Derivative Prices with Uncertain Expected returns

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Abstract

The optimal conditions of mean reversion speed for log-return of a stock is derived and approximate solutions are obtained. A value of a derivative under the initial measure is compared with the value under minimum variance measure. Moreover, the results provide an efficient way to simulate an underlying asset so that more accurate sensitivity analysis can be performed.

1. Introduction

It is not too much to say that pricing, hedging and synthesizing of an asset are everything in financial market. Specially, trader or risk manager lay emphasis on hedging and synthesizing of their positions. These days many assets traded in over-the-counter market are customized and hybrid so that there are hardly closed-form formulae for pricing and hedging. Usual ways to handle such an asset are numerical methods like trees, partial differential equations and simulations. It seems that Monte Carlo simulations are more extensively used than the other two methods even though each method has both advantages and disadvantages in stability as well as efficiency. To pursue efficiency in calculating Greeks using Monte Carlo simulation, one can recycle the random numbers which are already occurred to price the value. However for such a case, the Greeks may be underestimated than as used to be. Or one can generate random numbers one more time to value the position fully even though it is costly. In this case a Greek like delta can have the opposite sign which means that a hedger should have their position clear and make a new opposite position. This problem has been worse in according to the number of underlying assets increased and the structure of payoffs complicated. Indeed this situation can happen in real financial market.

The aim of this article is to provide a technical method to reduce potential trade-off between stability and efficiency when Monte Carlo simulations are used in delta hedging. The basic idea accomplishing the goal is to determine a control variable minimizing the Radon-Nikodym derivative which measures the sensitivity of two measures. This is in line with selecting an optimal measure from a class of probability measure by perturbing the initial measure in incomplete markets as in Rouge and El Karaoui (2000). Schellhorn (2004) applied the same idea for an interest rate whose dynamic follows Ornstein-Uhlenbeck process. Linearity of OU process allows the analytic tractability easy. However such a process is not appropriate for stocks, so we try to extend the method with the stock which is governed by a lognormal process.

One advantage of our model is that stochastic control problem is transformed into a system of ordinary differential equations which make implementation of the target measure possible. (Here the initial measure is used to price the value of an derivative and the target measure is for Greeks calculation.)

The rest of the paper is organized as follows. Section 2 sets up optimization problem and solves two kinds of variance minimization problem. Section 4 presents numerical results, and section 4 concludes.

2. Model

Let (Ω, F, P^I) be the complete filtered probability space and W^I a supported Brownian motion. It is assumed that the dynamic of a stock price process is initially a lognormal process

$$\frac{dS_t^*}{S_t^*} = rdt + \sigma dW_t^I \tag{1}$$

where r is a constant risk-free interest rate and σ a constant volatility of the stock returns. The scaled stock price denominated by the current price, $S_t = S_t^* / S_0^*$, follows the same process as (1) but S_0 is normalized i.e., $S_0 = 1$. The terminal measure P^T supports a Brownian motion W^T and dW^T is perturbed from the initial Brownian motion by $a(t)\log S_t / \sigma$, i.e.,

$$dW_t^T = dW_t^I + \frac{a(t)\log S_t}{\sigma} dt \tag{2}$$

where $a(t)$ is a time-varying but deterministic function. It is noted that the terminal variance of the state variable S_t is the same as the initial one of the state variable S_t^*. Under the new measure P^T, the dynamic of stock price process is

$$\frac{dS_t}{S_t} = \left(r - a(t)\log S_t\right)dt + \sigma dW_t^T \tag{3}$$

and Ito's lemma shows that log returns of stock are mean reversion process below,

$$d\log S_t = \left(r - \frac{1}{2}\sigma^2 - a(t)\log S_t\right)dt + \sigma dW_t^T \tag{4}$$

so that $a(t)$ represents a mean reversion speed of stock's log return. Once $a(t)$ is specified, the terminal measure P^T is completely determined. When $a(t)=0$, the SDE (4) is the same as the dynamic of the log return under the initial measure. Since the terminal measure is stochastically changed from the initial measure by (2), the Radon-Nikodym derivative also moves stochastically and the second moment is calculated in the Lemma below.

