The Analytic Valuation of
American Average Rate Options

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Abstract

This article explores the analytic valuation of American average rate option based on the continuous arithmetic average in the Black-Scholes framework. Because there is no closed-form exact valuation formula for the average rate option with the arithmetic average, a very well-approximated arithmetic average density function is used for the valuation. The optimal exercise boundary and the values of American average rate options are compared with those of American plain vanilla options. Especially, this article shows that American average rate option can have two different optimal exercise boundaries depending on the current stock price. Numerical experiments are also performed to demonstrate the influence of the component factors on the values of American average rate options and to illustrate the accuracy and efficiency of the valuation formula.

1. Introduction

Asian options, or Average options, are one of the most popular path-dependent contingent claims in OTC markets whose payoff depends on the average price of the underlying asset during a certain time period. They are very useful for the traders who want to hedge the volatility risk of the underlying asset price or reduce exposure to sudden movements before the expiration date. They are commonly traded on currencies and commodity products and their volume has grown up rapidly.

Asian options can be classified by several criteria. According to the possibility of early exercise, American-style Asian options and European-style Asians option are classified. In the light of the payoff type, an average rate option or a fixed strike Asian option is the case where the underlying asset is replaced by the average in the payoff of a plain vanilla option, and an average strike option or a floating strike Asian option is the case where the exercise price is replaced by the average in the payoff of a plain vanilla option. In terms of the type of averaging, arithmetic average and geometric average are divided and according to the type of sampling, continuous-sampled average and discretely-sampled average are categorized. From the standpoint of the type of weighting of the observations, equally-weighted average and flexible average are divided. Finally, in view of the starting time of the contract, forward-starting Asian option and backward-starting Asian option are classified.

The theoretical Asian option valuation problem has been studied first by Ingersoll (1987). Kemna and Vorst (1990) have derived the first exact valuation formula for the geometric average Asian option. Angus (1999) has provided a general expression for European-style continuous geometric Average options. Turnbull and Wakeman (1991) and Levy (1992) have introduced a closed-form analytic approximation formula for valuing European-style arithmetic Asian options using Edgeworth series expansion and Wilkinson approximation, respectively. Curran

(1994) has presented an approximation based on the geometric mean price conditioning approach. Dewynne and Wilmott (1995) have derived the general partial differential equation of Asian option with discrete sampling of average. Milevsky and Posner (1998) have provided closed-form approximation formulas for arithmetic Asian options based on the Reciprocal Gamma distribution. Zhang (2001) has proposed a semi-analytic method for pricing and hedging continuously sampled arithmetic average rate options. Ju (2002) has provided an approximation formula for the characteristic function of the average rate with the Taylor expansion. Zhang (2003) has obtained an analytic solution in a series form solving the PDE with a perturbation method.

The theoretical researches on American-style Asian option valuation have been done as well. Hull and White (1993) and Chalasani, Jha, Egriboyun and Varikooty (1999) have suggested binomial methods to the pricing of American-style Asian options. Barraquand and Pudet (1996) have proposed Forward Shooting Grid (FSG) Method to price American-style Asian option. Zvan, Forsyth and Vetzal (1998) have developed stable numerical PDE techniques to price American-style Asian option. Wu, Kwok and Yu (1999) and Hansen and Jorgensen (2000) have derived an analytic valuation formula for an American Average Strike options.

Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992) have derived analytic Valuation formula for American options in different methods, in which they showed that the American option price is made up of the corresponding European Option price and an integral part representing the early exercise premium.

The purpose of this article is to derive the analytic approximate valuation formula for the American average rate option with continuous arithmetic average. The American-style Asian option Valuation formula is to be constructed using partial differential equation approach. The expression will help us to gain rich understanding and intuition for the whole composition and dynamics of

American-style Asian options.

This article is organized as follows. In the next section, the derivation of the analytic valuation formula for American Average Rate options with continuous geometric average is presented. A very well-approximated arithmetic average density function which was provided by Levy is used for the valuation. The optimal exercise boundary is presented in the solution process of the valuation formula. The analytic valuation representation is decomposed into the corresponding European average rate option valuation formula and the early exercise premium. Numerical results are presented to describe the properties of optimal exercise boundary and the influence of the component factors on the valuation. The article ends with summary and conclusion in the last section.

2. Valuation of American Arithmetic Average Rate Option

We shall first consider an American Arithmetic Average Rate option written on an underlying stock price S with expiration date T and strike price K. Assume the stock price dynamics follows a lognormal diffusion process

$$dS = (r - q)Sdt + \sigma SdW$$

where dW is a standard Brownian motion, r is the risk free interest rate, σ is the volatility, and q is the continuous dividend rate which is less than r. Throughout the article K, T, r, q and σ are all taken to be constant and greater than or equal to 0, unless otherwise noted.

To this date, there is no known closed-form representation for the valuation of Asian options based on the arithmetic average because it is impossible to describe mathematically the distribution of arithmetic average of the underlying asset which follows lognormal distribution. Turnbull and Wakeman (1991) have derived an approximation model for arithmetic average rate option pricing using the fact that the distribution under arithmetic average is approximately lognormal and they put forward the first and second moments of the average in order to price the option. Levy (1992) has derived another analytic approximation model which is suggested to give more accurate results than the Turnbull and Wakeman approximation using Wilkinson approximation approach. Levy (1992) has used the knowledge of the conditional distribution of a sum of correlated log normal random variables follows log normal distribution. He has shown that the arithmetic average density function which has been approximated in his work produced very accurate European average rate option values. The greatest error for the cases with volatility below 20% has been no more than 0.02% of the underlying asset price. This article develops the analytic valuation formula of the American average rate option under the basic assumption proposed by Levy's work.

