Analytic Approximations for Valuing Ratchet Caps in the LIBOR Market Model

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Abstract

This paper provides two analytic approximation formulas for pricing ratchet caps in the LIBOR market model. The approximate values of a ratchet caplet are represented as sums of Black's (1976) regular caplet prices. So, these pricing formulas are extremely fast and easily implemented. The formulas can be easily extended to incorporate multiple factors. Illustrative numerical examples are provided and comparisons with results from Monte-Carlo implementation of the LIBOR market model are presented.

Keywords: LIBOR market model; Monte Carlo simulation; Ratchet Caps; Gauss-Hermite Integration.

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The LIBOR market model or the BGM model, developed by Brace, Gatarek, and Musiela (1997), Jamshidian (1997), and Miltersen, Sandmann, and Sondermann (1997), has become an increasingly popular model for pricing interest rate derivatives over recent years. The pricing of interest rate derivatives in the LIBOR market model has usually resorted to Monte Carlo or finite difference methods as studied in Hull and White (2000), Hunter, Jackel and Joshi (2001), Pietersz, Pelsser and van Regenmortel (2002), Kurbanmuradov, Sabelfeld and Schoenmakers (2002) and others. While these numerical methods are flexible and yield accurate prices, they are very time consuming and expensive to use. The purpose of this paper is to develop accurate and computationally efficient approximation methods for pricing and hedging interest rate derivatives in the LIBOR market model. Although this paper focuses only on the ratchet caplet but the main idea in this paper will be easily applicable to other interest rate derivatives.

In this paper we suggest two approximation methods for valuing ratchet caps in the LIBOR market model. The results are surprisingly simple and the approximate values are represented as weighted sums of Black's (1976) regular caplet prices. The weights are from Gauss-Hermite quadrature formula which can be easily calculated from standard numerical packages. So, our approximation methods are extremely fast and easily implemented.

The outline of this paper is as follows. Section I introduces the ratchet caplet. The developments of the first and second approximation methods are contained in Sections II and IV, respectively. Multi-factor extensions of two approximation methods are described in Sections III and V, respectively. Comparisons with results from Monte Carlo simulation are presented in Section VI. We conclude the paper in Section VII.

I. Ratchet Cap

One of the most interesting nonstandard caps is a ratchet cap. This is like a regular cap except that the cap rate equals the LIBOR rate at the previous reset date plus a spread. Define $t_0 = 0$ and consider a ratchet cap with reset dates t_1, t_2, ..., t_n and a final payment date t_{n+1}. Suppose that the principal is L and the spread is s. Define R_k as the interest rate for the period between time t_k and t_{k+1} observed at time t_k expressed with a compounding period of $\delta_k = t_{k+1} - t_k$. The ratchet cap leads to a payoff at time t_{k+1} $(k = 1,2,\ldots,n)$ of

$$L\delta_k \max[R_k - (R_{k-1} + s),0]$$

By considering a world that is forward risk neutral with respect to a zero-coupon bond maturing at time t_{k+1}, the equivalent martingale measure result of Harrison and Kreps (1979) and Harrison and Pliska (1981) says that the price of a ratchet caplet that provides a payoff at time t_{k+1} is

$$L\delta_k P(t,t_{k+1})E[\max[R_k - (R_{k-1} + s),0]]$$

where $P(t,t_{k+1})$ is the price at time t of a zero-coupon bond with principal \$1 maturing at time t_{k+1} and E denotes the expected value in a forward risk neutral world with respect to $P(t,t_{k+1})$. By the law of iterated expectation, we have

$$L\delta_k P(t,t_{k+1})E[E[\max[R_k - (R_{k-1} + s),0] \mid R_{k-1}]] \tag{1}$$

We note the fact that if we know R_{k-1} then the ratchet caplet is exactly a regular caplet with a cap rate of $(R_{k-1} + s)$. Thus, if the conditional distribution of the rate R_k, given R_{k-1}, is assumed to be lognormal, then the conditional expectation in equation (1) can be evaluated using Black's (1976) model. This paper gives two analytic approximation formulas for the equation (1) by differently approximating the probability density function of R_{k-1} in a forward risk neutral world with respect to $P(t,t_{k+1})$.

II. First Approximation Method

Define $F_k(t)$ as the forward interest rate as seen at time t for the period between t_k and t_{k+1} expressed with a compounding period of $\delta_k = t_{k+1} - t_k$ and $\varsigma_k(t)$ as the volatility of $F_k(t)$ at time t. The processes followed by $F_k(t)$ and $F_{k-1}(t)$ in the forward risk neutral world with respect to $P(t, t_{k+1})$ are

$$\frac{dF_k(t)}{F_k(t)} = \varsigma_k(t)dz \tag{2}$$

$$\frac{dF_{k-1}(t)}{F_{k-1}(t)} = \frac{-\delta_k F_k(t)\varsigma_k(t)\varsigma_{k-1}(t)}{1+\delta_k F_k(t)}dt + \varsigma_{k-1}(t)dz \tag{3}$$

where dz is a Wiener process.

It is usual to assume that $\varsigma_k(t) = \Lambda_{k-m(t)}$ where $m(t) = \min\{k : t \le t_k\}$ is the index for the next reset date at time t. This means that $\varsigma_k(t)$ is a step function only of the number of whole accrual periods between the next reset date and time t_k. The Λ_i can be obtained iteratively by

$$\sigma_k^2 t_k = \sum_{i=1}^{k}\left[\Lambda_{k-i}^2\delta_{i-1}\right] \tag{4}$$

where σ_k is the Black (1976) volatility for the regular caplet that corresponds to the period between times t_k and t_{k+1}. Expressed in terms of the Λ_i's equations (2) and (3) are, respectively,

$$\frac{dF_k(t)}{F_k(t)} = \Lambda_{k-m(t)}dz$$

$$\frac{dF_{k-1}(t)}{F_{k-1}(t)} = \frac{-\delta_k F_k(t)\Lambda_{k-m(t)}\Lambda_{k-1-m(t)}}{1+\delta_k F_k(t)}dt + \Lambda_{k-1-m(t)}dz$$

or

$$d\ln F_k(t) = -\frac{1}{2}\Lambda_{k-m(t)}^2 dt + \Lambda_{k-m(t)}dz \tag{5}$$

$$d\ln F_{k-1}(t) = \left(\frac{-\delta_k F_k(t)\Lambda_{k-m(t)}\Lambda_{k-1-m(t)}}{1+\delta_k F_k(t)} - \frac{1}{2}\Lambda_{k-1-m(t)}^2\right)dt + \Lambda_{k-1-m(t)}dz \tag{6}$$

If we assume in the drift of $\ln F_{k-1}(t)$ in equation (6) that $F_k(t) = F_k(t_0)$ for $t_0 < t < t_{k-1}$, then the drift remains constant within each accrual period and we have

$$\ln F_k(t_k) = \ln F_k(t_0) + \sum_{j=0}^{k-1}\left[-\frac{1}{2}\Lambda_{k-j-1}^2\delta_j + \Lambda_{k-j-1}\varepsilon_j\sqrt{\delta_j}\right] \tag{7}$$

$$\ln F_{k-1}(t_{k-1}) = \ln F_{k-1}(t_0) + \sum_{j=0}^{k-2}\left[\left(\frac{-\delta_k F_k(t_0)\Lambda_{k-j-1}\Lambda_{k-j-2}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\Lambda_{k-j-2}^2\right)\delta_j + \Lambda_{k-j-2}\varepsilon_j\sqrt{\delta_j}\right] \tag{8}$$

where ε_j $(0 \le j \le k-1)$ are random samples from a standard normal distribution.

