STABILITY AND STRENGTH OF MODES IN ROTATING MACHINERY

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ABSTRACT

Campbell diagram (whirl speed chart) has long been an important tool in the design and operation of rotating machinery for engineers to understand which modes are likely to be excited by the excitation sources of interest, which speed regions are safe for operation, and so on. However, it may lead to erroneous interpretation on the role of modes in the response prediction, unless their relative stability and strength are clearly addressed. Modal damping (logarithmic decrement) determines the magnification factor, particularly near the corresponding resonant speed. Thus, only the lightly damped modes make significant contributions to the response of rotor near the corresponding modal frequencies, while the heavily damped modes may be ignored even in the Campbell diagram. For a general rotor system, which possesses both anisotropy and asymmetry, there exists an infinite number of modes so that the corresponding whirl speed chart is heavily crowded with modes. Some of them are the original forward and backward modes, but the majority of modes are associated with their modulated (with twice the rotational speed and the integer multiples) and conjugate modes, that may not be as serious as the original modes. One of the effective ways to utilize Campbell diagram is to indicate the modal strength depending upon each modal response contribution to the probable excitation sources. In this paper, modes are classified depending upon modal strength into two: strong and weak modes. The strong modes are defined such that they still remain even when the system anisotropy and asymmetry disappear. The weak modes are the modes that start appearing when the system anisotropy and asymmetry are introduced. The order of strength for the weak modes can be identified in consideration of the degree of the system anisotropy and asymmetry.

Keywords: Modal strength, Norm order of eigenvectors, Campbell diagram, Strong and weak modes, Directional frequency response functions, Periodically time-varying system

INTRODUCTION

Rotating machines nowadays are designed such that they can be safely operated beyond or passing through many critical speeds. Rotating machinery consists of many structural elements such as shaft, disk, blade, bearing/seal/damper, casing, and foundation. Not only each machine structure reveals its own local dynamic characteristics, but the whole machine as an assemblage of part structures also reveals global dynamic characteristics. The dynamic properties of most common interest in rotating machinery typically include the critical speeds, stability of modes and forced response. The critical speeds of a rotor are defined as the rotational speeds at which the associated modal frequencies, rigid or flexible, coincide with the probable excitation sources of paramount interest. Perhaps one of the most convenient graphical presentations is known as the Campbell diagram, which is often referred to as the whirl speed chart [1], where the whirl speeds, or equivalently the modal frequencies, and the probable excitation sources are plotted against the rotational speed, as illustrated in Figs. 1 and 2. Campbell diagram is helpful for design and practice engineers to judge on the margin of safe operation in the design as well as field operation processes. The Campbell diagram has been popularly adopted in the design of rotors with bladed disks such as turbines, where the blade natural frequencies and the excitation lines associated with the blade passing frequency and the integer multiples are easily identified as shown in Fig. 1 [2,3]. However, its usage is limited in the sense that it does not provide practice engineers with the essential information such as the stability and forced response of the actual rotor system, particularly when the system possesses both stationary and rotating asymmetry. In other words, it does not tell us about which critical speeds have to be considered seriously in design and operation – the severity of the rotor response at each critical speed. For the critical speeds of an isotropic rotor system associated with unbalance excitation, the backward whirl speeds are traditionally indicated by broken lines, in order to indicate the less importance of the backward critical speeds in the unbalance response of the isotropic rotor system, as illustrated in Fig. 2 [4].

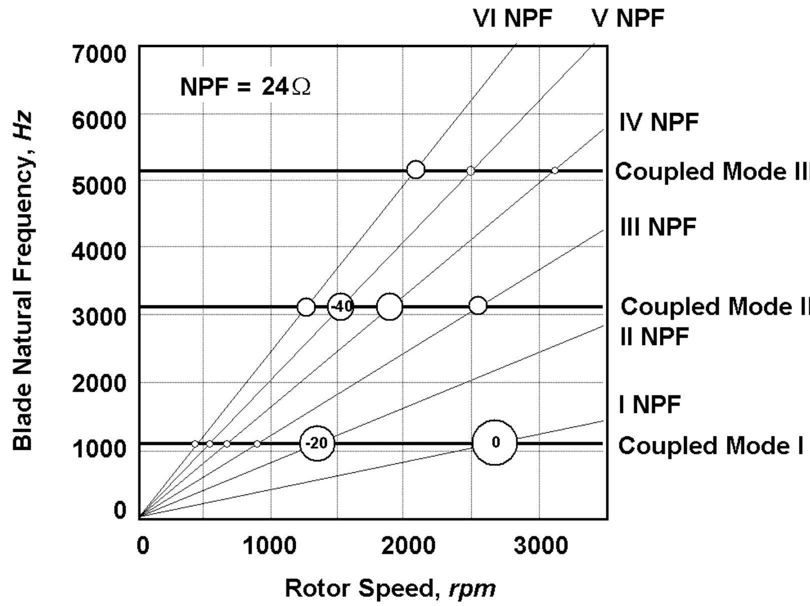


Fig. 1. Turbine blade and Campbell diagram: 24 nozzles [2, 3]; The size of circle is roughly scaled to the logarithmic response magnitude in dB at the free end of blade.

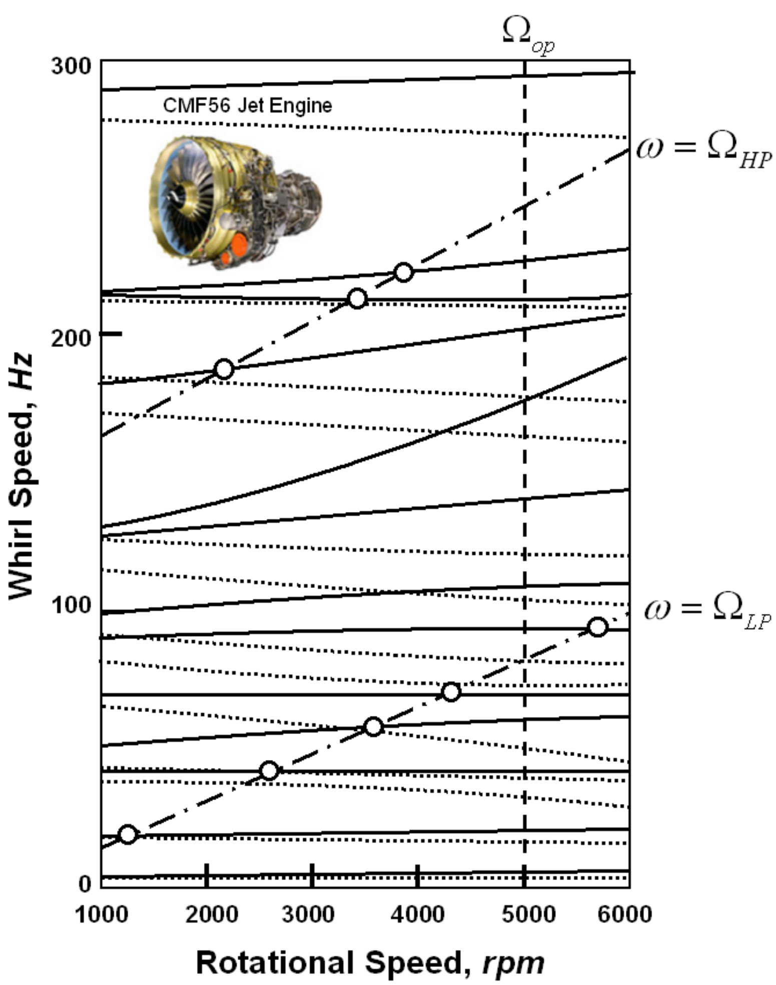


Fig. 2. Campbell Diagram for Twin Spool Jet Engine: Solid and broken lines indicate forward and backward whirl modes, respectively. $\Omega_{HP} = 1.25\Omega_{LP} + 146(Hz)$ **[4]**

One method of accommodating the stability information, which has been well adopted by many scholars in the past, is simply to add the information on the modal damping as well as frequency for each mode in the Campbell diagram, as shown in Fig. 3 [5]. Note that, in most practical applications if the system becomes unstable, it is usually the first forward mode whirl which yields the instability while the remaining modes remain stable [5]. The relative stability based on the modal damping can sure be addressed, but as the number of modes increases, the additional

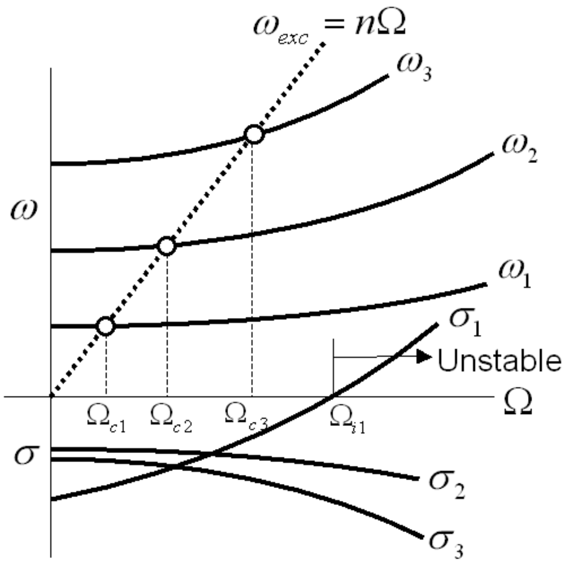


Fig. 3. Whirl Speed Map [5]

information complicates the understanding of the plot, obscuring the essence of information.