From the Levy's approach, the first two moments of arithmetic Average price of the underlying asset for the period [t, T], $A_{t,T}$, are used to derive the log normal distribution of arithmetic average. The average price dynamics at current time t is represented by

$$dA_{t,T} = \mu(t) A_{t,T} dt + \sigma_a dW , \tag{1}$$

where

$$\mu(t) = \frac{(r-q)(T-t)e^{(r-q)(T-t)} - e^{(r-q)(T-t)} + 1}{(T-t)(e^{(r-q)(T-t)} - 1)} ,$$

$$\sigma_a^2(t) = \frac{e^{(2(r-q)+\sigma^2)(T-t)} - e^{(r-q)(T-t)}}{\dfrac{e^{(2(r-q)+\sigma^2)(T-t)} - 1}{2(r-q)+\sigma^2} - \dfrac{e^{(r-q)(T-t)} - 1}{r-q}} - 2\frac{(r-q)e^{(r-q)(T-t)}}{e^{(r-q)(T-t)} - 1} .$$

With the results above, we can derive the following Black-Scholes-like Partial Differential Equation for an European average rate option $P_e(S,t)$ from the basic assumption of No-arbitrage.

$$\frac{1}{2}\frac{\partial^2 P_e}{\partial A_{t,T}{}^2}\sigma_a{}^2(t)A_{t,T}{}^2 + \frac{\partial P_e}{\partial A_{t,T}}\mu(t)A_{t,T} + \frac{\partial P_e}{\partial t} - rP_e = 0 \tag{2}$$

with initial and boundary conditions

$$P_e(A_{0,T},T) = \max(K - A_{0,T},0), \tag{3}$$

$$P_e(0,t) = Ke^{-r(T-t)} \tag{4}$$

$$P_e(A_{0,t},t) \approx 0, \quad as \quad A_{0,t} \approx \infty. \tag{5}$$

The Equation (2) and (3) are also applied to the American average rate options case. But we should consider one more condition for American rate option options for all the life-span.

We already know that the characteristics of the early exercise feature of American options lead to the condition that during the life American options must be worth at least their corresponding intrinsic values, namely, $max(A_{0,T}\text{-}K,0)$ for a call and $max(K\text{-}\ A_{0,T},0)$ for a put. To represent this constraint for an average rate put option, we should introduce a new expression to the original problem,

$$P_a(A_{0,t},t) \geq \max(K - A_{0,t},0) \tag{6}$$

where $P_a(A_{0,t},t)$ is American average rate put option value.

Moreover, the governing equation, Equation (2) is not always true. For the stopping region ($0<S<S_c(t)=B(t)$), of American average rate put option, the equality is not any longer valid. ($S_c(t)$ or $B(t)$ is the critical stock price at time t.)

$$\frac{1}{2}\frac{\partial^2 P_e}{\partial A_{t,T}{}^2}\sigma_a{}^2(t)A_{t,T}{}^2 + \frac{\partial P_e}{\partial A_{t,T}}\mu(t)A_{t,T} + \frac{\partial P_e}{\partial t} - rP_e < 0\,. \tag{7}$$

The governing equation (2) for European average rate option and equation (3) for American average rate option can be transformed into the basic heat or diffusion equation (8) and (9), respectively (see Appendix),

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{8}$$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} < 0\,. \tag{9}$$

We can make formulation of the original problem in diffusion equation form separately for exercise region and holding region ($S_c(t)=B(t)<S$), to understand the structure more clearly.

With the governing Equations (9), we reformulate the partial differential equation problem of American average rate option pricing into the following heat or diffusion equation problem:

$$\left[\begin{array}{ll} \dfrac{\partial u}{\partial \tau} - \dfrac{\partial^2 u}{\partial x^2} = Q(x,\tau), & -\infty < x \le b(\tau),\;\; \tau > 0,\;\; Q(x,\tau) \ge 0 \\ \dfrac{\partial u}{\partial \tau} - \dfrac{\partial^2 u}{\partial x^2} = 0, & b(\tau) < x < \infty,\;\; \tau > 0 \end{array}\right], \qquad \begin{array}{r} (10a) \\ (10b) \end{array}$$

$$u(x,0) = u_o(x) = \max(K^* e^{-x/2+\tilde{\beta}(0)} - e^{x/2-\tilde{\alpha}(0)+\tilde{\beta}(0)},0), \tag{11}$$

$$u(x,\tau) \approx 0 \qquad as \quad x \approx \infty, \tag{12}$$

$$u(x,\tau) = \left[\begin{array}{lll} 0 & if & 0 < \theta - \phi < 1 \\ 1 & if & \theta - \phi = 1 \\ \infty & if & \theta - \phi > 1 \end{array}\right] \quad as \quad x \approx -\infty, \tag{13}$$

$$u(x,\tau) \ge \max(K^* e^{-x/2+\tau/4+\tilde{\beta}(\tau)} - e^{x/2+\tau/4-\tilde{\alpha}(\tau)+\tilde{\beta}(\tau)},0). \tag{14}$$

Equation (10a) is for exercise region and Equation (10b) is for holding region. We can gain the final valuation formula from solving the above problems.