Let $X = \ln F_k(t_k)$ and $Y = \ln F_{k-1}(t_{k-1})$. Then a two dimensional random variable (X,Y) has a bivariate normal distribution with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ_{XY} where

$$\mu_X = \ln F_k(t_0) + \sum_{j=0}^{k-1}\left[-\frac{1}{2}\Lambda_{k-j-1}^2\delta_j\right]$$

$$\mu_Y = \ln F_{k-1}(t_0) + \sum_{j=0}^{k-2}\left[\left(\frac{-\delta_k F_k(t_0)\Lambda_{k-j-1}\Lambda_{k-j-2}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\Lambda_{k-j-2}^2\right)\delta_j\right]$$

$$\sigma_X^2 = \sum_{j=0}^{k-1}\left[\Lambda_{k-j-1}^2\delta_j\right]$$

$$\sigma_Y^2 = \sum_{j=0}^{k-2}\left[\Lambda_{k-j-2}^2\delta_j\right]$$

$$\rho_{XY} = \frac{1}{\sigma_X\sigma_Y}\sum_{j=0}^{k-2}\Lambda_{k-j-1}\Lambda_{k-j-2}\delta_j$$

Using equation (4) and simplifying yields

$$\sigma_X^2 = \sigma_k^2 t_k$$

$$\sigma_Y^2 = \sigma_{k-1}^2 t_{k-1}$$

$$\mu_X = \ln F_k(t_0) - \frac{1}{2}\sigma_X^2$$

$$\mu_Y = \ln F_{k-1}(t_0) - \frac{\delta_k F_k(t_0)\rho_{XY}\sigma_X\sigma_Y}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sigma_Y^2$$

Therefore, the expectation in equation (1) is

$$E\left[E\left[\max\left[F_k(t_k)-F_{k-1}(t_{k-1})-s,0\right]\mid F_{k-1}(t_{k-1})\right]\right]=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\max(e^x-e^y-s,0)f(x\mid y)dx\right\}f_Y(y)dy \qquad (9)$$

where the conditional probability density function $f(x\mid y)$ of X given $Y=y$ is normal with mean $\mu_X+\rho_{XY}\dfrac{\sigma_X}{\sigma_Y}(y-\mu_Y)$ and variance $\sigma_X^2(1-\rho_{XY}^2)$ and the marginal probability density function $f_Y(y)$ is normal with mean μ_Y and variance σ_Y^2. The inner integral in equation (9) can be explicitly evaluated as

$$\int_{-\infty}^{+\infty}\max(e^x-e^y-s,0)f(x\mid y)dx=E(e^X\mid y)N(d_1)-KN(d_2)$$

where $N(\bullet)$ is the standard normal distribution function and

$$d_1=\frac{\ln\left[E(e^X\mid y)/K\right]+\sigma_X^2(1-\rho_{XY}^2)/2}{\sigma_X\sqrt{1-\rho_{XY}^2}}$$

$$d_2=\frac{\ln\left[E(e^X\mid y)/K\right]-\sigma_X^2(1-\rho_{XY}^2)/2}{\sigma_X\sqrt{1-\rho_{XY}^2}}=d_1-\sigma_X\sqrt{1-\rho_{XY}^2}$$

$$E(e^X\mid y)=\exp\left[\mu_X+\rho_{XY}\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)+\frac{\sigma_X^2}{2}(1-\rho_{XY}^2)\right]$$

$$=F_k(t_0)\exp\left[\rho_{XY}\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)-\frac{1}{2}\sigma_X^2\rho_{XY}^2\right]$$

$$K=e^y+s$$

so that

$$E\left[E\left[\max\left[F_k(t_k)-F_{k-1}(t_{k-1})-s,0\right]\mid F_{k-1}(t_{k-1})\right]\right]=\int_{-\infty}^{+\infty}\left\{E(e^X\mid y)N(d_1)-KN(d_2)\right\}f_Y(y)dy$$

Note that $E(e^X\mid y)$ is the conditional expectation of $F_k(t_k)$ given $F_{k-1}(t_{k-1})$. If we change the variable of integration by writing

$$z=\frac{1}{\sqrt{2}}\frac{y-\mu_Y}{\sigma_Y}$$

then

$$E\left[E\left[\max\left[F_k(t_k) - F_{k-1}(t_{k-1}) - s, 0\right] \mid F_{k-1}(t_{k-1})\right]\right] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} \left[E(e^X \mid z)N(d_1) - KN(d_2)\right] e^{-z^2} dz \tag{10}$$

where

$$d_1 = \frac{\ln\left[E(e^X \mid z)/K\right] + \sigma_X^2 (1 - \rho_{XY}^2)/2}{\sigma_X \sqrt{1 - \rho_{XY}^2}}$$

$$d_2 = d_1 - \sigma_X \sqrt{1 - \rho_{XY}^2}$$

$$E(e^X \mid z) = F_k(t_0) \exp\left[\sqrt{2} \rho_{XY} \sigma_X z - \frac{1}{2} \sigma_X^2 \rho_{XY}^2\right]$$

$$K = \exp(\mu_Y + \sqrt{2} \sigma_Y z) + s$$

The integral in equation (10) can be computed using various standard numerical integration techniques. The one best suited to evaluating the integral in equation (10) is Gauss-Hermite quadrature. By using N-point Gauss-Hermite quadrature formula, the approximated value, $\hat{c}_k$, of a ratchet caplet that provides a payoff at time t_{k+1} is

$$\hat{c}_k = L\delta_k P(t, t_{k+1}) \int_{-\infty}^{+\infty} g(z) e^{-z^2} dz \approx L\delta_k P(t, t_{k+1}) \sum_{i=1}^{N} w_i g(z_i) \tag{11}$$

where

$$g(z) = \frac{1}{\sqrt{\pi}} \left[E(e^X \mid z)N(d_1) - KN(d_2)\right]$$

and d_1, d_2, $E(e^X \mid z)$ and K are as defined in equation (10). The weights w_i $(1 \le i \le N)$ and abscissas z_i $(1 \le i \le N)$ of the Gauss-Hermite quadrature formula can be calculated using GAUHER function in the Numerical Recipes program given by Press *et al* (1992). Note that the calculation of one ratchet caplet prices using N-point Gauss-Hermite quadrature formula with equation (11) requires N evaluations of Black's (1976) formula.