Lemma 2.1: Let $\xi(T)$ be the Radon-Nikodym derivative of the terminal measure with respect to the initial measure at time horizon T, i.e.,

$$\xi(T) = \left.\frac{dP^T}{dP^I}\right|_T .$$

Then the second moment is given by

$$E^I\left[\xi^2(T)\right] = \exp\left(\int_0^T \sigma^2\left(\frac{r}{\sigma^2}g(t) - \frac{1}{2}g(t) + f(t) + \frac{1}{2}g(t)^2\right)dt\right) \tag{5}$$

where

$$\frac{df(t)}{dt} = 4a(t)f(t) - 2\sigma^2 f(t)^2 - \frac{a(t)^2}{\sigma^2}, \tag{6}$$

$$\frac{dg(t)}{dt} = 2a(t)g(t) + \left(\sigma^2 + 2r\right)f(t) - 2\sigma^2 f(t)g(t), \tag{7}$$

$$f(T) = g(T) = 0.$$

Proof : see Appendix.

Lemma shows the Radon-Nikodym derivative is an exponential function satisfying a system of ordinary differential equations. A Riccardi equation (6) can not be analytically solvable unless $a(t)$ is a constant. The next theorem provides one way to select control variable $a(t)$ minimizing variance of the Radon-Nikodym derivative over the whole time interval. The average variance constraint problem is set up to solve as follows;
for some constant M_A,

$$\min_{a(t)} E^I\left[\xi(T)^2\right] \tag{8}$$

s.t. $$E^T\left[\int_0^T (\log S_t)^2 dt\right] \le M_A, \tag{9}$$

$$\frac{dS_t}{S_t} = (r - a(t)\log S_t)dt + \sigma dW_t^T, \tag{10}$$

$$S_0 = 1.$$

Theorem 2.1 : The solution $a(t)$ of average variance constraint problem is given by

$$a(t) = \frac{\sigma^2}{2\mu_1}\left(4\mu_1 f(t) + 2\mu_2 g(t) - \mu_3 v(t)\right) \tag{11}$$

where $f(t), g(t)$ are given by (6)-(7) and v, μ_1, μ_2 and μ_3 solve the following boundary value problem: for some value $\lambda \ge 0$,

$$\int_0^T v(t)dt \le M_A,$$

$$\frac{dv(t)}{dt} = -2a(t)v(t) + \sigma^2, \qquad v(0) = 0, \tag{12}$$

$$\frac{d\mu_1(t)}{dt} = \sigma^2 - 4\mu_1(t)\left(a(t) - \sigma^2 f(t)\right) + \mu_2(t)\left(2\sigma^2 g(t) - 2r + \sigma^2\right), \qquad \mu_1(T) = 0, \tag{13}$$

$$\frac{d\mu_2(t)}{dt} = \left(r - \frac{1}{2}\sigma^2 - \sigma^2 g(t)\right) - 2\mu_2(t)\left(a(t) - \sigma^2 f(t)\right), \qquad \mu_2(T) = 0, \tag{14}$$

$$\frac{d\mu_3(t)}{dt} = -\lambda + 2a(t)\mu_3(t), \qquad \mu_3(T) = 0. \tag{15}$$

Proof: It is enough to show that $a(t)$ satisfying (8) minimizes $\log E^I\left[\xi(T)^2\right]$ instead of (5) under the same constraints. Let λ be an Lagrange multiplier of inequality constraint (9) and $v(t) = E^T\left[(\log S_t)^2\right]$. Solving the SDE (10), $\log S_t$ is a Gaussian process below,

$$\log S_t = \exp\left(-\int_0^t a(u)du\right)\left[\int_0^t \exp\left(\int_0^u a(s)ds\right)\left(r - \frac{1}{2}\sigma^2\right)du + \int_0^t \exp\left(\int_0^u a(s)ds\right)\sigma dW^T(s)\right].$$

The dynamic of the second moment of logarithm of S_t is

$$\frac{dv(t)}{dt}=\frac{dE^T\left[(\log S_t)^2\right]}{dt}=\frac{d}{dt}\left(\exp\left(-2\int_0^t a(u)du\right)\int_0^t t\exp\left(2\int_0^u a(s)ds\right)du\right)$$

$$=-2a(t)v(t)+\sigma^2.$$

Applying Pontryagin Maximum Principle (see Intriligator 2002) to the equivalent maximization problem, Hamiltonian becomes

$$H=-\sigma^2\left(\frac{r}{\sigma^2}g(t)-\frac{1}{2}g(t)+f(t)+\frac{1}{2}g^2(t)\right)-\lambda v(t)+\mu_1\frac{df(t)}{dt}+\mu_2\frac{dg(t)}{dt}+\mu_3\frac{dv(t)}{dt}$$

so that the solution of Hamiltonian equation $\partial H(t)/\partial a=0$ is given by the formula (11) and conditions (12)-(15) also follow. □