Forced responses, including the most common unbalance responses, of a rotor at critical speeds essentially tell us about what actually happens with the rotor in operation subject to known excitation forces. Figure 1 is a typical Campbell diagram with the severity of the forced responses at critical speeds marked by the size of circles [2]. It succeeded in revealing clearly which critical speeds are important. However, the forced responses never represent the real rotor characteristics, unless the accurate quantitative information of all excitation forces, say the precise unbalance distribution, are available, which is an impractical, if not impossible, requirement.

Fortunately, the rotordynamic modeling and analysis have been quite successful in the past, since the rotordynamic modeling based on FEM or TMM is relatively simple in nature compared with other complicated structures and the parameter uncertainties are rarely encountered. From the rotordynamic analysis, we can obtain useful modal information such as the modal damping and frequency, and, above all, the modal vector. Based on the modal information, we can simulate forced response for excitation forces given. Often we assume the nature of excitation forces and freely simulate all probable situations. However, the forced response varies as the excitation force is changed. It means that, unless the exact information of excitation force is given, the forced response represents one realization of innumerable situations encountered in practice.

Modal damping certainly has to do with the relative stability of mode. Presence of modes with positive damping at a rotational speed indicates the unstable free response of the rotor at that speed, the response becoming large as the linearity assumption allows. Modes of light damping contribute more to the transient response than modes of heavy damping. However, it is not completely correct to say that modes of light damping contribute more to the steady-state response than modes of heavy damping. In fact, the magnification factor near the modal frequency for the harmonic response is inversely proportional to modal damping, but the response is also proportional to the residue that is a product of the modal and adjoint vectors as well as the force itself. Thus, the modal vector, whose importance is often forgotten, should be accounted whenever the severity of the response is addressed [6]. In this paper, a new method of presenting the modal strength in the Campbell diagram is proposed, which is based on the norm of the associated modal vector.

1. MODAL ANALYSIS OF ANISOTROPIC OR ASYMMETRIC ROTOR [6]

Rotors can be classified into four types: isotropic rotor with both rotating and stationary symmetry, anisotropic rotor with not stationary, but rotating symmetry, asymmetric rotor with not rotating, but stationary symmetry, and general rotor without both stationary and rotating symmetry. The equation of motion for types of rotor systems other than general rotor can be written, using the complex (stationary for anisotropic rotors and rotating for asymmetric rotors) coordinates, as [6]

$$\mathbf{M_f}\ddot{\mathbf{p}}(t) + \mathbf{C_f}\dot{\mathbf{p}}(t) + \mathbf{K_f}\mathbf{p}(t) + \left\{\Delta\mathbf{M_b}\ddot{\bar{\mathbf{p}}}(t) + \Delta\mathbf{C_b}\dot{\bar{\mathbf{p}}}(t) + \Delta\mathbf{K_b}\bar{\mathbf{p}}(t)\right\} = \mathbf{g}(t)\,, \tag{1}$$

where $\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{K}_i$ denote the complex valued $N\times N$ generalized mass, damping and stiffness matrices, respectively; the subscripts $i = \mathbf{f}, \mathbf{b}$ refer to the mean and deviatoric values, respectively;

$\mathbf{p}(t) = \mathbf{y}(t) + j\mathbf{z}(t)$ and $\mathbf{g}(t) = \mathbf{f}_{\mathrm{y}}(t) + j\mathbf{f}_{\mathrm{z}}(t)$ are the $N\times1$ complex response and input vectors, respectively; $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are the real valued response vectors, and, $\mathbf{f}_{\mathrm{y}}(t)$ and $\mathbf{f}_{\mathrm{z}}(t)$ are the real valued input vectors, in the direction of Y and Z (ξ and η) in the stationary (rotating) coordinates, forming a plane perpendicular to the bearing axis, respectively, as shown in Fig. 4; Ω is the rotational speed of the shaft; N is the dimension of the complex coordinate vector; j is the imaginary number; the bar indicates the complex conjugate; the terms preceded by Δ imply the first-order perturbation matrices due to presence of asymmetry. Note here that the parenthesized terms appear due to the loss of symmetry in the rotor or stator part. Note that the system matrices, including the effect of the gyroscopic moment, internal damping, and fluid-film bearing characteristics, may be dependent upon the rotational speed. However, they become constant for given rotational speed.

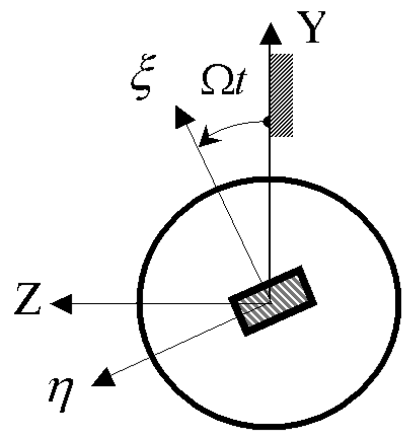


Fig. 4. Stationary and rotating coordinate system for a simple general rotor.

Equation (1) can be rewritten as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{2}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{\mathbf{f}} & \Delta\mathbf{M}_{\mathbf{b}} \\ \Delta\bar{\mathbf{M}}_{\mathbf{b}} & \bar{\mathbf{M}}_{\mathbf{f}} \end{bmatrix}_{2N\times2N} \mathbf{C} = \begin{bmatrix} \mathbf{C}_{\mathbf{f}} & \Delta\mathbf{C}_{\mathbf{b}} \\ \Delta\bar{\mathbf{C}}_{\mathbf{b}} & \bar{\mathbf{C}}_{\mathbf{f}} \end{bmatrix}_{2N\times2N} \mathbf{K} = \begin{bmatrix} \mathbf{K}_{\mathbf{f}} & \Delta\mathbf{K}_{\mathbf{b}} \\ \Delta\bar{\mathbf{K}}_{\mathbf{b}} & \bar{\mathbf{K}}_{\mathbf{f}} \end{bmatrix}_{2N\times2N} \mathbf{q}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \bar{\mathbf{p}}(t) \end{Bmatrix}_{2N\times1}, \quad \mathbf{f}(t) = \begin{Bmatrix} \mathbf{g}(t) \\ \bar{\mathbf{g}}(t) \end{Bmatrix}_{2N\times1}.$$

Assuming the solution form of $\mathbf{q}(t) = \mathbf{u}_c e^{\lambda t}$, one obtains the sets of homogeneous equations associated with equation (2) as [6,7]

$$\mathbf{D}(\lambda_r^i)\mathbf{u}_{cr}^i = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{v}}_{cr}^{iT}\mathbf{D}(\lambda_r^i) = \mathbf{0}^T, \; r = \pm1, \pm2, \ldots, \pm N, \; i = B, F, \tag{3}$$

where the lambda matrix of degree two is given by

$$\mathbf{D}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} = \begin{bmatrix} \mathbf{D}_{\mathrm{f}}(\lambda) & \Delta\tilde{\mathbf{D}}_{\mathrm{r}}(\lambda) \\ \Delta\mathbf{D}_{\mathrm{r}}(\lambda) & \tilde{\mathbf{D}}_{\mathrm{f}}(\lambda) \end{bmatrix}, \tag{4a}$$

with

$$\begin{aligned} &\mathbf{D}_{\mathrm{f}}(\lambda) = \lambda^2\mathbf{M}_{\mathbf{f}} + \lambda\mathbf{C}_{\mathbf{f}} + \mathbf{K}_{\mathbf{f}}, \quad \Delta\mathbf{D}_{\mathrm{r}}(\lambda) = \lambda^2\Delta\bar{\mathbf{M}}_{\mathbf{r}} + \lambda\Delta\bar{\mathbf{C}}_{\mathbf{r}} + \Delta\bar{\mathbf{K}}_{\mathbf{r}}, \\ &\tilde{\mathbf{D}}_{\mathrm{f}}(\lambda) = \lambda^2\bar{\mathbf{M}}_{\mathbf{f}} + \lambda\bar{\mathbf{C}}_{\mathbf{f}} + \bar{\mathbf{K}}_{\mathbf{f}}, \quad \Delta\tilde{\mathbf{D}}_{\mathrm{r}}(\lambda) = \lambda^2\Delta\mathbf{M}_{\mathbf{r}} + \lambda\Delta\mathbf{C}_{\mathbf{r}} + \Delta\mathbf{K}_{\mathbf{r}}, \end{aligned} \tag{4b}$$

and the right and left latent vectors take the form of

$$\mathbf{u}_c = \left\{\mathbf{u}^T \quad \hat{\mathbf{u}}^T\right\}^T, \quad \bar{\mathbf{v}}_c = \left\{\bar{\mathbf{v}}^T \quad \hat{\bar{\mathbf{v}}}^T\right\}^T \tag{4c}$$

The latent roots (eigenvalues) λ are determined from the characteristic polynomial of order $4N$

$$\left|\mathbf{D}(\lambda)\right| = 0 \tag{5}$$

Here, the pair of eigenvalues, equal in subscript value but different in sign of subscript, are dependent upon each other; they are complex conjugate pairs, as will be shown later. And the superscripts B and F implicitly refer to the backward and forward modes, respectively [6].