III. First Approximation Method with Several Factors

The first approximation method in the previous section can be easily extended to incorporate several independent factors. Suppose that there are p factors and $\varsigma_{k,q}(t)$ is the component of the volatility of $F_k(t)$ attributable to the qth factor. The processes followed by $F_k(t)$ and $F_{k-1}(t)$ in the forward risk neutral world with respect to $P(t,t_{k+1})$ are

$$\frac{dF_k(t)}{F_k(t)} = \sum_{q=1}^{p} \varsigma_{k,q}(t)dz_q \tag{12}$$

$$\frac{dF_{k-1}(t)}{F_{k-1}(t)} = \frac{-\delta_k F_k(t)\sum_{q=1}^{p}\varsigma_{k,q}(t)\varsigma_{k-1,q}(t)}{1+\delta_k F_k(t)}dt + \sum_{q=1}^{p}\varsigma_{k-1,q}(t)dz_q \tag{13}$$

where dz_q $(1 \le q \le p)$ are independent Wiener processes.

Define $\lambda_{i,q}$ as the qth component of the volatility when there are i accrual periods between the next reset date and time t_k. The $\lambda_{i,q}$'s and Λ_i's are related with

$$\Lambda_i^2 = \sum_{q=1}^{p}\lambda_{i,q}^2 \tag{14}$$

We approximate the drift in equation (13) by setting $F_k(t) = F_k(t_0)$ for $t_0 < t < t_{k-1}$. Then the drift remains constant within each accrual period and we have

$$\ln F_k(t_k) = \ln F_k(t_0) + \sum_{j=0}^{k-1}\left[-\frac{1}{2}\sum_{q=1}^{p}\lambda_{k-j-1,q}^2\delta_j + \sum_{q=1}^{p}\lambda_{k-j-1,q}\varepsilon_{j,q}\sqrt{\delta_j}\right] \tag{15}$$

$$\ln F_{k-1}(t_{k-1}) = \ln F_{k-1}(t_0) + \sum_{j=0}^{k-2}\left[\left(\frac{-\delta_k F_k(t_0)\sum_{q=1}^{p}\lambda_{k-j-1,q}\lambda_{k-j-2,q}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sum_{q=1}^{p}\lambda_{k-j-2,q}^2\right)\delta_j + \sum_{q=1}^{p}\lambda_{k-j-2,q}\varepsilon_{j,q}\sqrt{\delta_j}\right] \tag{16}$$

where $\varepsilon_{j,q}$ $(0 \le j \le k-1,\ 1 \le q \le p)$ are random samples from a standard normal distribution.

Let $X = \ln F_k(t_k)$ and $Y = \ln F_{k-1}(t_{k-1})$. Then a two dimensional random variable (X,Y) has a bivariate normal distribution with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ_{XY} where

$$\mu_X = \ln F_k(t_0) + \sum_{j=0}^{k-1}\left[-\frac{1}{2}\sum_{q=1}^{p}\lambda_{k-j-1,q}^2\delta_j\right]$$

$$\mu_Y = \ln F_{k-1}(t_0) + \sum_{j=0}^{k-2}\left[\left(\frac{-\delta_k F_k(t_0)\sum_{q=1}^{p}\lambda_{k-j-1,q}\lambda_{k-j-2,q}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sum_{q=1}^{p}\lambda_{k-j-2,q}^2\right)\delta_j\right]$$

$$\sigma_X^2 = \sum_{j=0}^{k-1}\left[\sum_{q=1}^{p}\lambda_{k-j-1,q}^2\delta_j\right] = \sum_{j=0}^{k-1}\left[\Lambda_{k-j-1}^2\delta_j\right] = \sigma_k^2 t_k$$

$$\sigma_Y^2 = \sum_{j=0}^{k-2}\left[\sum_{q=1}^{p}\lambda_{k-j-2,q}^2\delta_j\right] = \sum_{j=0}^{k-2}\left[\Lambda_{k-j-2}^2\delta_j\right] = \sigma_{k-1}^2 t_{k-1}$$

$$\rho_{XY} = \frac{1}{\sigma_X\sigma_Y}\sum_{j=0}^{k-2}\left[\sum_{q=1}^{p}\lambda_{k-j-1,q}\lambda_{k-j-2,q}\delta_j\right]$$

The means μ_X and μ_Y are simplified as

$$\mu_X = \ln F_k(t_0) - \frac{1}{2}\sigma_X^2$$

$$\mu_Y = \ln F_{k-1}(t_0) - \frac{\delta_k F_k(t_0)\rho_{XY}\sigma_X\sigma_Y}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sigma_Y^2$$

Note that σ_X^2, σ_Y^2, and μ_X are independent of the number of factors whereas ρ_{XY} and μ_Y depend on the number of factors. The remaining analysis is same as that in the previous section. Therefore, the approximated value of a ratchet caplet with several factors can be calculated using equation (11) with the adjustments only of ρ_{XY} and μ_Y.

IV. Second Approximation Method

For a more accurate approximation, we reconsider the drift of $\ln F_{k-1}(t)$ in equation (6). If we assume in the drift of $\ln F_{k-1}(t)$ in equation (6) that $F_k(t) = F_k(t_0)$ for $t_0 < t < t_1$ and $F_k(t) = F_k(t_1)$ for $t_1 < t < t_{k-1}$, then equation (8) becomes

$$\ln F_{k-1}(t_{k-1}) = \ln F_{k-1}(t_0) + \left(\frac{-\delta_k F_k(t_0)\Lambda_{k-1}\Lambda_{k-2}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\Lambda_{k-2}^2 \right)\delta_0 + \Lambda_{k-2}\varepsilon_0\sqrt{\delta_0}$$

$$+ \sum_{j=1}^{k-2}\left[\left(\frac{-\delta_k F_k(t_1)\Lambda_{k-j-1}\Lambda_{k-j-2}}{1+\delta_k F_k(t_1)} - \frac{1}{2}\Lambda_{k-j-2}^2 \right)\delta_j + \Lambda_{k-j-2}\varepsilon_j\sqrt{\delta_j} \right]$$

where

$$\ln F_k(t_1) = \ln F_k(t_0) + \left[-\frac{1}{2}\Lambda_{k-1}^2\delta_0 + \Lambda_{k-1}\varepsilon_0\sqrt{\delta_0} \right]$$

and ε_j $(0 \le j \le k-1)$ are random samples from a standard normal distribution.