While Theorem 2.1 has an advantage that a stochastic control problem is changed into a system of ordinary differential equations, it states only a necessary condition since the Hessian of the minimized Hamiltonian is not positive-definite. So the solution described in Theorem 2.1 is a suboptimal solution. It is worthy to mention that it has a problem to implement $a(t)$ since the initial condition for $v(t)$ is given while all other equations start with terminal conditions. So start any value for $v(T)$ and find $v(0)$ finally. If $v(0)$ is not 0, it needs to adjust $v(T)$ and iterate the procedure. The next proposition finds another suboptimal solution which approximates the variance $Var^I[\xi(T)]$. It is possible that all terms multiplied by $f(t)$ or $g(t)$ in (6) and (7) turn out to be small when the variance of $\xi(T)$ is small.

Proposition 2.1: Suppose that

(1) $a(t)\geq\dfrac{2\sigma^2 g(t)-(2r+\sigma^2)}{2g(t)}f(t)$ and

(2) $a(t)\geq\dfrac{2\sigma^2\left(2f(T-t)+2g(T-t)^2-g(T-t)\right)}{2}$ hold.

Then the variance has an upper bound below

$$Var^I\left[\xi^2(T)\right]\leq Var^I_{app}\left[\xi^2(T)\right]$$

where

$$Var^I_{app}\left[\xi^2(T)\right]=\exp\left(\int_0^T\sigma^2 w(t)dt\right)-1 \tag{16}$$

$$\frac{dw(t)}{dt} = \frac{a(t)^2}{\sigma^2} \tag{17}$$

$$w(0) = 0. \tag{18}$$

Proof: Condition (1) guarantees that $g'(t) \geq 0$ for all t by (7). Since $g(T) = 0$ at the terminal time, $g(t) \leq 0$ on the whole time interval so that

$$\min_{a(t)} E^I\left[\xi^2(T)\right] \leq \min_{a(t)} \exp\left(\int_0^T \sigma^2\left(-\frac{1}{2}g(t) + f(t) + \frac{1}{2}g(t)^2\right)dt\right).$$

Hence it suffices to show that

$$\exp\left(\int_0^T \sigma^2\left(-\frac{1}{2}g(t) + f(t) + \frac{1}{2}g(t)^2\right)dt\right) \leq \exp\left(\int_0^T \sigma^2 z(t)dt\right). \tag{19}$$

Let $x(t) = a(T-t)^2/\sigma^2$ and $y(t) = -g(T-t)/2 + f(T-t) + g(T-t)^2/2$. Then

$$\begin{aligned}
\frac{dw(t)}{dt} &\equiv G(x(t), z(t)) = x(t) \\
\frac{dy(t)}{dt} &\equiv F(x(t), y(t)) = \frac{1}{2}g'(T-t) - g'(T-t)g(T-t) - f'(T-t) \\
&= \left[a(T-t)g(T-t) + \frac{1}{2}(2r+\sigma^2)f(T-t)\right] \\
&\quad - 2g(T-t)\left[a(T-t)g(T-t) + \frac{1}{2}(2r+\sigma^2)f(T-t)\right] \\
&\quad + f(T-t)\left[-4a(T-t) - \sigma^2 g(T-t) + 2\sigma^2 g^2(T-t) + 2\sigma^2 f(T-t)\right] + x(t) \\
&\leq x(t) \\
&= \frac{dw(t)}{dt}
\end{aligned}$$

The inequality step follows since the first three terms are negative by hypotheses (1) and (2). Since $y(0) = z(0)$, Sturm Comparison Theorem says that

$$\int_0^T \sigma^2 y(t)dt \leq \int_0^T \sigma^2 w(t)dt.$$

which asserts (19). □

Using integration by parts, (16) in Proposition 2.1 can be restated below:

$$Var_{app}^I\left[\xi^2(T)\right] = \exp\left(\int_0^T (T-t)a(t)^2 dt\right) - 1. \tag{20}$$

Corollary 2.1 The approximated average variance constraint problem has the solution as follows:

$$a(t) = -\frac{\mu_3(t)v(t)}{T-t}$$

where $v(t)$, $\mu_3(t)$ satisfy (12), (15), respectively.