$$
\begin{aligned}
P_a(S,t) = & \left(K - \frac{t}{T}A_{0,t}\right)e^{-r(T-t)}N(-d_2) \\
& - \frac{T-t}{T}Se^{-r(T-t)}\frac{e^{(r-q)(T-t)}-1}{(r-q)(T-t)}N(-d_1) \\
& + \int_{u=t}^{T} e^{-r(T-u)}\left(rK - (1+r)\frac{t}{u}A_{0,t}\right)\left(N(d_5) - N(d_6)\right)du \\
& - \int_{u=t}^{T} e^{\alpha(u)-r(T-u)}\left(-\frac{t}{u^2} + \left(1-\frac{t}{u}\right)\left(\alpha'(u)+r\right)\right)\left(N(d_3) - N(d_4)\right)du
\end{aligned}
\tag{15}
$$

where

$$
\alpha'(u) = \frac{e^{(r-q)(T-u)}\left(1-(r-q)(T-u)\right)-1}{(T-u)\left(e^{(r-q)(T-u)}-1\right)}
$$

$$
d_1 = \frac{\log\frac{T-t}{T}S + \alpha(t) - \ln\left(K - \frac{t}{T}A(0,t)\right) + \gamma(t)}{\sqrt{2\gamma(t)}}, \quad d_2 = d_1 - \sqrt{2\gamma(t)}
$$

$$
d_3 = \frac{\log\frac{u-t}{u}S + \alpha(u) - \ln\left(B_1(u) - \frac{t}{u}A(0,t)\right) + \gamma(u)}{\sqrt{2\gamma(u)}}, \quad d_5 = d_3 - \sqrt{2\gamma(u)}
$$

$$
d_4 = \frac{\log\frac{u-t}{u}S + \alpha(u) - \ln\left(B_2(u) - \frac{t}{u}A(0,t)\right) + \gamma(u)}{\sqrt{2\gamma(u)}}, \quad d_6 = d_4 - \sqrt{2\gamma(u)}
$$

The first term in Equation (15) is the corresponding European average rate put option price while the second term represents the early exercise premium. Following the same process as the American average rate put option valuation formula, we can obtain the valuation formula for American average rate call option $C_a(S,t)$ as follows :

$$
\begin{aligned}
C_a(S,t) = {} & \frac{T-t}{T} S e^{-r(T-t)} \frac{e^{(r-q)(T-t)}-1}{(r-q)(T-t)} N(d_1) \\
& -\left(K - \frac{t}{T} A_{0,t}\right) e^{-r(T-t)} N(d_2) \\
& + \int_{u=t}^{T} e^{\alpha(u)-r(T-u)} \left(-\frac{t}{u^2} + \left(1-\frac{t}{u}\right)\left(\alpha'(u)+r\right)\right)\left(N(d_3)-N(d_4)\right) du \\
& - \int_{u=t}^{T} e^{-r(T-u)} \left(rK - (1+r)\frac{t}{u} A_{0,t}\right)\left(N(d_5)-N(d_6)\right) du
\end{aligned}
\tag{16}
$$

Like the American average rate put option case, the first term in Equation (16) is the European average rate all option price while the second term represents the early exercise premium.

Equations (15) and (16) give us the information that the American average rate option valuation formula consist of the European Option Valuation formula and early exercise premium. So, we can conclude that the existence of the analytic European option valuation formula is the necessary condition for the existence of the analytic American Option valuation formula like the plain vanilla option case.

The early exercise premium can be interpreted as the profits originated from an instantaneous arbitrage opportunity as soon as the early exercise property is endowed all of a sudden at time t to the holder of a European average rate option which is an American average rate option from time immediately posterior to t to the expiration date. Therefore, we only have to add up the amount of profits to the exercise region to preclude arbitrage opportunity. That is, we need to supplement the necessary amount of value to keep the American average rate option value be equal to the intrinsic value of the option in the exercise region.

3. Optimal Exercise Boundary

Like the optimal exercise boundary of American plain vanilla options, the optimal exercise boundary of American average rate options should be determined in the solution process of the valuation formula. The optimal exercise boundary *B(t)* of American average rate put option is implicitly determined by the following equation:

$$
\begin{aligned}
K - A_{0,t} &= \left(K - \frac{t}{T} A_{0,t} \right) e^{-r(T-t)} N(-d_2) \\
&\quad - \frac{T-t}{T} S e^{-r(T-t)} \frac{e^{(r-q)(T-t)} - 1}{(r-q)(T-t)} N(-d_1) \\
&\quad + \int_{u=t}^{T} e^{-r(T-u)} \left(rK - (1+r) \frac{t}{u} A_{0,t} \right) \big(N(d_5) - N(d_6) \big) du \\
&\quad - \int_{u=t}^{T} e^{\alpha(u) - r(T-u)} \left(-\frac{t}{u^2} + \left(1 - \frac{t}{u} \right) \big(\alpha'(u) + r \big) \right) \big(N(d_3) - N(d_4) \big) du
\end{aligned}
$$

This shape of the optimal exercise boundary is very similar to that of the American plain vanilla option. But the solution for the above implicit equation can be two, one or zero depending on the average price. If the solutions are two, the American average rate option takes two different optimal exercise boundaries for the life-span. If the solution is one, the option takes only one optimal exercise boundary which is the same case as the American plain vanilla option. Lastly, if the solution is zero, then there is no optimal exercise boundary like the American plain vanilla call option with zero dividend rates.

4. Numerical Experiments

In this section, the numerical experiments are performed to evaluate the analytic valuation formula of the previous section. The critical average values consisting of the optimal exercise boundary and the values of American average rate options are provided by the numerical implementation.