The latent vectors, obtained from equation (3), are normalized so as to satisfy the bi-orthonormality condition given by

$$(\lambda_r^i + \lambda_s^k)\bar{\mathbf{v}}_{cs}^{k\,T}\mathbf{M}\mathbf{u}_{cr}^i + \bar{\mathbf{v}}_{cs}^{k\,T}\mathbf{C}\mathbf{u}_{cr}^i = \delta_{sr}^{ki}, \quad r, s = \pm1, \ldots \pm N; \; i, k = B, F \tag{6a}$$

or, for $r = s, i = k$,

$$\bar{\mathbf{v}}_c^T[\mathbf{D}'(\lambda)]\mathbf{u}_c = \{\bar{\mathbf{v}}^T \quad \hat{\bar{\mathbf{v}}}^T\}\begin{bmatrix} \mathbf{D}_\mathbf{f}'(\lambda) & \Delta\tilde{\mathbf{D}}_\mathbf{r}'(\lambda) \\ \Delta\mathbf{D}_\mathbf{r}'(\lambda) & \tilde{\mathbf{D}}_\mathbf{f}'(\lambda) \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \hat{\mathbf{u}} \end{bmatrix} = 1 \tag{6b}$$

where

$$\mathbf{D}'(\lambda) = \frac{d}{d\lambda}\mathbf{D}(\lambda) = 2\lambda\mathbf{M} + \mathbf{C}, \tag{6c}$$

and the Kronecker delta is defined as

$$\delta_{sr}^{ki} = \begin{cases} 1; \text{ when } i = k \text{ and } r = s \\ 0; \qquad \text{otherwise.} \end{cases} \tag{6d}$$

Since the eigensolution takes the form of

$$\mathbf{q}(t) = \left\{\mathbf{p}^T(t) \quad \bar{\mathbf{p}}^T(t)\right\}^T = \left\{\mathbf{u}^T \quad \hat{\mathbf{u}}^T\right\}^T e^{\lambda t}, \tag{7a}$$

it holds, for each eigensolution,

$$\mathbf{p}(t) = \mathbf{u}_r^i e^{\lambda_r^i t}, \ \bar{\mathbf{p}}(t) = \bar{\mathbf{u}}_s^k e^{\bar{\lambda}_s^k t} = \hat{\mathbf{u}}_r^i e^{\lambda_r^i t} \tag{7b}$$

or, equivalently,

$$\bar{\mathbf{u}}_s^k = \hat{\mathbf{u}}_r^i \quad \text{and} \quad \bar{\lambda}_s^k = \lambda_r^i. \tag{7c}$$

In order to satisfy the above two relations (7c), it should hold

$$s = -r, \ i = k\ . \tag{8}$$

From the first set of conditions, it can be shown that the eigenvalues and right latent vectors, which are associated with the pair of positive and negative subscripts, satisfy the relations given by

$$\lambda_r^i, \lambda_{-r}^i (= \bar{\lambda}_r^i)\ , \quad \mathbf{u}_{cr}^i = \begin{Bmatrix} \mathbf{u} \\ \hat{\mathbf{u}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{u}_r \\ \bar{\mathbf{u}}_{-r} \end{Bmatrix}^i, \ \mathbf{u}_{c-r}^i = \begin{Bmatrix} \mathbf{u} \\ \hat{\mathbf{u}} \end{Bmatrix}_{-r}^i = \begin{Bmatrix} \bar{\hat{\mathbf{u}}} \\ \bar{\mathbf{u}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{u}_{-r} \\ \bar{\mathbf{u}}_r \end{Bmatrix}^i, \tag{9}$$
$$r = \pm 1, \pm 2, \ldots \pm N\ ,\ i = B, F\ .$$

Similarly, one can obtain the relation between the left latent vectors as

$$\mathbf{v}_{\mathbf{c}r}^{\ i} = \begin{Bmatrix} \mathbf{v} \\ \hat{\mathbf{v}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{v}_r \\ \bar{\mathbf{v}}_{-r} \end{Bmatrix}^i, \mathbf{v}_{\mathbf{c}-r}^{\ i} = \begin{Bmatrix} \mathbf{v} \\ \hat{\mathbf{v}} \end{Bmatrix}_{-r}^i = \begin{Bmatrix} \bar{\hat{\mathbf{v}}} \\ \bar{\mathbf{v}} \end{Bmatrix}_r^i = \begin{Bmatrix} \mathbf{v}_{-r} \\ \bar{\mathbf{v}}_r \end{Bmatrix}^i, \quad r = \pm 1, \pm 2, \ldots \pm N\ ,\ i = B, F\ . \tag{10}$$

Note that the eigenvalue λ_{-r}^i can be derived as the complex conjugate of λ_r^i and that the corresponding latent vectors can also be derived from each other as given in equations (9) and (10).

The complex response vector $\mathbf{p}(t)$ can then be expanded in terms of the right latent vectors as

$$\mathbf{p}(t) = \sum_{i=B,F}\sum_{r=-N}^{N}{}'\{\mathbf{u}_r^i \eta_r^i(t)\}\,, \tag{11a}$$

where the principal coordinates $\eta_r^i(t)$ satisfy the 4N sets of modal equations given by

$$\dot{\eta}_r^i(t) - \lambda_r^i \eta_r^i(t) = \bar{\mathbf{v}}_r^{iT}\mathbf{g}(t) + \hat{\bar{\mathbf{v}}}_r^{iT}\bar{\mathbf{g}}(t)\,, \quad r = \pm 1, \pm 2, \ldots \pm N\ ,\ i = B, F \tag{11b}$$

$$\eta_r^i(t) = \int e^{\lambda_r^i (t-\tau)}\left\{\bar{\mathbf{v}}_r^{iT}\mathbf{g}(\tau) + \hat{\bar{\mathbf{v}}}_r^{iT}\bar{\mathbf{g}}(\tau)\right\}d\tau\ . \tag{11c}$$

Here, $\sum_{r=-N}^{N}{}'$ is the summation operator excluding $r=0$. Note here that the modal response $\eta_r^i(t)$ is proportional to the modal forces $\bar{\mathbf{v}}_r^{iT}\mathbf{g}(t)$ and $\hat{\bar{\mathbf{v}}}_r^{iT}\bar{\mathbf{g}}(t)$, which, in turn, are directly related to the adjoint modal vectors $\bar{\mathbf{v}}_r^{iT}$ and $\hat{\bar{\mathbf{v}}}_r^{iT}$. From equation (11), we can derive the input-output relation in the frequency domain as

$$\mathbf{P}(j\omega)=\begin{bmatrix}\mathbf{H_{gp}} & \mathbf{H_{\hat{g}p}}\end{bmatrix}\begin{Bmatrix}\mathbf{G}(j\omega)\\ \hat{\mathbf{G}}(j\omega)\end{Bmatrix} \tag{12a}$$

where

$$\begin{aligned}\mathbf{H_{gp}}(j\omega)&=\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left[\frac{\mathbf{u}\bar{\mathbf{v}}^T}{j\omega-\lambda}\right]_r^i=\sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{\mathbf{u}_r^i\bar{\mathbf{v}}_r^{i\,T}}{j\omega-\lambda_r^i}+\frac{\mathbf{u}_{-r}^i\bar{\mathbf{v}}_{-r}^{i\;T}}{j\omega-\lambda_{-r}^i}\right],\\ \mathbf{H_{\hat{g}p}}(j\omega)&=\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left[\frac{\mathbf{u}\hat{\bar{\mathbf{v}}}^T}{j\omega-\lambda}\right]_r^i=\sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{\mathbf{u}_r^i\mathbf{v}_{-r}^{i\;T}}{j\omega-\lambda_r^i}+\frac{\mathbf{u}_{-r}^i\mathbf{v}_r^{i\,T}}{j\omega-\lambda_{-r}^i}\right].\end{aligned} \tag{12b}$$

Here $\mathbf{P}(j\omega)$, $\mathbf{G}(j\omega)$ and $\hat{\mathbf{G}}(j\omega)$ are the Fourier transforms of $\mathbf{p}(t)$, $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively, and, $\mathbf{H_{gp}}$ and $\mathbf{H_{\hat{g}p}}$ are referred to as the normal directional frequency response matrix (n-dFRM) and the reverse directional frequency response matrix (r-dFRM), respectively.