Let $X = \ln F_k(t_k)$, $Y = \ln F_{k-1}(t_{k-1})$ and $A = \ln F_k(t_1)$. We note that the conditional distribution of Y given A is normal. Thus a two dimensional random variable (X,Y) given $A = a$ has a bivariate normal distribution with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ_{XY} where

$$\mu_X = a + \sum_{j=1}^{k-1}\left[-\frac{1}{2}\Lambda_{k-j-1}^2\delta_j \right]$$

$$\mu_Y = b + \sum_{j=1}^{k-2}\left[\left(\frac{-\delta_k e^a \Lambda_{k-j-1}\Lambda_{k-j-2}}{1+\delta_k e^a} - \frac{1}{2}\Lambda_{k-j-2}^2 \right)\delta_j \right]$$

$$\sigma_X^2 = \sum_{j=1}^{k-1}\left[\Lambda_{k-j-1}^2\delta_j \right]$$

$$\sigma_Y^2 = \sum_{j=1}^{k-2}\left[\Lambda_{k-j-2}^2\delta_j \right]$$

$$\rho_{XY} = \frac{1}{\sigma_X\sigma_Y}\sum_{j=1}^{k-2}\Lambda_{k-j-1}\Lambda_{k-j-2}\delta_j$$

$$b = \ln F_{k-1}(t_0) + \left(\frac{-\delta_k F_k(t_0)\Lambda_{k-1}\Lambda_{k-2}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\Lambda_{k-2}^2 \right)\delta_0 + \frac{\Lambda_{k-2}}{\Lambda_{k-1}}\left(a - \ln F_k(t_0) + \frac{1}{2}\Lambda_{k-1}^2\delta_0 \right)$$

The variable b is the value of $\ln F_{k-1}(t_1)$ given $A = a$. Using equation (4) and simplifying yields

$$\sigma_X^2 = \sigma_k^2 t_k - \Lambda_{k-1}^2 \delta_0$$

$$\sigma_Y^2 = \sigma_{k-1}^2 t_{k-1} - \Lambda_{k-2}^2 \delta_0$$

$$\mu_X = a - \frac{1}{2}\sigma_X^2$$

$$\mu_Y = b - \frac{\delta_k e^a \rho_{XY}\sigma_X\sigma_Y}{1+\delta_k e^a} - \frac{1}{2}\sigma_Y^2$$

Therefore, the expectation in equation (1) is

$$E\big[E[\max[F_k(t_k) - F_{k-1}(t_{k-1}) - s),0] \mid F_{k-1}(t_{k-1}), F_k(t_1)]\big]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{ \int_{-\infty}^{+\infty} \max(e^x - e^y - s,0) f(x \mid y,a)dx \right\} f(y \mid a) f_A(a) dyda \tag{17}$$

where the first conditional probability density function $f(x \mid y,a)$ of X given $Y = y$ and $A = a$ is normal with mean $\mu_X + \rho_{XY}\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$ and variance $\sigma_X^2(1-\rho_{XY}^2)$, the second conditional probability density function $f(y \mid a)$ is normal with mean μ_Y and variance σ_Y^2, and the marginal density function $f_A(a)$ is normal with mean $\mu_A = \ln F_k(t_0) - \frac{1}{2}\sigma_A^2$ and variance $\sigma_A^2 = \Lambda_{k-1}^2\delta_0$.

The inner integral in equation (17) can be explicitly evaluated as

$$\int_{-\infty}^{+\infty} \max(e^x - e^y - s,0) f(x \mid y,a)dx = E(e^X \mid y,a)N(d_1) - KN(d_2)$$

where $N(\bullet)$ is the standard normal distribution function and

$$d_1 = \frac{\ln\left[E(e^X \mid y,a)/K\right] + \sigma_X^2(1-\rho_{XY}^2)/2}{\sigma_X\sqrt{1-\rho_{XY}^2}}$$

$$d_2 = d_1 - \sigma_X\sqrt{1-\rho_{XY}^2}$$

$$E(e^X \mid y,a) = \exp\left[\mu_X + \rho_{XY}\frac{\sigma_X}{\sigma_Y}(y-\mu_Y) + \frac{\sigma_X^2}{2}(1-\rho_{XY}^2)\right]$$

$$= e^{a} \exp\left[\rho_{XY} \frac{\sigma_X}{\sigma_Y}(y - \mu_Y) - \frac{1}{2}\sigma_X^2 \rho_{XY}^2\right]$$

$$K = e^{y} + s$$

so that

$$E\left[E\left[\max\left[F_k(t_k) - F_{k-1}(t_{k-1}) - s, 0\right] \mid F_{k-1}(t_{k-1}), F_k(t_1)\right]\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{E(e^{X} \mid y, a)N(d_1) - KN(d_2)\right\} f(y \mid a) f_A(a)\,dy\,da$$

If we change the variables of integration by writing

$$z = \frac{1}{\sqrt{2}}\frac{y - \mu_Y}{\sigma_Y} \qquad \text{and} \qquad h = \frac{1}{\sqrt{2}}\frac{a - \mu_A}{\sigma_A}$$

then

$$E\left[E\left[\max\left[F_k(t_k) - F_{k-1}(t_{k-1}) - s, 0\right] \mid F_{k-1}(t_{k-1}), F_k(t_1)\right]\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\pi}\left[E(e^{X} \mid z, h)N(d_1) - KN(d_2)\right] e^{-z^2} e^{-h^2}\,dz\,dh \tag{18}$$

where

$$d_1 = \frac{\ln\left[E(e^{X} \mid z, h)/K\right] + \sigma_X^2(1 - \rho_{XY}^2)/2}{\sigma_X\sqrt{1 - \rho_{XY}^2}}$$

$$d_2 = d_1 - \sigma_X\sqrt{1 - \rho_{XY}^2}$$

$$E(e^{X} \mid z, h) = \exp(\mu_A + \sqrt{2}\sigma_A h)\exp\left[\sqrt{2}\rho_{XY}\sigma_X z - \frac{1}{2}\sigma_X^2\rho_{XY}^2\right]$$

$$K = \exp(\mu_Y + \sqrt{2}\sigma_Y z) + s$$

The one best suited to evaluating the integral in equation (18) is Gauss-Hermite quadrature. By using N-point Gauss-Hermite quadrature formula, the approximated value, $\hat{c}_k$, of a ratchet caplet that provides a payoff at time t_{k+1} is

$$\hat{c}_k = L\delta_k P(t, t_{k+1})\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(z, h)e^{-z^2} e^{-h^2}\,dz\,dh \approx L\delta_k P(t, t_{k+1})\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j g(z_i, h_j) \tag{19}$$

where

$$g(z,h) = \frac{1}{\pi}\left[E(e^X \mid z,h)N(d_1) - KN(d_2)\right]$$

and d_1, d_2, $E(e^X \mid z,h)$ and K are as defined in equation (18). The weights w_i $(1 \le i \le N)$ and abscissas z_i, h_i $(1 \le i \le N)$ of the Gauss-Hermite quadrature formula can be calculated using GAUHER function in the Numerical Recipes program. Note that the calculation of one ratchet caplet prices using N-point Gauss-Hermite quadrature formula with equation (19) requires N^2 evaluations of Black's (1976) formula.