Several numerical techniques to implement the valuation formula can be considered but the numerical integration method provides the most accurate solutions in American option valuation problems. The time interval [t, T] is divided into N sub-intervals and the critical average values are calculated at each time points from the expiration date T to the current time t. As the number of sub-intervals N increases, the error of integration is reduced and the more accurate results can be obtained. Of course, this makes the problem of increasing the time complexity.

Figure 1 shows the critical average values consisting of the optimal exercise boundary of American average rate put option for the special case of t=0. In this figure, time to expiration of the option is 1 year, risk-free interest rate is 10% and the volatility is 20%. For this case, only one optimal exercise boundary exists. The shape of the optimal exercise boundary is very similar with that of American plain vanilla put option. The optimal exercise boundary shows the continuous and monotonous decreasing movement as time to expiration increases. And the optimal exercise boundary of the American average rate put option is always higher than that of American plain vanilla put option. This is due to the fact that the value of average rate option is always less than the value of plain vanilla option. Figure 2 shows the optimal exercise boundaries when volatility has different values. The optimal exercise boundary of the average value based on the asset with a higher volatility is always lower than that based on the asset with a lower volatility. This is also due to the fact that the value of average rate option with a higher volatility is always higher than the value of average rate option with a lower volatility. Figure 3 shows the values of American average rate option, European average rate option, American plain vanilla option and European plain vanilla option at t=0. The value of American plain vanilla option is always higher than or equal to the value of American average rate option.

Table 1 reports the results of the values of the American average rate options for the representative set of parameters for the special case of t=0. To access the accuracy of the analytic valuation formula, the results from implicit finite difference method with 10,000×10,000 grid points to the corresponding partial differential equation are regarded as the true option values and are used as the benchmark. The results of Table 1 provides the information that the values of the American average rate put options increase gradually as the interest rate increases, as time to expiration increases, and as the volatility increases. These characteristics can also be observed in the values of the European average rate put options.

To evaluate the performance of the valuation model, three error statistics are used in Table 1. Mean of absolute error (MAE) and Mean of relative error (MRE) is used to measure the accuracy of the analytic valuation formula and maximum absolute error (MaxAE) is used to measure the maximum possible error. Table 1 shows that the numerical results of the analytic valuation formula has 0.067% and 0.046% in terms of the mean of relative error and 0.0019 and 0.0007 in terms of the mean of absolute error for the cases of N=100 and N=1,000, respectively. Moreover, the measure of maximum absolute error is as small as 0.0115 and 0.0047. These observations lead us to confirm the correctness of the analytic valuation formula.

The execution times in several numbers of sub-intervals in time are reported on Table 2. As the number of sub-intervals increases, the execution time of the numerical integration process of the analytic valuation increases exponentially.
Considering the error statistics and execution time for the cases of N=100 and N=1,000 in the numerical integration process, we can find out that more divisions of time axis does not lead to much more accurate results in option values. When we increase the number of sub-intervals by 10 times, the error is reduced to around a half, but the execution times increases by about 100 times. We can conclude that numerical integration process with only 100 sub-intervals makes accurate results enough to be taken.

Detailed numerical values of American average rate options from Forward Shooting Grid (FSG) method of Barraquand & Pudet (1996) are given in Table 3 to be compared with the results from the analytic valuation formula derived in the previous section. All the statistics of MRE, MAE, and MaxAE mention that the analytic valuation formula is much accurate than the FSG method when all the conditions are the same.

Figure 4 shows a very interesting result which distinguishes the American average rate option from the other options. Unlike the plain vanilla option, the American average rate option can take two optimal exercise boundaries – upper optimal exercise boundary and lower optimal exercise boundary - depending on the current stock price. The lower optimal exercise boundary increases continuously and monotonically and the upper optimal exercise boundary decreases continuously and monotonically. Finally, the optimal exercise boundaries meet each other at some time in a point and disappear from the point of contact. This means that the value of American average rate option comes to be higher than the early exercise payoff whatever the average value has. The early exercise privilege makes no advantage to the option holders for that period.

Table 4 reports the results of the values of the American average rate options for the representative set of parameters for more general cases than Table 1. In this table, we can observe the effect of the component factors on the option values. Focusing on each factor with other factors equal, long time to expiration, small current average value, small risk-free interest rate, small current time, big volatility and small current stock value make the average rate put option values higher than ever in American case as well as in European case. We can detect the fact in Table 3 that big exercise price makes the average rate put option value higher.

When parameters have the value of T=0.5, A=100, r=0.05, t=0.25, σ=0.2, q=0.04, and K=100 in Table 4, we can find that the value of American average rate put option value is equal to the European average rate put option. This means that

European average rate option value is always higher than the early exercise payoff as time goes on. It is already known that the phenomena of the option value being always higher than the early exercise payoff can be observed in the American plain vanilla call option. When parameters have the value of T=1.0, A=110, r=0.1, t=0.75, σ=0.2, q=0.01, and K=100 in Table 4, we can find that the American average rate put option value as well as the European average rate put option has the value of 0. If the current value is put back to 0.5, the average rate put option becomes to have a positive value. This correspond to the case that the current stock price has little possible to make the average value till the expiration date drop under the exercise price as explained in the previous section.