2. STABILITY AND STRENGTH OF MODES

The modal solution of rotor systems can be summarized as

$$\begin{gathered}\lambda_r^i=\sigma_r^i+j\omega_r^i,\quad \lambda_{-r}^i(=\bar{\lambda}_r^i)=\sigma_r^i-j\omega_r^i,\quad \mathbf{u}_r^i,\quad \mathbf{u}_{-r}^i(=\hat{\bar{\mathbf{u}}}_r^i),\quad \mathbf{v}_r^i,\quad \mathbf{v}_{-r}^i(=\hat{\bar{\mathbf{v}}}_r^i),\\ r=\pm1,\pm2,\ldots\pm N,\ i=B,F.\end{gathered} \tag{13}$$

where σ_r^i and ω_r^i are the modal damping and frequency, respectively. The stability of mode is determined by the modal damping σ_r^i. If σ_r^i is positive (negative), the mode becomes unstable (stable). For the stable system, all modes should be negative damped. Thus, in order to indicate the overall system instability, plotting of the modal damping associated with the most unstable modes is enough. For stable rotor systems, the modal damping theoretically indicates the relative stability or the stability margin, but the actual forced response may not be directly related with the relative stability. The forced response $\mathbf{p}(t)$ is made up with modal vector weighted modal responses $\mathbf{u}_r^i\eta_r^i(t)$ as in Eq.(11a). And the modal responses $\eta_r^i(t)$ are again related to adjoint vectors $\mathbf{v}_{-r}^i(=\hat{\bar{\mathbf{v}}}_r^i)$. Thus the contribution of each mode to the forced response should be determined based on the norm of modal and adjoint vectors. Using the results of the perturbation of eigenvalue problem [8]

$$\lambda_r^i=\lambda_{r0}^i+\Delta\lambda_{r1}^i \tag{14a}$$

we obtain, for $r > 0$,

$$\left\|\mathbf{D_f}(\lambda_r^i)\right\|=\left\|\mathbf{D_f}(\lambda_{r0}^i+\Delta\lambda_{r1}^i)\right\|\sim O(\Delta),\ \left\|\tilde{\mathbf{D}}_\mathbf{f}(\lambda_{-r}^i)\right\|=\left\|\mathbf{D_f}(\lambda_{-r0}^i+\Delta\lambda_{-r1}^i)\right\|\sim O(\Delta). \tag{14b}$$

From equations (3) and (14), we can derive the relations between modal vectors, for $r > 0$, as

$$\hat{\mathbf{u}}_r^i=-\tilde{\mathbf{D}}_\mathbf{f}^{-1}(\lambda_r^i)\Delta\mathbf{D}_\mathrm{r}(\lambda_r^i)\mathbf{u}_r^i\quad\text{and}\quad \mathbf{u}_{-r}^i=-\mathbf{D}_\mathbf{f}^{-1}(\lambda_{-r}^i)\Delta\tilde{\mathbf{D}}_\mathrm{r}(\lambda_{-r}^i)\hat{\mathbf{u}}_{-r}^i \tag{15a}$$

or, the relations between the norms of modal vectors as

$$\left\|\hat{\mathbf{u}}_r^i\right\| \approx O(\Delta)\left\|\mathbf{u}_r^i\right\| \quad \text{and} \quad \left\|\mathbf{u}_{-r}^i\right\| \approx O(\Delta)\left\|\hat{\mathbf{u}}_{-r}^i\right\| \approx O(\Delta)\left\|\mathbf{u}_r^i\right\| \tag{15b}$$

where $O(\Delta)$ means the perturbation of the first order. Here, Note that the similar relations to equations (15) can be readily obtained with the adjoint vectors. Using the results in Eq. (15), we can derive, letting $\left\|\mathbf{u}_r^i\right\| \approx \left\|\mathbf{v}_r^i\right\| \approx O(1)$,

$$\begin{aligned}
\left\|\mathbf{H}_{\mathbf{gp}}\left(j\omega\right)\right\| &\leq \sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{\left\|\mathbf{u}_r^i\right\|\left\|\overline{\mathbf{v}}_r^{iT}\right\|}{\left|j\omega-\lambda_r^i\right|}+\frac{\left\|\mathbf{u}_{-r}^i\right\|\left\|\overline{\mathbf{v}}_{-r}^{i\;T}\right\|}{\left|j\omega-\lambda_{-r}^i\right|}\right] \approx \sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{O(1)}{\left|j\omega-\lambda_r^i\right|}+\frac{O(\Delta^2)}{\left|j\omega-\lambda_{-r}^i\right|}\right], \\
\left\|\mathbf{H}_{\hat{\mathbf{g}}\mathbf{p}}\left(j\omega\right)\right\| &\leq \sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{\left\|\mathbf{u}_r^i\right\|\left\|\mathbf{v}_{-r}^{i\;T}\right\|}{j\omega-\lambda_r^i}+\frac{\left\|\mathbf{u}_{-r}^i\right\|\left\|\mathbf{v}_r^{iT}\right\|}{j\omega-\lambda_{-r}^i}\right] \approx \sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{O(\Delta)}{j\omega-\lambda_r^i}+\frac{O(\Delta)}{j\omega-\lambda_{-r}^i}\right].
\end{aligned} \tag{16}$$

Note that $O(1)$ means the norm order is independent of the perturbation Δ . It can be concluded from equations (15) and (16) that [9]

1. The norm of the original forward and backward modal and adjoint vectors is $O(1)$, independent of the perturbation Δ due to presence of asymmetry in the rotor system.
2. The norm of the conjugate forward and backward modal and adjoint vectors is $O(\Delta)$, directly proportional to the perturbation. These additional modes tend to vanish as the system asymmetry diminishes, never contributing to the forced responses.
3. The modes associated with the vector norm of $O(1)$ are referred to as the strong modes, whereas the modes of vector norm of order less than and equal to $O(\Delta)$ are referred to as the weak modes. The strong modes are almost independent, in contribution to forced response, of the degree of system asymmetry.
4. The ratio of the residue value of the n-dFRF between the original (strong) and conjugate (weak) modes becomes $O(\Delta^2)$. It implies that the strong modes, which are associated with the mean property of the rotor, are easily detected, but the weak modes, which are associated with the deviatoric property of the rotor, are hardly detected in the normal dFRF unless the degree of asymmetry becomes prominent.
5. The residue value of the r-dFRF for all, strong and weak, modes becomes the order of Δ in magnitude. And the r-dFRF tends to vanish, as the asymmetry of the rotor decreases. Thus the magnitude of the r-dFRF, relative to that of the n-dFRF for strong modes, is a good indicator for presence of asymmetry in the rotor.

3. MODAL ANALYSIS OF GENERAL ROTOR

General rotors normally possess both stationary and rotating asymmetry, leading to a complicated periodically time-varying equation of motion. The modal analysis of such system has already been developed, but the detailed procedure will not be treated because the derivation is mathematically involved [10,11]. Instead, the main feature of the analysis will be briefly demonstrated with a simple, but general, rotor model in the following section.

4. SIMPLE GENERAL ROTOR: USE OF $4N \times 4N$ REDUCED ORDER MATRIX

This section illustrates the modal analysis procedure using modulated coordinates with a simple general rotor system (with N = 1) shown in Fig. 4. The equation of motion for the system can be written as

$$\ddot{\mathrm{p}}(t)+(2\zeta - \mathrm{j}\alpha\Omega)\dot{\mathrm{p}}(t)+\mathrm{p}(t)+\delta \mathrm{e}^{\mathrm{j}2\Omega t}\overline{\mathrm{p}}(t)+\Delta\overline{\mathrm{p}}(t)=\mathrm{g}(t) \tag{17}$$

where δ and Δ represent the degree of asymmetry and anisotropy, respectively, and α represents the gyroscopic effect. The periodically time-varying linear differential equation (17) can be converted to an equivalent time-invariant linear differential matrix-vector equation of infinite dimension [10,11]. Although the reduced matrix-vector equation of order 6*N* or higher turns out to be computationally efficient in calculation of the eigensolutions, we now demonstrate the norm order analysis of eigenvectors with the reduced matrix of order 4*N*, for simplicity but still without loss of generality [10,11]. The $4N \times 4N$ reduced order matrix equation of motion may be written as

$$\tilde{\mathbf{M}}\,\ddot{\tilde{\mathbf{p}}}(t)+\tilde{\mathbf{C}}\,\dot{\tilde{\mathbf{p}}}(t)+\tilde{\mathbf{K}}\,\tilde{\mathbf{p}}(t)=\tilde{\mathbf{g}}(t) \tag{18}$$

where

$$\tilde{\mathbf{p}}=\left\{\mathrm{p}^T_{;-1}(t)\quad \overline{\mathrm{p}}^T_{;0}(t)\quad \mathrm{p}^T_{;0}(t)\quad \overline{\mathrm{p}}^T_{;-1}(t)\right\}^T,\quad \tilde{\mathbf{g}}=\left\{\mathrm{g}^T_{;-1}(t)\quad \overline{\mathrm{g}}^T_{;0}(t)\quad \mathrm{g}^T_{;0}(t)\quad \overline{\mathrm{g}}^T_{;-1}(t)\right\}^T$$

$$\tilde{\mathbf{M}}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \tilde{\mathbf{C}}=\begin{bmatrix}2\varsigma+j(4-\alpha)\Omega&0&0&0\\0&2\varsigma+j\alpha\Omega&0&0\\0&0&2\varsigma-j\alpha\Omega&0\\0&0&0&2\varsigma-j(4-\alpha)\Omega\end{bmatrix},\ \tilde{\mathbf{K}}=\begin{bmatrix}1-2(2-\alpha)\Omega^2+j4\varsigma\Omega&\delta&0&0\\\delta&1&\Delta&0\\0&\Delta&1&\delta\\0&0&\delta&1-2(2-\alpha)\Omega^2-j4\varsigma\Omega\end{bmatrix}$$

introducing the modulated complex coordinate and force, $\mathrm{p}_{;n}$ and $\mathrm{g}_{;n}$, n being an arbitrary integer, defined as

$$\mathrm{p}_{;n}(t)\equiv \mathrm{p}(t)e^{j2n\Omega t},\quad \mathrm{g}_{;n}(t)\equiv \mathrm{g}(t)e^{j2n\Omega t}\ .$$