V. Second Approximation Method with Several Factors

The second approximation method in the previous section can also be easily extended to incorporate several independent factors. If we approximate the drift in equation (13) by setting $F_k(t) = F_k(t_0)$ for $t_0 < t < t_1$ and $F_k(t) = F_k(t_1)$ for $t_1 < t < t_{k-1}$, then equation (16) becomes

$$\ln F_{k-1}(t_{k-1}) = \ln F_{k-1}(t_0) + \left(\frac{-\delta_k F_k(t_0)\sum_{q=1}^{p}\lambda_{k-1,q}\lambda_{k-2,q}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sum_{q=1}^{p}\lambda_{k-2,q}^2 \right)\delta_0 + \sum_{q=1}^{p}\lambda_{k-2,q}\varepsilon_{0,q}\sqrt{\delta_0}$$

$$+ \sum_{j=1}^{k-2}\left[\left(\frac{-\delta_k F_k(t_1)\sum_{q=1}^{p}\lambda_{k-j-1,q}\lambda_{k-j-2,q}}{1+\delta_k F_k(t_1)} - \frac{1}{2}\sum_{q=1}^{p}\lambda_{k-j-2,q}^2 \right)\delta_j + \sum_{q=1}^{p}\lambda_{k-j-2,q}\varepsilon_{j,q}\sqrt{\delta_j}\right]$$

where

$$\ln F_k(t_1) = \ln F_k(t_0) - \frac{1}{2}\sum_{q=1}^{p}\lambda_{k-1,q}^2\delta_0 + \sum_{q=1}^{p}\lambda_{k-1,q}\varepsilon_{0,q}\sqrt{\delta_0}$$

and $\varepsilon_{j,q}$ $(0 \le j \le k-1,\ 1 \le q \le p)$ are random samples from a standard normal distribution.

Let $X = \ln F_k(t_k)$, $Y = \ln F_{k-1}(t_{k-1})$, $A = \ln F_k(t_1)$, and $B = \ln F_{k-1}(t_1)$. Then a two dimensional random variable (A,B) has a bivariate normal distribution with means μ_A and μ_B, variances σ_A^2

and σ_B^2, and correlation coefficient ρ_{AB} where

$$\mu_A = \ln F_k(t_0) - \frac{1}{2}\sum_{q=1}^{p} \lambda_{k-1,q}^2 \delta_0$$

$$\mu_B = \ln F_{k-1}(t_0) + \left(\frac{-\delta_k F_k(t_0) \sum_{q=1}^{p} \lambda_{k-1,q}\lambda_{k-2,q}}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sum_{q=1}^{p} \lambda_{k-2,q}^2 \right)\delta_0$$

$$\sigma_A^2 = \sum_{q=1}^{p} \lambda_{k-1,q}^2 \delta_0$$

$$\sigma_B^2 = \sum_{q=1}^{p} \lambda_{k-2,q}^2 \delta_0$$

$$\rho_{AB} = \frac{1}{\sigma_A \sigma_B} \sum_{q=1}^{p} \lambda_{k-1,q}\lambda_{k-2,q}\delta_0$$

The means μ_A and μ_B are simplified as

$$\mu_A = \ln F_k(t_0) - \frac{1}{2}\sigma_A^2$$

$$\mu_B = \ln F_{k-1}(t_0) - \frac{\delta_k F_k(t_0)\rho_{AB}\sigma_A\sigma_B}{1+\delta_k F_k(t_0)} - \frac{1}{2}\sigma_B^2$$

And a two dimensional random variable (X,Y) given $A = a$ has a bivariate normal distribution with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ_{XY} where

$$\mu_X = a + \sum_{j=1}^{k-1}\left[-\frac{1}{2}\sum_{q=1}^{p} \lambda_{k-j-1,q}^2 \delta_j \right]$$

$$\mu_Y = E[B \mid A = a] + \sum_{j=1}^{k-2}\left[\left(\frac{-\delta_k e^a}{1+\delta_k e^a} \sum_{q=1}^{p} \lambda_{k-j-1,q}\lambda_{k-j-2,q} - \frac{1}{2}\sum_{q=1}^{p} \lambda_{k-j-2,q}^2 \right)\delta_j \right]$$

$$\sigma_X^2 = \sum_{j=1}^{k-1}\left[\sum_{q=1}^{p} \lambda_{k-j-1,q}^2 \delta_j \right]$$

$$\sigma_Y^2 = \sum_{j=1}^{k-2}\left[\sum_{q=1}^{p} \lambda_{k-j-2,q}^2 \delta_j \right] + Var[B \mid A = a]$$

$$\rho_{XY} = \frac{1}{\sigma_X \sigma_Y} \sum_{j=1}^{k-2} \left[\sum_{q=1}^{p} \lambda_{k-j-1,q} \lambda_{k-j-2,q} \delta_j \right]$$

$$E[B \mid A = a] = \mu_B + \rho_{AB} \frac{\sigma_B}{\sigma_A} (a - \mu_A)$$

$$Var[B \mid A = a] = \sigma_B^2 (1 - \rho_{AB}^2)$$

The means μ_X and μ_Y are simplified as

$$\mu_X = a - \frac{1}{2} \sigma_X^2$$

$$\mu_Y = E[B \mid A = a] + \frac{1}{2} Var[B \mid A = a] - \frac{\delta_k e^a \rho_{XY} \sigma_X \sigma_Y}{1 + \delta_k e^a} - \frac{1}{2} \sigma_Y^2$$

Note that σ_X^2 and μ_X are independent of the number of factors whereas ρ_{XY}, σ_Y^2, and μ_Y depend on the number of factors. The remaining analysis is same as that in the previous section. Therefore, the approximated value of a ratchet caplet with several factors can be calculated using equation (19) with the adjustments only of ρ_{XY}, σ_Y^2, and μ_Y.

VI. Numerical Results

To validate the accuracy of our approximation methods, a comparison was conducted with the data originally given by Hull [(2002), p.579 – 583]. The principal is \$100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is assumed to be flat at 5% per annum with continuous compounding. The caplet volatilities are humped as in Table I. The arithmetic average of the volatilities between one and ten years is 16.71%. Table I also shows volatility components in two- and three-factor models. For the practical usage of our approximation methods, one must decide on the size of N in the Gauss-Hermite integration. The convergences of our two approximation methods are illustrated in Table II. The results are for the tenth ratchet caplet that provides a payoff at time 11 years. Similar results are obtained for other caplets, other term structures, and other volatility structures. Table II shows that the convergences of our two approximation methods are extremely fast. More experiments show that 6-point Gauss-Hermite integration is sufficient to obtain convergence to the sixth decimal place for the most cases. We choose Hull and White's (2000) Monte Carlo implementation in the rolling forward risk neutral world as our benchmark for the true values. The rolling forward risk neutral world is a world that is always forward risk neutral with respect to a bond maturing at the next reset date. The true values are based on 500,000 Monte Carlo simulations incorporating the antithetic variable technique. To simulate the path followed by the forward rates more precisely, we divide each accrual period into 8 small time intervals. The error measure that we report is root mean squared relative error (RMSE). RMSE is defined by

$$RMSE = \sqrt{\frac{1}{n-1}\sum_{k=2}^{n} e_k^2} \qquad \text{where} \qquad e_k = \frac{\hat{c}_k - c_k}{c_k}$$

is the relative error, c_k is the "true" value of the ratchet caplet that provides a payoff at time t_{k+1} calculated from Monte Carlo simulation, and $\hat{c}_k$ is the approximated ratchet caplet value.