Proof: Applying Pontryagin Maximum Principle to minimizing of the equation (20) once again, the result is obtained. □

Instead of minimizing average variance on the whole time interval, the variance at terminal time can be minimized as follows;

$$\min_{a(t)} E^I\left[\xi(T)^2\right]$$

$$\text{s.t.} \qquad E^T\left[(\log S_T)^2\right] \le M_T,$$

$$\frac{dS_t}{S_t} = (r - a(t)\log S_t)dt + \sigma dW_t^T,$$

$$S_0 = 1.$$

The solution of this problem is similar with the one in Theorem 2.1. In this case the terminal variance condition is specifically given.

The analog of Theorem 2.1 for terminal variance constraint problem is

Theorem 2.2 A necessary condition for terminal variance constraint problem is

$$a(t) = \frac{\sigma^2}{2\mu_1}\left(4\mu_1 f(t) + 2\mu_2 g(t) - \mu_3 v(t)\right)$$

where $f(t), g(t)$ are given by (6)-(7) and v, μ_1, μ_2 and μ_3 solve the following boundary value problems:

$$\frac{dv(t)}{dt} = -2a(t)v(t) + \sigma^2, \qquad v(T) = M_T \quad (21)$$

$$\frac{d\mu_1(t)}{dt} = \sigma^2 - 4\mu_1(t)\left(a(t) - \sigma^2 f(t)\right) + \mu_2(t)\left(2\sigma^2 g(t) - 2r + \sigma^2\right), \qquad \mu_1(T) = 0,$$
$$\frac{d\mu_2(t)}{dt} = \left(r - \frac{1}{2}\sigma^2 - \sigma^2 g(t)\right) - 2\mu_2(t)\left(a(t) - \sigma^2 f(t)\right), \qquad \mu_2(T) = 0,$$
$$\frac{d\mu_3(t)}{dt} = 2a(t)\mu_3(t), \qquad \mu_3(T) = 0.$$

Solving terminal variance constraint problem is easier than solving average variance constraint problem since Lagrangian multiplier λ disappears. Since the control parameter a is a constant for terminal variance constraint problem, (6) and (7) will be explicitly solved so that Radon-Nikodym derivative given by (2) is specifically calculated.

Corollary 2.2 The terminal variance constraint problem has the solution as follows:

$$a = \frac{\sigma^2}{M_T}.$$

Proof : Terminal value of the solution of Equation (21) has the upper bound

$$v(T) = \frac{\sigma^2}{a}\left(1 - e^{-2aT}\right) \leq \frac{\sigma^2}{a}.$$

Hence the minimum is attained when $v(T) = M_T$. □

Since the control variable $a(t)$ is chosen for each constraint problem, the terminal measure P^T and Radon-Nikodym derivative are completely determined so that pricing of derivatives is possible as in theorem next.

Theorem 2.3 Under the measure P^T**,** the price of a derivative whose payoff $V(T)$ at T is

$$V(0) = E^T\left[\frac{e^{-rT}}{\xi(T)} V(T)\right] \tag{22}$$

where $\xi(T)$ is a Radon-Nikydym derivative defined by (5).

Proof : Let $\theta(t) = -a(t)\log S_t / \sigma$ be a market price of risk. Since the drift and diffusion of (3) is adapted to the filtration, Girsanov theorem shows that $Z(T)V(T)$ is a martingale where

$$Z(T) = \exp\left(-\int_0^T \theta(s) dW_s^T - \int_0^T \left(r + \frac{1}{2}\theta(s)^2 \right) ds \right). \tag{23}$$

Then the price of the derivative is

$$V(0) = E^T\left[Z(T)V(T)\right] = E^T\left[\frac{e^{-rT}}{\xi(T)} V(T) \right]. \tag{24}$$

3. Application to Greeks

4. Conclusion

This article is mainly contributed to show that the variance of the Radon-Nikodym derivative is the exponential of the time-integral of the solution of a system of ordinary differential equations. Average variance constraint problem and terminal variance constraint problem are solved. Using the control variables which are solutions of minimization problems, Greeks can be efficiently calculated without generating random numbers twice. However it is unfortunate that the solutions satisfy only a necessary condition so are suboptimal.