The associated latent value problem then becomes

$$\tilde{\mathbf{D}}(\lambda)\mathbf{u}_c=\tilde{\mathbf{0}}\quad\text{and}\quad \overline{\mathbf{v}}_c^T\tilde{\mathbf{D}}(\lambda)=\tilde{\mathbf{0}}^T \tag{19a}$$

where the lambda matrix of degree two is given by

$$\tilde{\mathbf{D}}(\lambda)=\lambda^2\tilde{\mathbf{M}}+\lambda\tilde{\mathbf{C}}+\tilde{\mathbf{K}}=\begin{bmatrix}b(\lambda)&\delta&0&0\\\delta&\hat{a}(\lambda)&\Delta&0\\0&\Delta&a(\lambda)&\delta\\0&0&\delta&\hat{b}(\lambda)\end{bmatrix} \tag{19b}$$

with

$$\begin{aligned}
a(\lambda)&=\lambda^2+(2\varsigma-j\alpha\Omega)\lambda+1=(\lambda-\lambda^{\mathrm{o}F}_{1(0)})(\lambda-\lambda^{\mathrm{o}B}_{1(0)})\\
b(\lambda)&=\left(\lambda+j2\Omega-\lambda^{\mathrm{o}F}_{1(0)}\right)\left(\lambda+j2\Omega-\lambda^{\mathrm{o}B}_{1(0)}\right)=(\lambda-\lambda^{\mathrm{o}F}_{1(-1)})(\lambda-\lambda^{\mathrm{o}B}_{1(-1)})\\
\hat{a}(\lambda)&=\lambda^2+(2\varsigma+j\alpha\Omega)\lambda+1=(\lambda-\lambda^{\mathrm{o}F}_{-1(0)})(\lambda-\lambda^{\mathrm{o}B}_{-1(0)})\\
\hat{b}(\lambda)&=\left(\lambda-j2\Omega-\lambda^{\mathrm{o}F}_{-1(0)}\right)\left(\lambda-j2\Omega-\lambda^{\mathrm{o}B}_{-1(0)}\right)=(\lambda-\lambda^{\mathrm{o}F}_{-1(1)})(\lambda-\lambda^{\mathrm{o}B}_{-1(1)})
\end{aligned} \tag{19c}$$

Here, $\lambda^{\mathrm{o}i}_{r(m)}$, $i=B,F;\ r=\pm1;\quad m=0,\mp1$, are the eigenvalues of the associated isotropic system, i.e. the simple rotor with $\delta=\Delta=0$. The characteristic equation associated with equation (19) reduces to

$$\left|\tilde{\mathbf{D}}(\lambda)\right|=\left\{\hat{a}(\lambda)b(\lambda)-\delta^2\right\}\left\{a(\lambda)\hat{b}(\lambda)-\delta^2\right\}-\hat{b}(\lambda)b(\lambda)\Delta^2=\prod_{\substack{m=0,-1\\ i=B,F}}\left(\lambda-\lambda^i_{1(m)}\right)\left(\lambda-\overline{\lambda}^i_{1(m)}\right)=0 \tag{20}$$

Note here that it holds, from equation (19c), introducing the definition of $O(\delta,\Delta)=O(\delta)+O(\Delta)$,

$$a\left(\lambda^i_{r(m)}\right) \sim \begin{cases} O\left(\delta,\Delta\right) & \text{for} \quad r(m)=1(0) \\ O\left(1\right) & \textit{otherwise} \end{cases}, \qquad b\left(\lambda^i_{r(m)}\right) \sim \begin{cases} O\left(\delta,\Delta\right) & \text{for} \quad r(m)=1(-1) \\ O\left(1\right) & \textit{otherwise} \end{cases}$$
$$\hat{a}\left(\lambda^i_{r(m)}\right) \sim \begin{cases} O\left(\delta,\Delta\right) & \text{for} \quad r(m)=-1(0) \\ O\left(1\right) & \textit{otherwise} \end{cases}, \qquad \hat{b}\left(\lambda^i_{r(m)}\right) \sim \begin{cases} O\left(\delta,\Delta\right) & \text{for} \quad r(m)=-1(+1) \\ O\left(1\right) & \textit{otherwise} \end{cases} \tag{21}$$

The latent value problem can be formulated as

$$\tilde{\mathbf{D}}(\lambda^i_{r(m)})\mathbf{u}_{cr(m)} = \begin{bmatrix} b & \delta & 0 & 0 \\ \delta & \hat{a} & \Delta & 0 \\ 0 & \Delta & a & \delta \\ 0 & 0 & \delta & \hat{b} \end{bmatrix} \begin{Bmatrix} u^i_{r(m);-1} \\ \hat{u}^i_{r(m);0} \\ u^i_{r(m);0} \\ \hat{u}^i_{r(m);-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{22a}$$

$$\overline{\mathbf{v}}^T_{cr(m)}\tilde{\mathbf{D}}(\lambda^i_{r(m)}) = \left\{ \overline{v}^i_{r(m);-1} \quad \overline{\hat{v}}^i_{r(m);0} \quad \overline{v}^i_{r(m);0} \quad \overline{\hat{v}}^i_{r(m);-1} \right\} \begin{bmatrix} b & \delta & 0 & 0 \\ \delta & \hat{a} & \Delta & 0 \\ 0 & \Delta & a & \delta \\ 0 & 0 & \delta & \hat{b} \end{bmatrix} = \left\{ 0 \quad 0 \quad 0 \quad 0 \right\} \tag{22b}$$

The eight eigenvalues and the corresponding right and left latent vectors are related to each other, i.e. for $i = F, B$,

$$\begin{array}{ccc} \text{eigenvalue} & \text{left latent vector} & \text{right latent vector} \end{array}$$
$$\left\{ \begin{array}{ccc} \lambda^i_{1(0)} & (\overline{v}_{;-1}, \overline{\hat{v}}_{;0}, \overline{v}_{;0}, \overline{\hat{v}}_{;-1})^{iT}_{1(0)} & (u_{;-1}, \hat{u}_{;0}, u_{;0}, \hat{u}_{;-1})^{iT}_{1(0)} \\ \lambda^i_{-1(0)} = \overline{\lambda}^i_{1(0)} & (\overline{v}_{;-1}, \overline{\hat{v}}_{;0}, \overline{v}_{;0}, \overline{\hat{v}}_{;-1})^{iT}_{-1(0)} & (u_{;-1}, \hat{u}_{;0}, u_{;0}, \hat{u}_{;-1})^{iT}_{-1(0)} \\ \lambda^i_{1(-1)} = \lambda^i_{1(0)} - j2\Omega & (\overline{v}_{;-1}, \overline{\hat{v}}_{;0}, \overline{v}_{;0}, \overline{\hat{v}}_{;-1})^{iT}_{1(-1)} & (u_{;-1}, \hat{u}_{;0}, u_{;0}, \hat{u}_{;-1})^{iT}_{1(-1)} \\ \lambda^i_{-1(+1)} = \overline{\lambda}^i_{1(-1)} = \overline{\lambda}^i_{1(0)} + j2\Omega & (\overline{v}_{;-1}, \overline{\hat{v}}_{;0}, \overline{v}_{;0}, \overline{\hat{v}}_{;-1})^{iT}_{-1(+1)} & (u_{;-1}, \hat{u}_{;0}, u_{;0}, \hat{u}_{;-1})^{iT}_{-1(+1)} \end{array} \right. \tag{23}$$