Table III reports results of Monte Carlo simulation and our two approximation methods for one-, two-, and three-factor models. Two approximated values are computed using 6-point Gauss-Hermite quadrature formula with equations (11) and (19), respectively. In table III, we first notice that the ratchet caplet prices of Monte Carlo and our approximation methods are very close, suggesting the usefulness of our approximation methods. Table III also shows that, as we expect, the second approximation method is more accurate than the first approximation method. The most dramatic comparison is that of computing times. In Monte Carlo simulation, total computing times for pricing 9 ratchet caplets on a 1.8 GHz Pentium PC are 282, 427, and 485 seconds for the one-, two-, and three-factor models, respectively. The computing times of our two approximation methods are less than hundredth of one second. If we use 100-point Gauss-Hermite quadrature formula rather than 6-point, then total computing times for pricing 9 ratchet caplets are 0.003 and 0.312 seconds for the first and second approximation methods, respectively. Note that the computing times of our two approximation methods are independent of the number of factors.

Tables IV through VI contain a sensitivity analysis of the ratchet caplet values for a range of parameter values. Table IV shows the valuation when the term structure is flat at 20%. Table V shows the valuation when the average volatility is 30%. Table VI shows the valuation when the term structure is flat at 20% and the average volatility is 30%. The general observations drawn from Table III also hold here. Figure 1 graphs RMS errors of our two approximation methods with different values of the interest rate. Figure 2 graphs RMS errors of our two approximation methods with different values of the average volatility. We can see that pricing errors are insensitive for the interest rates and slightly larger errors occur when the volatilities are high.

VII. Conclusion

In this paper, we have presented two analytic approximation formulas for pricing ratchet caps in the LIBOR market model. The results are surprisingly simple and the approximate values of a ratchet caplet are represented as weighted sums of Black's (1976) regular caplet prices. The weights are from Gauss-Hermite quadrature formula which can be easily calculated from standard numerical packages. So, these approximation formulas are extremely fast and easily implemented. The formulas can be easily extended to incorporate multiple factors. Comparisons with results from Monte-Carlo simulation show that, for the volatility and interest rate environments that are typically encountered in the market, our approximation methods give very accurate results. Although this paper focuses only on the ratchet caplet but the main idea in this paper will be easily applicable to other interest rate derivatives.

REFERENCES

Black, F., 1976, "The Pricing of Commodity Contracts," *Journal of Financial Economics* 3, 167-179.

Brace, A., D. Gatarek, and M. Musiela, 1997, "The Market Model of Interest Rate Dynamics," *Mathematical Finance* 7, 127-155.

Harrison, J. M. and D. M. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Market," *Journal of Economic Theory* 20, 381-408.

Harrison, J. M. and S. R. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications* 11, 215-260.

Hull, J., 2002, *Options, Futures, and Other Derivatives* 5th edition, Prentice-Hall.

Hull, J. and A. White, 2000, "Forward Rate Volatilities, Swap Rate Volatilities, and Implementation of the LIBOR Market Model," *Journal of Fixed Income* 10(2), 46-62.

Hunter, C.J., P. Jackel, and M.S. Joshi, 2001, "Drift Approximations in a Forward-Rate-Based LIBOR Market Model," Working Paper.

Jamshidian, F., 1997, "LIBOR and Swap Market Models and Measures," *Finance and Stochastics* 1, 293-330.

Kurbanmuradov, O., K, Sabelfeld, and J. Schoenmakers, 2002, "Lognormal Approximations to LIBOR Market Models," *Journal of Computational Finance* 6, 69-100.

Miltersen, K., K. Sandmann, and D. Sondermann, 1997, "Closed Form Solutions for Term Structure Derivatives with Lognormal Interest Rates," *Journal of Finance* 52, 409-430.

Pietersz, R., A. Pelsser, and M. van Regenmortel, 2002, "Fast Drift Approximated Pricing in the BGM Model," Working Paper.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, 1992, *Numerical Recipes in C: The Art of Scientific Computing* 2nd edition, Cambridge University Press.

Table I: Volatility Data

The "humped" volatility data originally given by Hull [(2002), p.579 – 583]. The arithmetic average of the volatilities between one and ten years is 16.71%.

Year (k)	σ_k (%)	One factor	Two factors		Three factors		
		Λ_{k-1} (%)	$\lambda_{k-1,1}$ (%)	$\lambda_{k-1,2}$ (%)	$\lambda_{k-1,1}$ (%)	$\lambda_{k-1,2}$ (%)	$\lambda_{k-1,3}$ (%)
1	15.50	15.50	14.10	-6.45	13.65	-6.62	3.19
2	18.25	20.64	19.52	-6.70	19.28	-7.02	2.25
3	17.91	17.21	16.78	-3.84	16.72	-4.06	0.00
4	17.74	17.22	17.11	-1.96	16.98	-2.06	-1.98
5	17.27	15.25	15.25	0.00	14.85	0.00	-3.47
6	16.79	14.15	14.06	1.61	13.95	1.69	-1.63
7	16.30	12.98	12.65	2.89	12.61	3.06	0.00
8	16.01	13.81	13.06	4.48	12.90	4.70	1.51
9	15.76	13.60	12.36	5.65	11.97	5.81	2.80
10	15.54	13.40	11.63	6.65	10.97	6.66	3.84

Table II: Convergence of Gauss-Hermite Integration

The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat at 5% per annum with continuous compounding and the caplet volatilities are as in Table I. The approximated values are for the tenth ratchet caplet that provides a payoff at time 11 years.

N	One factor		Two factors		Three factors	
	A1[1)]	A2[2)]	A1[1)]	A2[2)]	A1[1)]	A2[2)]
1	0.135494	0.135405	0.143924	0.144066	0.153344	0.153334
2	0.151293	0.151573	0.159000	0.159512	0.167570	0.167961
3	0.151651	0.151871	0.159303	0.159755	0.167815	0.168150
4	0.151653	0.151872	0.159304	0.159755	0.167815	0.168150
5	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
6	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
7	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
8	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
9	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
10	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
20	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
50	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149
100	0.151653	0.151872	0.159304	0.159755	0.167815	0.168149

(1) First approximated values are computed using N-point Gauss-Hermite quadrature formula with equation (11)

(2) Second approximated values are computed using N-point Gauss-Hermite quadrature formula with equation (19).