5. Summary and Conclusions

This article has presented the analytic valuation formula for American average rate option with the continuous arithmetic average which is composed of the corresponding European average rate option and early exercise premium. We use the result of Levy (1992) that the distribution of an arithmetic average is well-approximated by the log normal distribution when the underlying asset follows the log normal diffusion process. Like the optimal exercise boundary of American average rate options which should be determined in the solution process of the valuation formula has been developed. This shape of the optimal exercise boundary is very similar to that of the American plain vanilla option. Also, the optimal exercise boundary of American average rate options is closer to the early exercise price than that of American plain vanilla options. From this property of the optimal exercise boundary, the values of the American average rate options are always lower than that of the American plain vanilla options. This is an interesting results considering that the curve of the European average option value cuts across the curve of the European plain vanilla option value.

We need to concentrate on the fact that unlike the American plain vanilla, the American average rate option can take two different optimal exercise boundaries depending on the current stock value. At the beginning time of the average rate option contract, namely t=0, the optimal exercise boundary is just one. But if the option holder loses some opportunities to exercise the American average rate optimally as time passes, then the American average rate option can take two different optimal exercise boundaries, which is a substantially different characteristic of the American average rate options.

Because the arithmetic density function is approximated based on the Levy's work, the accuracy cannot be guaranteed for large volatility. More precise distribution could improve the accuracy of the valuation of Asian options. Closed-form analytic approximation formula for American options could be more efficient for calculating the option values.

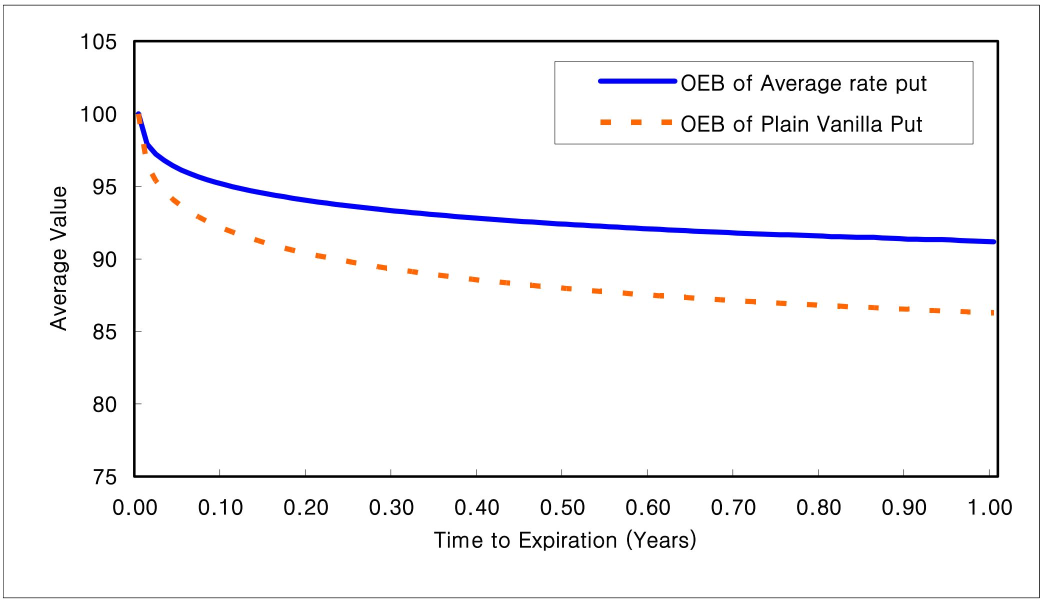

FIGURE 1. Comparison of the Optimal Exercise Boundary(OEB) of American average rate put option and American plain vanilla option (S=50, K=50, T=1.0, t=0.0, A=0.0, r=0.1, q=0.0, σ=0.2)

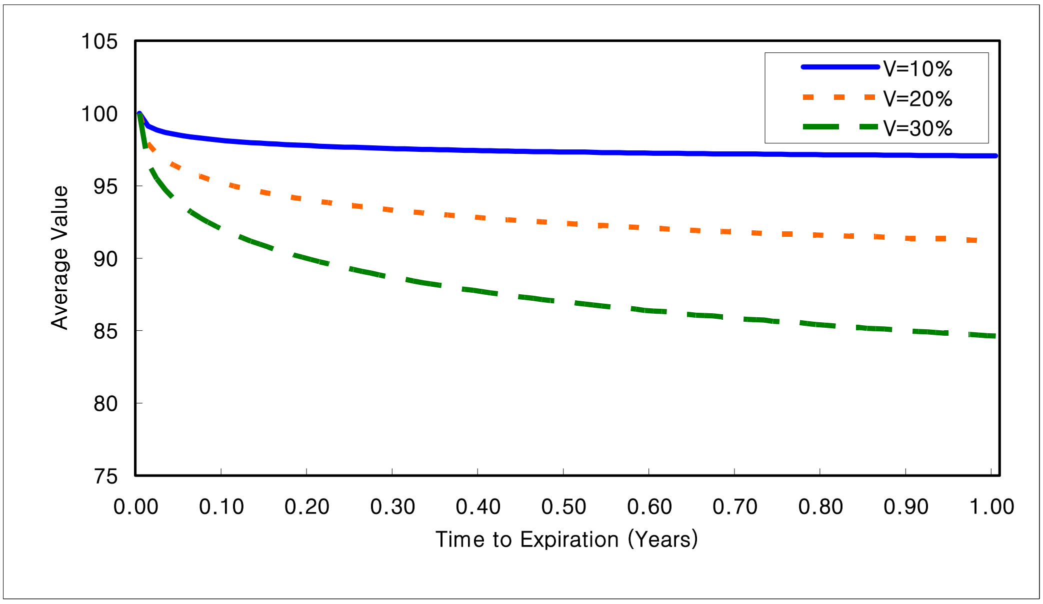

FIGURE 2. Comparison of the Optimal Exercise Boundary(OEB) of American average rate put option with different volatilities. (S=50, K=50, T=1.0, t=0.0, A=0.0, r=0.1, q=0.0.)