By removing the row with the elements of order less than 1 from the lambda matrix $\tilde{\mathbf{D}}(\lambda^i_{r(m)})$, we can derive: for $r(m)=1(0)$ where $a(\lambda^i_{r(m)}) \sim O(\delta,\Delta)$,

$$\begin{Bmatrix} u^i_{r(m);-1} \\ \hat{u}^i_{r(m);0} \\ \hat{u}^i_{r(m);-1} \end{Bmatrix} = \begin{bmatrix} b & \delta & 0 \\ \delta & \hat{a} & 0 \\ 0 & 0 & \hat{b} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -\Delta \\ -\delta \end{Bmatrix} u^i_{r(m);0} \cong \frac{1}{\hat{a}b\hat{b}} \begin{Bmatrix} \delta\Delta\hat{b} \\ -\Delta b\hat{b} \\ -\delta\hat{a}b \end{Bmatrix} u^i_{r(m);0} \quad \to \begin{Bmatrix} \left\| u^i_{r(m);-1} \right\| \\ \left\| \hat{u}^i_{r(m);0} \right\| \\ \left\| \hat{u}^i_{r(m);-1} \right\| \end{Bmatrix} \sim \begin{Bmatrix} O(\delta\Delta) \\ O(\Delta) \\ O(\delta) \end{Bmatrix} \left\| u^i_{r(m);0} \right\| \tag{24a}$$

for $r(m)=-1(0)$ where $\hat{a}(\lambda^i_{r(m)}) \sim O(\delta,\Delta)$,

$$\begin{Bmatrix} u^i_{r(m);-1} \\ u^i_{r(m);0} \\ \hat{u}^i_{r(m);-1} \end{Bmatrix} = \begin{bmatrix} b & 0 & 0 \\ 0 & a & \delta \\ 0 & \delta & \hat{b} \end{bmatrix}^{-1} \begin{Bmatrix} -\delta \\ -\Delta \\ 0 \end{Bmatrix} \hat{u}^i_{r(m);0} \cong \frac{1}{ab\hat{b}} \begin{Bmatrix} -\delta a\hat{b} \\ -\Delta b\hat{b} \\ \delta\Delta b \end{Bmatrix} \hat{u}^i_{r(m);0} \quad \to \begin{Bmatrix} \left\| u^i_{r(m);-1} \right\| \\ \left\| u^i_{r(m);0} \right\| \\ \left\| \hat{u}^i_{r(m);-1} \right\| \end{Bmatrix} \sim \begin{Bmatrix} O(\delta) \\ O(\Delta) \\ O(\delta\Delta) \end{Bmatrix} \left\| \hat{u}^i_{r(m);0} \right\| \tag{24b}$$

for $r(m)=1(-1)$ where $b(\lambda^i_{r(m)}) \sim O(\delta,\Delta)$,

$$\begin{Bmatrix} \hat{u}^i_{r(m);0} \\ u^i_{r(m);0} \\ \hat{u}^i_{r(m);-1} \end{Bmatrix} = \begin{bmatrix} \hat{a} & \Delta & 0 \\ \Delta & a & \delta \\ 0 & \delta & \hat{b} \end{bmatrix}^{-1} \begin{Bmatrix} -\delta \\ 0 \\ 0 \end{Bmatrix} u^i_{r(m);-1} \cong \frac{-\delta}{\hat{a}a\hat{b}} \begin{Bmatrix} a\hat{b} \\ -\Delta\hat{b} \\ \delta\Delta \end{Bmatrix} u^i_{r(m);-1} \quad \to \begin{Bmatrix} \left\| \hat{u}^i_{r(m);0} \right\| \\ \left\| u^i_{r(m);0} \right\| \\ \left\| \hat{u}^i_{r(m);-1} \right\| \end{Bmatrix} \sim \begin{Bmatrix} O(\delta) \\ O(\delta\Delta) \\ O(\delta^2\Delta) \end{Bmatrix} \left\| u^i_{r(m);-1} \right\| \tag{24c}$$

for $r(m) = -1(1)$, where $\hat{b}(\lambda^i_{r(m)}) \sim O(\delta, \Delta)$,

$$\begin{Bmatrix} u^i_{r(m);-1} \\ \hat{u}^i_{r(m);0} \\ u^i_{r(m);0} \end{Bmatrix} = \begin{bmatrix} b & \delta & 0 \\ \delta & \hat{a} & \Delta \\ 0 & \Delta & a \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ -\delta \end{Bmatrix} \hat{u}^i_{r(m);-1} \cong \frac{-\delta}{\hat{a}ab} \begin{Bmatrix} \delta\Delta \\ -\Delta b \\ \hat{a}b \end{Bmatrix} \hat{u}^i_{r(m);-1} \qquad \rightarrow \begin{Bmatrix} \left\| u^i_{r(m);-1} \right\| \\ \left\| \hat{u}^i_{r(m);0} \right\| \\ \left\| u^i_{r(m);0} \right\| \end{Bmatrix} \sim \begin{Bmatrix} O(\delta^2\Delta) \\ O(\delta\Delta) \\ O(\delta) \end{Bmatrix} \left\| \hat{u}^i_{r(m);-1} \right\| \tag{24d}$$

Note that the similar relations to equations (24) hold for the left latent vectors. Substituting the above relations (24) into the bi-orthonormality condition

$$\bar{\mathbf{v}}^{iT}_{cr} \left[\frac{d}{d\lambda} \tilde{\mathbf{D}}(\lambda) \right]_{\lambda=\lambda^i_{r(m)}} \mathbf{u}^i_{cr} = \bar{\mathbf{v}}^{iT}_{cr} [2\lambda^i_{r(m)} \tilde{\mathbf{M}} + \tilde{\mathbf{C}}] \mathbf{u}^i_{cr} = \left\{ \bar{v}^i_{r(m);-1} \quad \bar{\hat{v}}^i_{r(m);0} \quad \bar{v}^i_{r(m);0} \quad \bar{\hat{v}}^i_{r(m);-1} \right\} \begin{bmatrix} b' & 0 & 0 & 0 \\ 0 & \breve{a}' & 0 & 0 \\ 0 & 0 & a' & 0 \\ 0 & 0 & 0 & \breve{b}' \end{bmatrix} \begin{Bmatrix} u^i_{r(m);-1} \\ \hat{u}^i_{r(m);0} \\ u^i_{r(m);0} \\ \hat{u}^i_{r(m);-1} \end{Bmatrix} = 1 \tag{25a}$$

we can derive

$$\left\| \bar{v}^i_{r(m);-1} u^i_{r(m);-1} \right\| + \left\| \bar{\hat{v}}^i_{r(m);0} \hat{u}^i_{r(m);0} \right\| + \left\| \bar{v}^i_{r(m);0} u^i_{r(m);0} \right\| + \left\| \bar{\hat{v}}^i_{r(m);-1} \hat{u}^i_{r(m);-1} \right\| \sim O(1) \tag{25b}$$

or,

$\left\| u^i_{r(m);0} \right\| \sim O(1)$, $\left\| \bar{v}^i_{r(m);0} \right\| \sim O(1)$, for $r(m)$ = 1(0); $\left\| \hat{u}^i_{r(m);0} \right\| \sim O(1)$, $\left\| \bar{\hat{v}}^i_{r(m);0} \right\| \sim O(1)$, for $r(m)$ = -1(0); (25c)

$\left\| u^i_{r(m);-1} \right\| \sim O(1)$, $\left\| \bar{v}^i_{r(m);-1} \right\| \sim O(1)$, for $r(m)$ = 1(-1); $\left\| \hat{u}^i_{r(m);-1} \right\| \sim O(1)$, $\left\| \bar{\hat{v}}^i_{r(m);-1} \right\| \sim O(1)$, for $r(m)$ = -1(+1)

In equation (25a), ' denotes the differentiation with respect to λ. Table 1 summarizes the results for the right latent vectors obtained from the $6N \times 6N$ reduced order matrix equation, where the results from equations (24) and (25) are shaded. Note that the column associated with $\left\| u^i_{r(m);0} \right\|$ indicates the modal strength; the modes, whose orthonormalized modal vectors are order of 1 (less than 1) in magnitude are referred to as 'strong (weak) modes.' The weak modes tend to vanish as the degree of anisotropy and asymmetry diminishes. Referring to Table 1, we can conclude that the modes associated with $\lambda^{B,F}_{1(0)}$ are the strong modes, the rest of modes being the weak modes. In particular, the pair of modes associated with $\lambda^{B,F}_{-1(1)}$ are the weakest modes, in the sense that they are vulnerable to the presence of either asymmetry or anisotropy and that they are not likely to be easily captured in practice.

Table 1. Modal strength obtained from $6N \times 6N$ reduced order matrix equation.
(Shaded area shows the results from $4N \times 4N$ reduced order matrix equation.)