Table III: Valuation of Ratchet Caplets (Average Rate = 5%, Average Volatility = 16.71%)

The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat at 5% per annum with continuous compounding and the caplet volatilities are as in Table I. RMSE is the root mean squared relative error.

Caplet start time	One factor			Two factors			Three factors		
(years)	True[1)]	A1[2)]	A2[3)]	True[1)]	A1[2)]	A2[3)]	True[1)]	A1[2)]	A2[3)]
2	0.205	0.206	0.206	0.208	0.207	0.207	0.210	0.209	0.209
3	0.202	0.201	0.201	0.205	0.204	0.205	0.208	0.208	0.208
4	0.194	0.194	0.194	0.198	0.198	0.199	0.204	0.204	0.204
5	0.186	0.187	0.187	0.192	0.192	0.193	0.199	0.199	0.199
6	0.179	0.179	0.179	0.186	0.185	0.186	0.194	0.193	0.193
7	0.173	0.172	0.172	0.180	0.178	0.179	0.187	0.187	0.187
8	0.166	0.165	0.165	0.174	0.172	0.173	0.182	0.181	0.181
9	0.160	0.158	0.159	0.167	0.166	0.166	0.176	0.174	0.175
10	0.153	0.152	0.152	0.160	0.159	0.160	0.169	0.168	0.168
RMSE		0.005291	0.004555		0.005358	0.003153		0.005246	0.004029

(1) The true values are based on 500,000 Monte Carlo simulations incorporating the antithetic variable technique. Each accrual period is divided into 8 equal subintervals.

(2) First approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (11).

(3) Second approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (19).

Table IV: Valuation of Ratchet Caplets (Average Rate = 20%, Average Volatility = 16.71%)

The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat at 20% per annum with continuous compounding and the caplet volatilities are as in Table I. RMSE is the root mean squared relative error.

Caplet start time (years)	One factor			Two factors			Three factors		
	True(1)	A1(2)	A2(3)	True(1)	A1(2)	A2(3)	True(1)	A1(2)	A2(3)
2	0.763	0.762	0.762	0.768	0.767	0.767	0.774	0.771	0.771
3	0.667	0.666	0.666	0.675	0.673	0.674	0.682	0.682	0.683
4	0.567	0.566	0.567	0.576	0.575	0.577	0.588	0.586	0.587
5	0.482	0.481	0.481	0.492	0.491	0.492	0.504	0.502	0.503
6	0.406	0.405	0.405	0.417	0.414	0.415	0.430	0.425	0.426
7	0.342	0.339	0.339	0.351	0.348	0.349	0.361	0.358	0.359
8	0.287	0.284	0.285	0.296	0.292	0.293	0.305	0.301	0.302
9	0.241	0.238	0.238	0.248	0.245	0.246	0.256	0.253	0.254
10	0.202	0.199	0.199	0.208	0.205	0.206	0.216	0.212	0.212
RMSE		0.008817	0.007054		0.009127	0.006287		0.009498	0.007420

(1) The true values are based on 500,000 Monte Carlo simulations incorporating the antithetic variable technique. Each accrual period is divided into 8 equal subintervals.

(2) First approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (11).

(3) Second approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (19).

Table V: Valuation of Ratchet Caplets (Average Rate = 5%, Average Volatility = 30%)

The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat at 5% per annum with continuous compounding and the average volatility is 30% with a humped volatility structure similar to that of Table I. RMSE is the root mean squared relative error.

Caplet start time	One factor			Two factors			Three factors		
(years)	True[1)]	A1[2)]	A2[3)]	True[1)]	A1[2)]	A2[3)]	True[1)]	A1[2)]	A2[3)]
2	0.443	0.442	0.442	0.446	0.445	0.445	0.450	0.448	0.448
3	0.437	0.435	0.436	0.443	0.441	0.442	0.449	0.448	0.448
4	0.425	0.422	0.424	0.433	0.430	0.432	0.444	0.440	0.442
5	0.412	0.409	0.410	0.423	0.419	0.421	0.436	0.431	0.433
6	0.400	0.394	0.396	0.412	0.405	0.408	0.427	0.419	0.421
7	0.388	0.379	0.381	0.401	0.391	0.394	0.414	0.406	0.408
8	0.377	0.366	0.369	0.391	0.379	0.382	0.405	0.394	0.397
9	0.366	0.352	0.354	0.378	0.366	0.368	0.394	0.381	0.383
10	0.356	0.339	0.341	0.368	0.352	0.355	0.384	0.368	0.370
RMSE		0.024556	0.020316		0.023326	0.017898		0.022780	0.018347

(1) The true values are based on 500,000 Monte Carlo simulations incorporating the antithetic variable technique. Each accrual period is divided into 8 equal subintervals.

(2) First approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (11).

(3) Second approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (19).

Table VI: Valuation of Ratchet Caplets (Average Rate = 20%, Average Volatility = 30%)

The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat at 20% per annum with continuous compounding and the average volatility is 30% with a humped volatility structure similar to that of Table I. RMSE is the root mean squared relative error.

Caplet start time	**One factor**			**Two factors**			**Three factors**		
(years)	**True[1)]**	**A1[2)]**	**A2[3)]**	**True[1)]**	**A1[2)]**	**A2[3)]**	**True[1)]**	**A1[2)]**	**A2[3)]**
2	1.464	1.460	1.460	1.474	1.467	1.467	1.484	1.475	1.475
3	1.321	1.310	1.314	1.334	1.323	1.327	1.347	1.338	1.342
4	1.158	1.139	1.149	1.174	1.154	1.166	1.193	1.173	1.184
5	1.009	0.985	0.993	1.026	1.001	1.010	1.047	1.021	1.029
6	0.871	0.841	0.849	0.888	0.857	0.866	0.909	0.875	0.884
7	0.747	0.713	0.720	0.762	0.727	0.735	0.778	0.745	0.751
8	0.640	0.604	0.613	0.655	0.617	0.626	0.669	0.632	0.640
9	0.550	0.511	0.518	0.560	0.522	0.530	0.574	0.535	0.542
10	0.471	0.432	0.438	0.480	0.442	0.448	0.491	0.452	0.459
RMSE		0.046237	0.037089		0.045252	0.035519		0.044600	0.035724