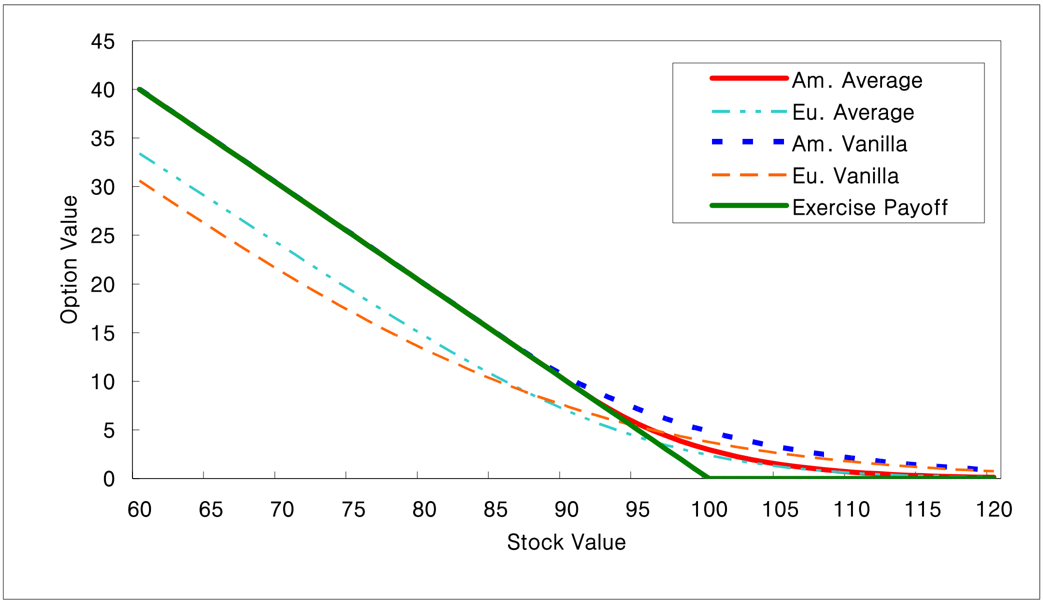

FIGURE 3. Comparison of the graph of American Average rate option, European Average rate option, American plain vanilla option and European plain vanilla option

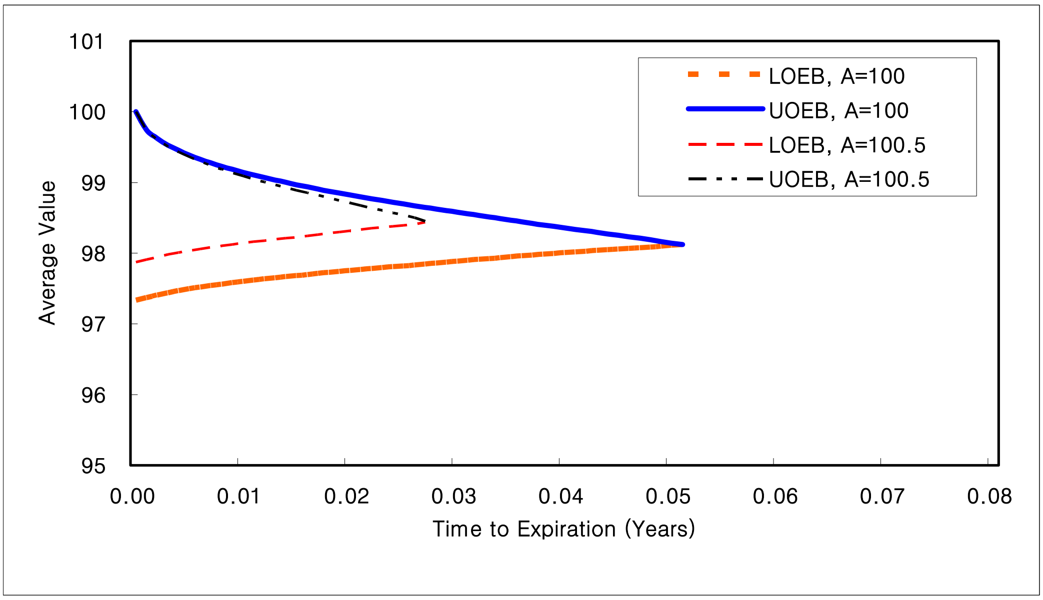

FIGURE 4. Lower Optimal Exercise Boundary(OEB) and Upper Optimal Exercise Boundary(OEB) of American average rate put option (S=100, K=100, T=1.0, t=0.5, r=0.1, q=0.0, σ=0.2)

