [19],[42]):

Theorem 6. *Let M be a compact $\mathbb{R}\mathbb{P}^2$ -manifold with $\chi(M) < 0$. Then there is a unique collection of disjoint simple closed geodesics C_i on M such that each component of the complement $M - \bigcup_i C_i$ is one of the following:*

- *an annulus covered by an affine half-space;*
- *the interior of a compact convex $\mathbb{R}\mathbb{P}^2$ -manifold of negative Euler characteristic.*

Furthermore each C_i bounds either one or two annuli.

Convexity is equivalent to an extension property, which has been used in several other contexts (Fried [18], Carrière [4], Shima-Yagi [41]).

Let $\bar{\Delta}$ denote the closed 2-simplex

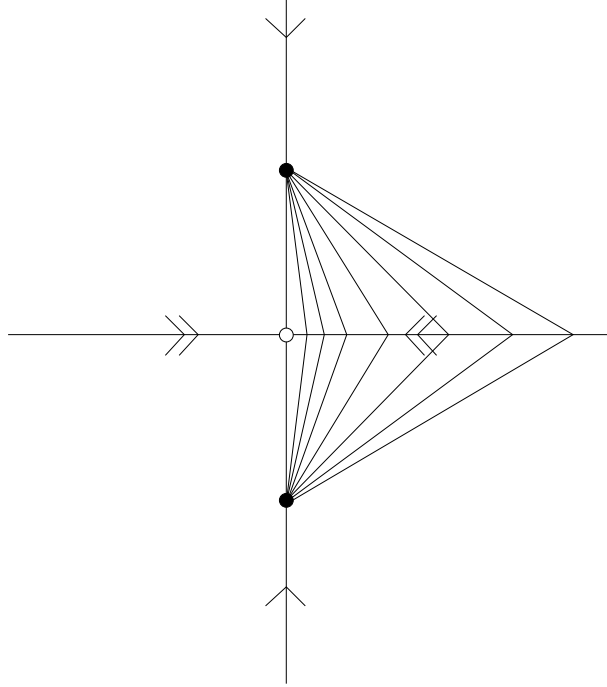
$$\{[x, y, z] \in \mathbb{R}\mathbb{P}^2 \mid x \geq 0, y \geq 0, z \geq 0\}$$

with the induced $\mathbb{R}\mathbb{P}^2$ -structure. Let $q \in \partial\bar{\Delta}$ denote the point $[0, 1, 1]$ which is an interior point of the edge defined by $x = 0$. Then $\Delta = \bar{\Delta} - \{q\}$ is an open subset of $\bar{\Delta}$.

Let M be an $\mathbb{R}\mathbb{P}^2$ -manifold. Then M is convex if and only if every projective map $\Delta \rightarrow M$ extends to a projective map $\bar{\Delta} \rightarrow M$. Thus non-convexity is expressed by the existence of a special kind of projective map.

For example, a π -annulus is not convex. Let M denote the π -annulus above, described in inhomogeneous coordinates on the affine plane \mathbb{R}^2 as $H/\langle T \rangle$ where H is the right half-space

$$H = \{(Y, Z) \in \mathbb{R}^2 \mid Z \geq 0, (Y, Z) \neq (0, 0)\}.$$

FIGURE 4. Nonconvexity of a π -annulus

Then the composition of the map

$$\begin{aligned} \Delta &\longrightarrow H \\ [x, y, z] &\longmapsto (x/(x+y+z), (y-z)/(x+y+z)) \end{aligned}$$

with the quotient projection $H \longrightarrow M = H/\langle T \rangle$ is a projective mapping $\Delta \longrightarrow M$ which does not extend to $\bar{\Delta}$. Compare Figure 4.

The idea of the admissible deformation theorem is that every inextendible projective map $\Delta \longrightarrow M$ can be replaced by one which is of the above type. Thus M is either convex or contains a π -annulus. After removing a π -annulus, one obtains a surface to which one applies the above argument inductively. After a finite number of steps, M is represented as a union of π -annuli and convex \mathbb{RP}^2 -manifolds. The uniqueness of this decomposition is proved in Choi [8],[9].

4. REPRESENTATIONS OF THE FUNDAMENTAL GROUP

The powerful theory developed by Hitchin ([27],[26]) gives precise topological information concerning the deformation space $X(S)$. In particular Hitchin [27] shows that $X(S)$ has exactly three connected components:

1. \mathcal{C}_0 , the component containing the class of the trivial representation;
2. \mathcal{C}_1 , the component consisting of classes of representations which do not lift to the double covering of $\mathbf{PGL}(3, \mathbb{R})$;
3. \mathcal{C}_2 , the component containing discrete faithful representations into $\mathrm{SO}(2, 1)$.

In particular the component \mathcal{C}_2 contains the Teichmüller space $\mathfrak{T}(\mathfrak{S})$ of S . By §2.2 the holonomy map \mathbf{hol} maps $\mathfrak{P}(S)$ bijectively onto an open subset in \mathcal{C}_2 ([22],[23]).

Furthermore, Hitchin shows that \mathcal{C}_2 is homeomorphic to \mathbb{R}^{16g-16} . This naturally led Hitchin to conjecture in [27] that $\mathcal{C}_2 = \mathfrak{P}(S)$:

Theorem 7 (Choi-Goldman [11]). *$\mathfrak{P}(S)$ is closed in \mathcal{C}_2 . Thus \mathbf{hol} is a diffeomorphism of $\mathfrak{P}(S)$ onto a connected component of $X(S)$.*

The proof is geometric and involves Hausdorff limits of the metric spaces associated to convex $\mathbb{R}\mathbb{P}^2$ -structures defined by the Hilbert metric. (See Kobayashi [31] for a discussion of intrinsic metrics and Choi [10] for another proof of Theorem 7.) $\mathbb{R}\mathbb{P}^2$ -structures whose holonomy lies in other components of $X(S)$ are given in Goldman [21]. (For example the holonomy representation may be a non-Fuchsian representation in $\mathrm{SO}(2, 1)$.) Curiously, all invariants derived from characteristic classes [20] of associated bundles fail to distinguish the components \mathcal{C}_0 and \mathcal{C}_2 .

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