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Appendix

Proof of Lemma

Define a new measure P^M so that the process defined below becomes a new Brownian motion under this measure,

$$dW_t^M = dW_t^T + \frac{a(t)\log S_t}{\sigma}dt .$$

Note that Radon-Nykodym derivatives between these measures are

$$\frac{dP^T}{dP^I} = \exp\left(-\int_0^T \frac{a(t)\log S_t}{\sigma} dW_t^I - \frac{1}{2}\int_0^T \left(\frac{a(t)\log S_t}{\sigma} \right)^2 dt \right),$$

and

$$\frac{dP^T}{dP^M} = \exp\left(-\int_0^T \frac{a(t)\log S_t}{\sigma} dW_t^M - \frac{1}{2}\int_0^T \left(\frac{a(t)\log S_t}{\sigma} \right)^2 dt \right)$$

so that

$$E^I\left[\xi^2(T)\right] = E^M\left[\left(\frac{dP^T}{dP^I}\right)^2 \frac{dP^I}{dP^M}\right] = E^M\left[\frac{dP^T}{dP^I}\frac{dP^T}{dP^M}\right] = E^M\left[\exp\left(\int_0^T x(t)dt\right)\right].$$

where

$$x(t) = \left(\frac{a(t)}{\sigma}\log S_t \right)^2 .$$

By Ito's lemma, the dynamics of new scaled process above is

$$dx(t) = \frac{a'(t)}{\sigma}(\log S_t)^2 dt + 2\frac{a(t)}{\sigma}\frac{1}{S_t}\log S_t dS_t$$

$$= \left(a(t)^2 + \left(2r - \sigma^2\right)\frac{a(t)}{\sigma}\sqrt{x(t)} + \left(\frac{2a'(t)}{a(t)} - 4a(t) \right) x(t) \right) dt + 2a(t)^2 \sqrt{x(t)} dW_t^M .$$

Integration by parts shows that

$$\int_t^T - x(u)\exp\left(\int_u^T x(s)ds\right)du = 1 - \exp\left(\int_u^T x(s)ds\right).$$

Taking a conditional expectation up to t under P^M and defining $h(x,t)$ as $E_t^M\left[\exp\left(\int_t^T x(s)ds\right)\right]$, we have an equation as follows:

$$h(x(t),t) = E_t^M\left[h(x(T),T)\right] - \int_t^T E_t^M\left[dh(x(s),s)\right].$$

A trial function is defined by

$$h(x(t),t) = \exp\left(A(\tau) + B(\tau)\sqrt{x(t)} + C(\tau)x(t)\right),$$

A system of simultaneous ordinary differential equation is derived below

$$A'(\tau) + \left(r - \frac{\sigma^2}{2}\right)\frac{a(t)}{\sigma}B(\tau) + a(t)^2 C(\tau) + \frac{a(t)^2}{2}B(\tau)^2 = 0,$$

$$-B'(\tau) + \frac{a'(t)}{2a(t)}B(\tau) - 2a(t)B(\tau) + \left(\frac{2r}{\sigma} - 1\right)a(t)C(\tau) + 2a(t)^2 B(\tau)C(\tau) = 0$$

$$-C'(\tau) + \left(\frac{a'(t)}{a(t)} - 4a(t)\right)C(\tau) + 2a(t)^2 C(\tau)^2 = 0.$$

Terminal conditions are $A(0) = B(0) = C(0) = 0$.

It is hardly to have a closed form solution since $a(t)$ is a time-varying function. By changing variables as $f(t) = a(t)^2 C(\tau)/\sigma^2$ and $g(t) = a(t)B(\tau)/\sigma$, a simpler system of ordinary differential equations is followed:

$$f'(t) = -\frac{a(t)^2}{\sigma^2} + 4a(t)f(t) - 2\sigma^2 f(\tau)^2, \qquad f(T) = 0$$

$$g'(t) = 2a(t)g(t) + \left(2r + \sigma^2\right)f(\tau) - 2\sigma f(t)g(t), \qquad g(T) = 0.$$

Since $A(0) = 0$,

$$A(T) = \int_0^T \sigma^2\left(\frac{r}{\sigma^2}g(t) - \frac{1}{2}g(t) + f(t) + \frac{1}{2}g(t)^2\right).$$

Using $x(0) = \log S_0 = 0$ and $E^I\left[\xi^2(T)\right] = h(x(0),0) = \exp(A(T))$, the result is held.□