$\lambda^{B,F}_{r(m)}$	$\left\| \hat{u}^i_{r(m);1} \right\|$ $\left\| \hat{v}^i_{r(m);1} \right\|$	$\left\| u^i_{r(m);-1} \right\|$ $\left\| v^i_{r(m);-1} \right\|$	$\left\| \hat{u}^i_{r(m);0} \right\|$ $\left\| \hat{v}^i_{r(m);0} \right\|$	$\left\| u^i_{r(m);0} \right\|$ $\left\| v^i_{r(m);0} \right\|$	$\left\| \hat{u}^i_{r(m);-1} \right\|$ $\left\| \hat{v}^i_{r(m);-1} \right\|$	$\left\| u^i_{r(m);1} \right\|$ $\left\| u^i_{r(m);1} \right\|$
1(0) ; $a(\lambda^{B,F}_{1(0)}) \sim O(\delta,\Delta)$	$O(\delta^2\Delta)$	$O(\delta\Delta)$	$O(\Delta)$	$O(1)$	$O(\delta)$	$O(\delta\Delta)$
-1(0); $\hat{a}(\lambda^{B,F}_{-1(0)}) \sim O(\delta,\Delta)$	$O(\delta\Delta)$	$O(\delta)$	$O(1)$	$O(\Delta)$	$O(\delta\Delta)$	$O(\delta\Delta^2)$
1(-1); $b(\lambda^{B,F}_{1(-1)}) \sim O(\delta,\Delta)$	$O(\delta\Delta)$	$O(1)$	$O(\delta)$	$O(\delta\Delta)$	$O(\delta^2\Delta)$	$O(\delta^2\Delta^2)$
-1(1); $\hat{b}(\lambda^{B,F}_{-1(1)}) \sim O(\delta,\Delta)$	$O(\delta^2\Delta^2)$	$O(\delta^2\Delta)$	$O(\delta\Delta)$	$O(\delta)$	$O(1)$	$O(\Delta)$
1(1); $c(\lambda^{B,F}_{1(1)}) \sim O(\delta,\Delta)$	$O(\delta^2\Delta^3)$	$O(\delta^2\Delta^2)$	$O(\delta\Delta^2)$	$O(\delta\Delta)$	$O(\Delta)$	$O(1)$
-1(-1); $\hat{c}(\lambda^{B,F}_{-1(-1)}) \sim O(\delta,\Delta)$	$O(1)$	$O(\Delta)$	$O(\delta\Delta)$	$O(\delta\Delta^2)$	$O(\delta^2\Delta^2)$	$O(\delta^2\Delta^3)$

The directional frequency responses can be expressed as

$$
\begin{aligned}
\left\|\mathrm{H}_{\mathrm{gp}}(j\omega)\right\| \le\ & \frac{\left\|\mathrm{u}^F_{1(0);0}\bar{\mathrm{v}}^F_{1(0);0}\right\|}{\left|j\omega-\lambda^F_{1(0)}\right|}+\frac{\left\|\mathrm{u}^B_{1(0);0}\bar{\mathrm{v}}^B_{1(0);0}\right\|}{\left|j\omega-\lambda^B_{1(0)}\right|}+\frac{\left\|\mathrm{u}^F_{1(-1);0}\bar{\mathrm{v}}^F_{1(-1);0}\right\|}{\left|j\omega-\lambda^F_{1(-1)}\right|}+\frac{\left\|\mathrm{u}^B_{1(-1);0}\bar{\mathrm{v}}^B_{1(-1);0}\right\|}{\left|j\omega-\lambda^B_{1(-1)}\right|}+\frac{\left\|\mathrm{u}^F_{-1(0);0}\bar{\mathrm{v}}^F_{-1(0);0}\right\|}{\left|j\omega-\bar{\lambda}^F_{1(0)}\right|}+\frac{\left\|\mathrm{u}^B_{-1(0);0}\bar{\mathrm{v}}^B_{-1(0);0}\right\|}{\left|j\omega-\bar{\lambda}^B_{1(0)}\right|}+\frac{\left\|\mathrm{u}^F_{-1(1);0}\bar{\mathrm{v}}^F_{-1(1);0}\right\|}{\left|j\omega-\bar{\lambda}^F_{1(-1)}\right|}+\frac{\left\|\mathrm{u}^B_{-1(1);0}\bar{\mathrm{v}}^B_{-1(1);0}\right\|}{\left|j\omega-\bar{\lambda}^B_{1(-1)}\right|} \\
\sim\ & \frac{O(1)}{\left|j\omega-\lambda^F_{1(0)}\right|}+\frac{O(1)}{\left|j\omega-\lambda^B_{1(0)}\right|}+\frac{O(\delta^2\Delta^2)}{\left|j\omega-\lambda^F_{1(-1)}\right|}+\frac{O(\delta^2\Delta^2)}{\left|j\omega-\lambda^B_{1(-1)}\right|}+\frac{O(\Delta^2)}{\left|j\omega-\bar{\lambda}^F_{1(0)}\right|}+\frac{O(\Delta^2)}{\left|j\omega-\bar{\lambda}^B_{1(0)}\right|}+\frac{O(\delta^2)}{\left|j\omega-\bar{\lambda}^F_{1(-1)}\right|}+\frac{O(\delta^2)}{\left|j\omega-\bar{\lambda}^B_{1(-1)}\right|} \qquad (26) \\
\left\|\mathrm{H}_{\hat{\mathrm{g}}\mathrm{p}}(j\omega)\right\| = \left\|\mathrm{H}_{\bar{\mathrm{g}};0\,\mathrm{p};0}(j\omega)\right\| \sim\ & \frac{O(\Delta)}{\left|j\omega-\lambda^F_{1(0)}\right|}+\frac{O(\Delta)}{\left|j\omega-\lambda^B_{1(0)}\right|}+\frac{O(\Delta\delta^2)}{\left|j\omega-\lambda^F_{1(-1)}\right|}+\frac{O(\Delta\delta^2)}{\left|j\omega-\lambda^B_{1(-1)}\right|}+\frac{O(\Delta)}{\left|j\omega-\bar{\lambda}^F_{1(0)}\right|}+\frac{O(\Delta)}{\left|j\omega-\bar{\lambda}^B_{1(0)}\right|}+\frac{O(\Delta\delta^2)}{\left|j\omega-\bar{\lambda}^F_{1(-1)}\right|}+\frac{O(\Delta\delta^2)}{\left|j\omega-\bar{\lambda}^B_{1(-1)}\right|} \\
\left\|\mathrm{H}_{\breve{\mathrm{g}}\mathrm{p}}(j\omega)\right\| = \left\|\mathrm{H}_{\mathrm{g};-1\,\mathrm{p};0}(j\omega)\right\| \sim\ & \frac{O(\delta\Delta)}{\left|j\omega-\lambda^F_{1(0)}\right|}+\frac{O(\delta\Delta)}{\left|j\omega-\lambda^B_{1(0)}\right|}+\frac{O(\delta\Delta)}{\left|j\omega-\lambda^F_{1(-1)}\right|}+\frac{O(\delta\Delta)}{\left|j\omega-\lambda^B_{1(-1)}\right|}+\frac{O(\delta\Delta)}{\left|j\omega-\bar{\lambda}^F_{1(0)}\right|}+\frac{O(\delta\Delta)}{\left|j\omega-\bar{\lambda}^B_{1(0)}\right|}+\frac{O(\delta^3\Delta)}{\left|j\omega-\bar{\lambda}^F_{1(-1)}\right|}+\frac{O(\delta^3\Delta)}{\left|j\omega-\bar{\lambda}^B_{1(-1)}\right|} \\
\left\|\mathrm{H}_{\tilde{\mathrm{g}}\mathrm{p}}(j\omega)\right\| = \left\|\mathrm{H}_{\bar{\mathrm{g}};1\,\mathrm{p};0}(j\omega)\right\| \sim\ & \frac{O(\delta)}{\left|j\omega-\lambda^F_{1(0)}\right|}+\frac{O(\delta)}{\left|j\omega-\lambda^B_{1(0)}\right|}+\frac{O(\delta^3\Delta^2)}{\left|j\omega-\lambda^F_{1(-1)}\right|}+\frac{O(\delta^3\Delta^2)}{\left|j\omega-\lambda^B_{1(-1)}\right|}+\frac{O(\delta\Delta^2)}{\left|j\omega-\bar{\lambda}^F_{1(0)}\right|}+\frac{O(\delta\Delta^2)}{\left|j\omega-\bar{\lambda}^B_{1(0)}\right|}+\frac{O(\delta)}{\left|j\omega-\bar{\lambda}^F_{1(-1)}\right|}+\frac{O(\delta)}{\left|j\omega-\bar{\lambda}^B_{1(-1)}\right|}
\end{aligned}
$$

The third and fourth dFRFs defined in equation (26) may be referred to as the modulated dFRFs, respectively [9,10]. Here, it can be concluded that:

1. $\mathrm{H}_{\mathrm{gp}}(j\omega)$ is useful to identify the strong modes,
2. $\mathrm{H}_{\hat{\mathrm{g}}\mathrm{p}}(j\omega)$ is a good indicator of degree of anisotropy, irrespective of presence of system asymmetry,
3. $\mathrm{H}_{\tilde{\mathrm{g}}\mathrm{p}}(j\omega)$ is a good indicator of degree of asymmetry, irrespective of presence of system anisotropy, and
4. $\mathrm{H}_{\breve{\mathrm{g}}\mathrm{p}}(j\omega)$ is very sensitive to the coupled effect of system anisotropy and asymmetry.