				Approximation		LEVY	FDM	
N				100	1,000	–	10,000	10,000
r	T	σ	S	American	American	European	American	European
0.05	0.25	0.1	95	4.9957	4.9997	4.4207	5.0000	4.4208
			100	0.9192	0.9187	0.8595	0.9187	0.8595
			105	0.0332	0.0332	0.0321	0.0332	0.0321
		0.2	95	5.2649	5.2638	5.0462	5.2638	5.0462
			100	2.0445	2.0440	1.9895	2.0440	1.9894
			105	0.5460	0.5459	0.5360	0.5459	0.5360
	0.50	0.1	95	4.9949	5.0000	4.0697	5.0000	4.0698
			100	1.1914	1.1904	1.0634	1.1905	1.0634
			105	0.1296	0.1295	0.1216	0.1295	0.1216
		0.2	95	5.6745	5.6725	5.3255	5.6729	5.3255
			100	2.7549	2.7539	2.6314	2.7541	2.6314
			105	1.1224	1.1221	1.0845	1.1221	1.0845
	0.75	0.1	95	4.9941	4.9999	3.8228	5.0000	3.8228
			100	1.3674	1.3658	1.1696	1.3664	1.1695
			105	0.2278	0.2275	0.2068	0.2276	0.2068
		0.2	95	6.0143	6.0115	5.5377	6.0126	5.5377
			100	3.2524	3.2508	3.0540	3.2514	3.0540
			105	1.5759	1.5752	1.5001	1.5754	1.5000
0.10	0.25	0.1	95	4.9908	4.9995	3.8248	5.0000	3.8248
			100	0.7519	0.7509	0.6237	0.7511	0.6236
			105	0.0196	0.0195	0.0180	0.0195	0.0180
		0.2	95	5.0894	5.0868	4.5426	5.0869	4.5426
			100	1.8322	1.8312	1.7064	1.8313	1.7064
			105	0.4549	0.4547	0.4338	0.4547	0.4338
	0.50	0.1	95	4.9890	5.0000	3.0708	5.0000	3.0708
			100	0.9101	0.9080	0.6557	0.9093	0.6557
			105	0.0695	0.0693	0.0581	0.0694	0.0581
		0.2	95	5.3066	5.3021	4.4634	5.3038	4.4634
			100	2.3695	2.3673	2.0961	2.3685	2.0961
			105	0.8940	0.8932	0.8167	0.8936	0.8167
	0.75	0.1	95	4.9885	4.9965	2.5470	5.0000	2.5470
			100	0.9965	0.9933	0.6287	0.9966	0.6287
			105	0.1136	0.1133	0.0864	0.1136	0.0864
		0.2	95	5.4886	5.4823	4.3718	5.4870	4.3718
			100	2.7150	2.7117	2.2893	2.7152	2.2893
			105	1.2140	1.2126	1.0640	1.2140	1.0640
Mean of Relative Error (%)				0.067%	0.046%	0.003%	0.000%	0.000%
Mean of Absolute Error				0.0019	0.0007	0.0000	0.0000	0.0000
Maximum of Absolute Error				0.0115	0.0047	0.0001	0.0000	0.0000

Table 1. Values of American average rate put options (K=100,t=0, q=0)

N	50	100	300	500	750	1,000	FDM
Time(sec)	0.186	0.766	6.715	18.102	40.704	70.636	857.213

Table 2. Execution Times

			Approximation		FSG		FDM	
σ	T	K	Amer	Euro	Amer	Euro	Amer	Euro
0.1	0.25	95	0.015	0.013	0.013	0.013	0.015	0.013
		100	0.752	0.624	0.832	0.626	0.751	0.624
		105	4.990	3.794	5.337	3.785	5.000	3.794
	0.50	95	0.056	0.047	0.051	0.046	0.056	0.047
		100	0.910	0.656	0.978	0.655	0.909	0.656
		105	4.988	3.048	5.287	3.039	5.000	3.049
	1.00	95	0.126	0.088	0.104	0.084	0.126	0.088
		100	1.052	0.584	1.079	0.577	1.054	0.584
		105	4.986	2.142	5.230	2.137	5.000	2.142
0.2	0.25	95	0.397	0.379	0.407	0.379	0.397	0.379
		100	1.832	1.706	2.066	1.716	1.831	1.706
		105	5.128	4.589	6.108	4.598	5.125	4.589
	0.50	95	0.803	0.734	0.820	0.731	0.803	0.734
		100	2.370	2.096	2.629	2.102	2.369	2.096
		105	5.382	4.539	6.338	4.552	5.378	4.539
	1.00	95	1.335	1.125	1.318	1.099	1.336	1.125
		100	2.969	2.390	3.181	2.369	2.971	2.390
		105	5.749	4.363	6.596	4.356	5.750	4.363
Mean of Relative Error (%)			0.12%	0.00%	9.11%	0.99%	0.00%	0.00%
Mean of Absolute Error			0.003	0.000	0.255	0.007	0.000	0.000
Maximum Absolute Error			0.014	0.000	0.983	0.026	0.000	0.000

Table 3. Comparison of the American average rate put options with FSG method(Barraquand and Pedet (1996)).