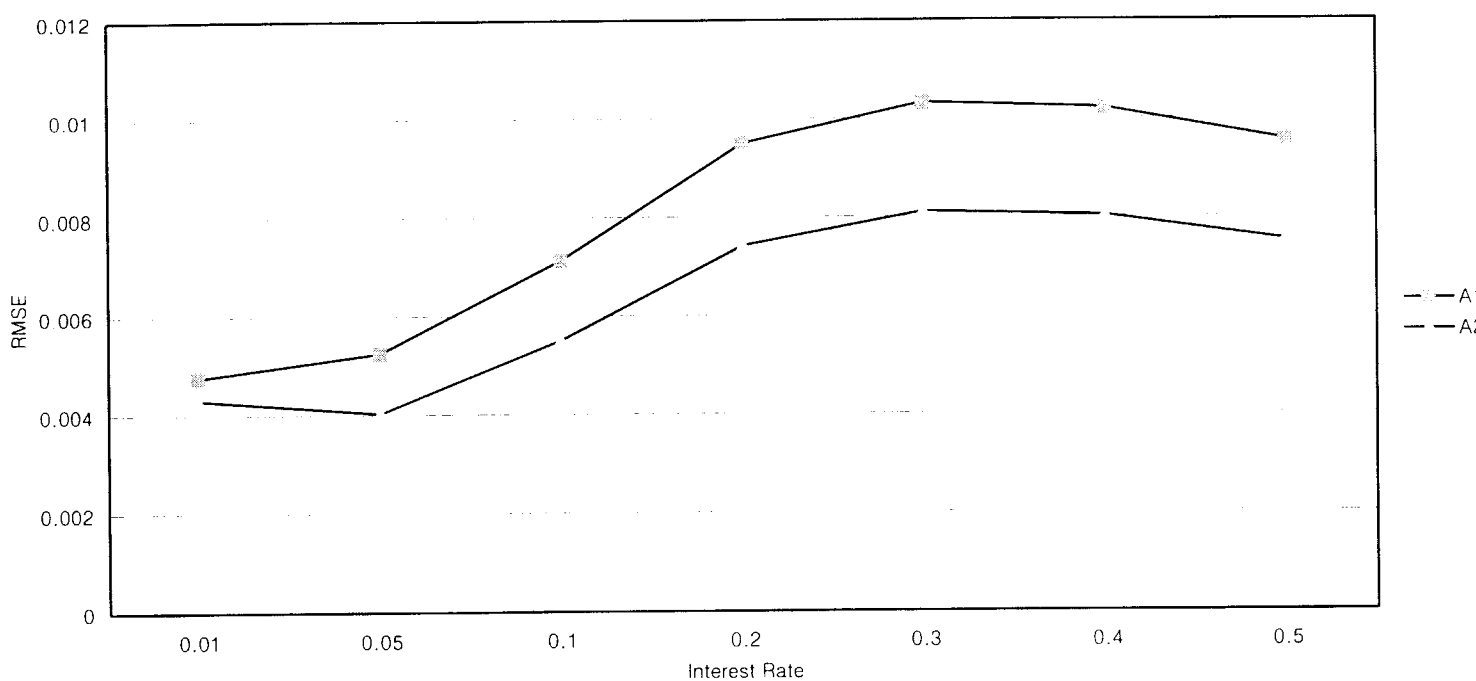
(1) The true values are based on 500,000 Monte Carlo simulations incorporating the antithetic variable technique. Each accrual period is divided into 8 equal subintervals.

(2) First approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (11).

(3) Second approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (19).

Figure 1: RMSE of Two Approximation Methods (Average Volatility = 16.71%)

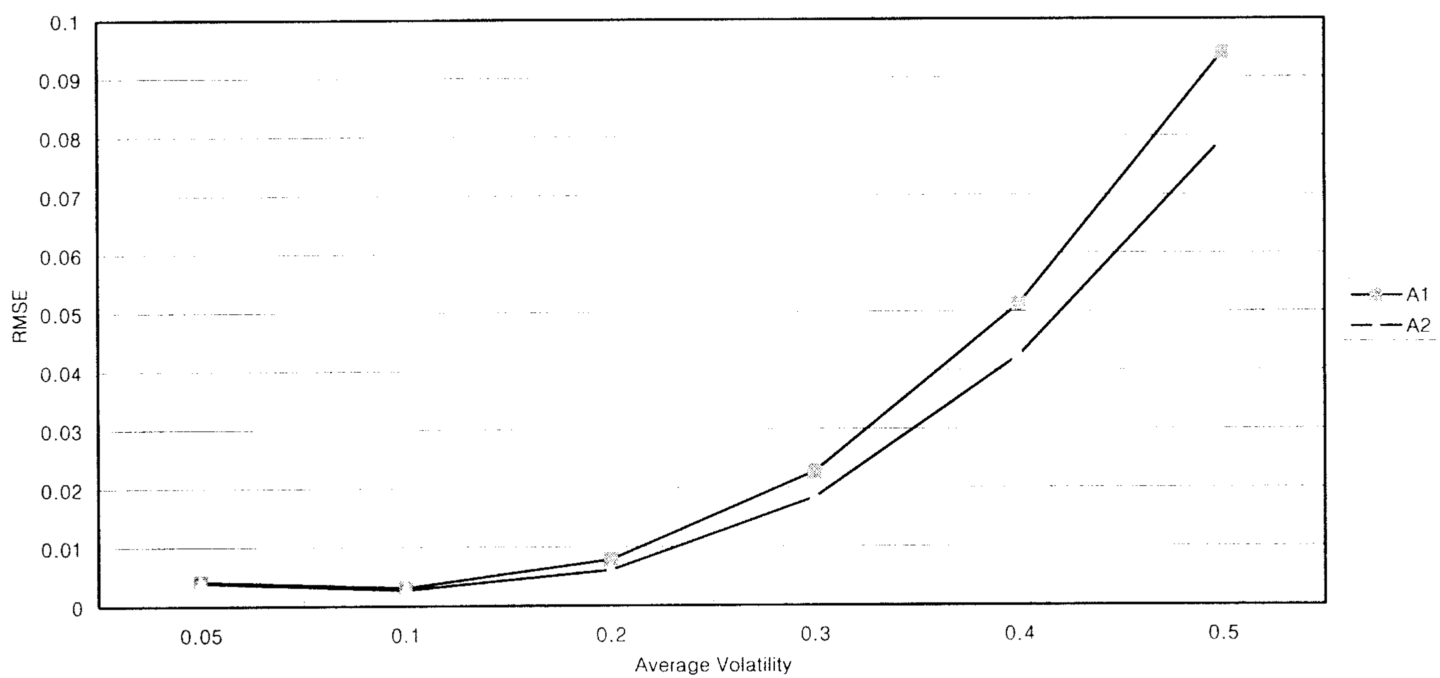
The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat and the caplet volatilities are as in Table I. RMSE is the root mean squared relative error.



(A1) First approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (11).
(A2) Second approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (19).

Figure 2: RMSE of Two Approximation Methods (Average Rate = 5%)

The principal is $100. The accrual periods are one year in length. The spread is 25 basis points. The term structure is flat at 5% per annum with continuous compounding and the caplet volatilities are humped similar to that of Table I. RMSE is the root mean squared relative error.



(A1) First approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (11).
(A2) Second approximated values are computed using 6-point Gauss-Hermite quadrature formula with equation (19).