							Approximation		LEVY	FDM	
N							100	1,000		10,000	10,000
T	A	r	t	σ	q	S	Amer.	Amer.	Euro.	Amer.	Euro
0.5	100	0.05	0.25	0.2	0.04	95	2.7074	2.7074	2.7074	2.7074	2.7074
0.5	100	0.05	0.25	0.2	0.04	100	1.1081	1.1081	1.1081	1.1081	1.1081
0.5	100	0.05	0.25	0.2	0.04	105	0.3124	0.3124	0.3124	0.3124	0.3124
0.5	100	0.1	0.25	0.1	0.01	95	1.9671	1.9671	1.9659	1.9671	1.9659
0.5	100	0.1	0.25	0.1	0.01	100	0.3338	0.3338	0.3324	0.3338	0.3324
0.5	100	0.1	0.25	0.1	0.01	105	0.0102	0.0102	0.0101	0.0102	0.0101
1.0	100	0.05	0.25	0.1	0.01	95	3.0763	3.0762	3.0621	3.0760	3.0621
1.0	100	0.05	0.25	0.1	0.01	100	0.9846	0.9845	0.9753	0.9844	0.9752
1.0	100	0.05	0.25	0.1	0.01	105	0.1835	0.1834	0.1811	0.1834	0.1811
1.0	100	0.05	0.25	0.2	0.01	95	4.3140	4.3141	4.3126	4.3140	4.3126
1.0	100	0.05	0.25	0.2	0.01	100	2.4037	2.4037	2.4024	2.4036	2.4023
1.0	100	0.05	0.25	0.2	0.01	105	1.1937	1.1938	1.1928	1.1937	1.1928
1.0	100	0.1	0.25	0.1	0.01	95	2.2789	2.2795	2.0677	2.2765	2.0678
1.0	100	0.1	0.25	0.1	0.01	100	0.6068	0.6063	0.5337	0.6056	0.5337
1.0	100	0.1	0.25	0.1	0.01	105	0.0872	0.0871	0.0773	0.0870	0.0772
1.0	100	0.1	0.25	0.1	0.02	95	2.3927	2.3930	2.2319	2.3908	2.2319
1.0	100	0.1	0.25	0.1	0.02	100	0.6624	0.6619	0.6019	0.6613	0.6019
1.0	100	0.1	0.25	0.1	0.02	105	0.1007	0.1006	0.0917	0.1005	0.0917
1.0	100	0.1	0.25	0.2	0.01	95	3.4657	3.4653	3.4169	3.4641	3.4169
1.0	100	0.1	0.25	0.2	0.01	100	1.8412	1.8408	1.8083	1.8401	1.8082
1.0	100	0.1	0.25	0.2	0.01	105	0.8677	0.8675	0.8499	0.8671	0.8499
1.0	100	0.1	0.25	0.2	0.02	95	3.5918	3.5915	3.5581	3.5906	3.5581
1.0	100	0.1	0.25	0.2	0.02	100	1.9264	1.9262	1.9028	1.9256	1.9028
1.0	100	0.1	0.25	0.2	0.02	105	0.9176	0.9175	0.9043	0.9172	0.9043
1.0	100	0.1	0.5	0.1	0.02	95	1.7177	1.7176	1.7095	1.7175	1.7095
1.0	100	0.1	0.5	0.1	0.02	100	0.4019	0.4018	0.3967	0.4017	0.3966
1.0	100	0.1	0.5	0.1	0.02	105	0.0398	0.0397	0.0390	0.0397	0.0390
1.0	105	0.1	0.25	0.1	0.01	95	1.4071	1.4074	1.3950	1.4067	1.3950
1.0	105	0.1	0.25	0.1	0.01	100	0.2953	0.2954	0.2914	0.2952	0.2914
1.0	105	0.1	0.25	0.1	0.01	105	0.0336	0.0336	0.0330	0.0336	0.0330
1.0	110	0.1	0.75	0.2	0.01	95	0.0000	0.0000	0.0000	0.0000	0.0000
1.0	110	0.1	0.75	0.2	0.01	100	0.0000	0.0000	0.0000	0.0000	0.0000
1.0	110	0.1	0.75	0.2	0.01	105	0.0000	0.0000	0.0000	0.0000	0.0000
1.0	110	0.1	0.5	0.2	0.01	95	0.3454	0.3454	0.3454	0.3454	0.3454
1.0	110	0.1	0.5	0.2	0.01	100	0.0963	0.0963	0.0963	0.0963	0.0963
1.0	110	0.1	0.5	0.2	0.01	105	0.0216	0.0216	0.0216	0.0216	0.0216
Mean of Relative Error (%)							0.046%	0.033%	0.003%	0.000%	0.000%
Mean of Absolute Error							0.0004	0.0004	0.0000	0.0000	0.0000
Maximum Absolute Error							0.0024	0.0030	0.0000	0.0000	0.0000

Table 4. Values of American average rate put options (K=100)

Appendix

Derivation of equation (8)

It is more convenient to work in traditional normalized coordinates with the stock price variable normalized by the strike price, the time variable normalized by the volatility and the strike price, the critical stock price variable normalized by the exercise price, the American Options price normalized by the exercise price respectively,

$$\alpha(t) = \int_t^T \mu(u)du = \ln \frac{e^{(r-q)(T-t)} - 1}{(r-q)(T-t)}$$

$$\beta(t) = \int_t^T r du = r(T-t).$$

$$\gamma(t) = \tau = \int_t^T \frac{1}{2}\sigma_a{}^2(u)du$$

$$= \frac{1}{2}\ln \frac{2}{(r-q+\sigma^2)}\left(\frac{e^{(2(r-q)+\sigma^2)(T-t)} - 1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)(T-t)} - 1}{r-q}\right) - \ln \frac{e^{(r-q)(T-t)} - 1}{r-q}$$

$$x = \ln A_{t,T} + \alpha(t)$$

$$K - A(0,T) = K - \left(\frac{t}{T}A_{0,t} + \frac{T-t}{T}A_{t,T}\right) = K^* - \frac{T-t}{T}A_{t,T}$$

$$\alpha(t) = \tilde{\alpha}(\tau), \quad \beta(t) = \tilde{\beta}(\tau), \quad b(\tau) = \ln \frac{B(t)}{K},$$

$$P_a(A_{t,T}, t) = e^{x/2 - \tau/4 + \tilde{\beta}(\tau)}u(x,\tau).$$

then, Equation (2) can be transformed into the basic heat or diffusion equation problem in thermal physics as follows,

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0$$

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