So far, for demonstration purpose, the $4N\times 4N$ reduced order matrix equation of motion has been treated, but it can be easily extended to higher order matrix equation of motion. As the reduced order increases, the accuracy of eigensolutions certainly improves. And it has been found that the $6N\times 6N$ reduced order matrix equation of motion gives sufficiently accurate eigensolutions of practical interest [11]. One of the benefits of the suggested norm order analysis is that the norm order of eigenvectors obtained from a lower order matrix equation remains unchanged as the

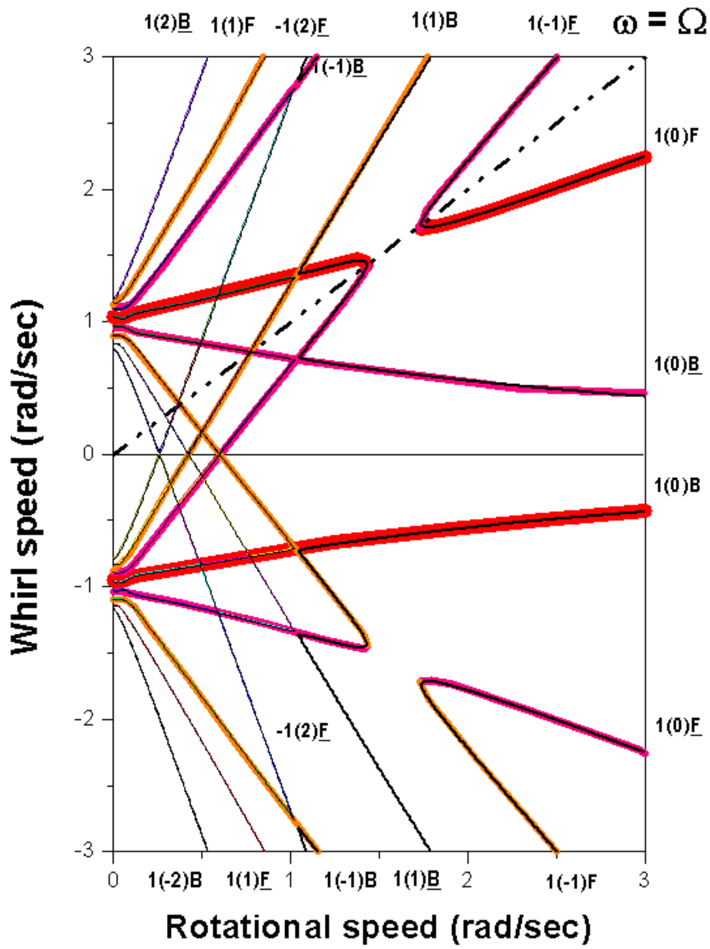
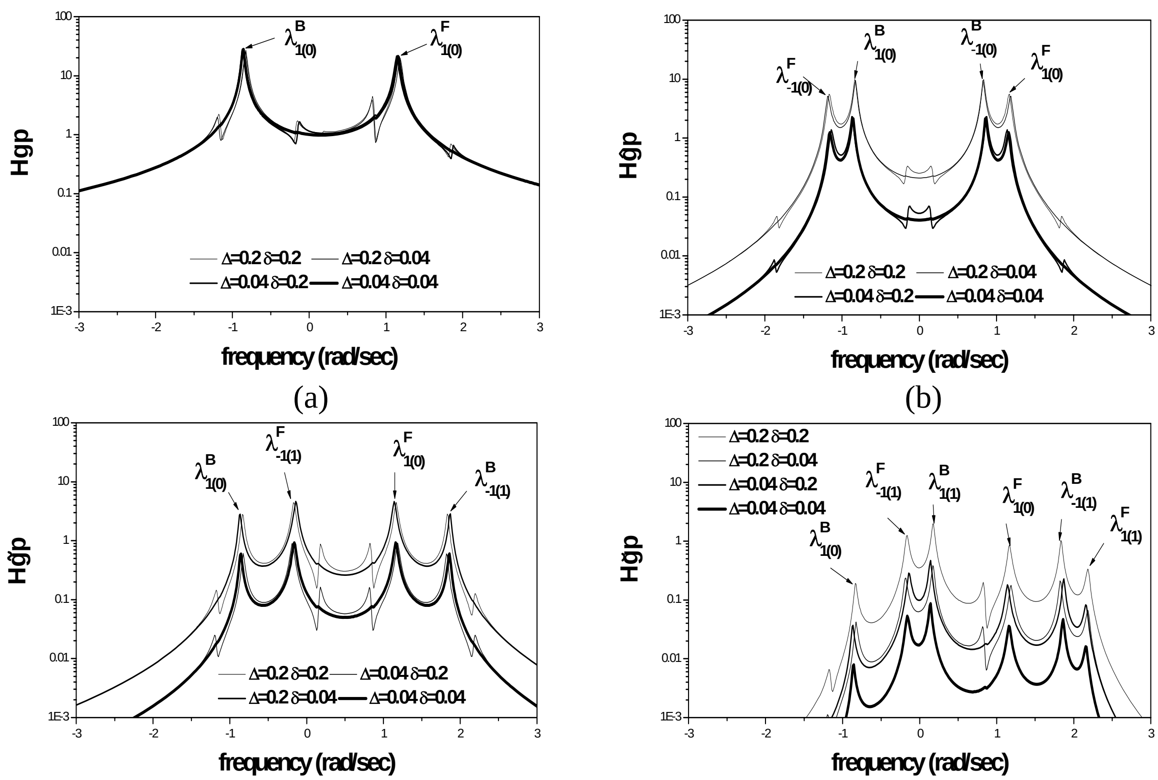


Fig. 5. Whirl speed chart for the simple gyroscopic rotor with both stationary and rotating asymmetry:
$\alpha = 0.6,\ \omega_n = 1,\ \varsigma = 0.02,\ \delta = \Delta = 0.2\ ,\Omega = 1$
($O(1)$, $O(\delta,\Delta)$, $O(\delta\Delta)$, — higher order)

matrix order increases, as shown in Table 1. In conclusion, the modal strength given in Table 1 is always valid, irrespective of the matrix order used for norm order analysis of eigenvectors.

Figure 5 shows the whirl speed chart for the simple general rotor with the parameters $\alpha = 0.6, \omega_n = 1, \varsigma = 0.02, \delta = \Delta = 0.2, \Omega = 1$. The natural frequencies were obtained from the $8N \times 8N$ reduced order matrix equation. Note that the modal strengths are indicated by the thickness of the corresponding natural frequencies, as the rotational speed is varied.

Figure 6 shows four types of dFRFs of the rotor with the degree of anisotropy and asymmetry varied. Note that the strong modes (1F, 1B), which are related with the associated isotropic system, are clearly observed in the n-dFRF (Fig. 6a), but the weak modes, which exist due to the deviatoric nature from system symmetry (isotropy), are hardly observed. The n-dFRF is not sensitive to the change in degree of asymmetry and anisotropy. On the other hand, the r-dFRF, as shown in Fig. 6b, is very sensitive to degree of anisotropy but insensitive to the asymmetry. The split of neighboring peaks in the r-dFRF is mainly due to the gyroscopic effect. Figure 6c shows that the modulated dFRF (m-dFRF) of index –1 is very sensitive to degree of asymmetry δ, but robust to the change of anisotropy. Note here that the r-dFRF and m-dFRF of index -1 are almost decoupled in the sense that the r-dFRF (the m-dFRF of index -1) remains almost unchanged due to the degree of asymmetry (anisotropy). On the other hand, as shown in Fig.6d, the m-dFRF of index 1 reflects the effect of both anisotropy and asymmetry, although the magnitude is only the order of the anisotropy times the asymmetry, i.e. $\sim O(\Delta\delta)$.



(c) (d)

Fig. 6. (a) $H_{gp}(j\omega)$**, (b)** $H_{\hat{g}p}(j\omega)$**, (c)** $H_{\tilde{g}p}(j\omega)$ **and (d)** $H_{\breve{g}p}(j\omega)$ **of the simple gyroscopic rotor with both stationary and rotating asymmetry:** $\alpha = 0.6, \omega_n = 1, \varsigma = 0.02, \Omega = 0.5$

CONCLUSIONS

A new effective method of improving the Campbell diagram by indicating the modal strengths is proposed, which is based on the norm analysis of the system eigenvectors of interest. The method is particularly useful for design and operation of general rotor systems with either stationary or rotating asymmetry. It is shown that, for the general rotor systems, modes can be classified into strong and weak modes. The strong modes are the modes that are likely to contribute significantly to the response of rotor to all possible excitation sources. The weak modes, which are less significant in response contribution than strong modes, can also be classified more in detail, depending upon the norm magnitude of the associated eigenvectors. It is demonstrated with a simple analysis rotor model that the order of strength for the weak modes can also be identified from dFRFs in consideration of the degree of the system anisotropy and asymmetry

REFERENCES

1. Dimentberg, F. M., *Flexural Vibrations of Rotating Shafts*, Butterworths, London (1961).
2. Rao, J. S., *Turbomachine Blade Vibration*, Wiley Eastern Ltd., New Delhi (1991).
3. Irretier, H., *Free and Force Vibrations of Turbine Blades*, Chapter 11.1, *Rotordynamics 2: Problems in Turbomachinery* (ed. by N. F. Rieger), Springer-Verlag (1988).
4. Lalanne, M., Ferraris, G., *Rotordynamics Prediction in Engineering*, John Wiley & Sons (1990).
5. Nelson, H. D., Crandall, S. H. C., *Analytic Prediction of Rotordynamic Response*, Chapter 2, *Handbook of Rotordynamics* (ed. by F. F. Ehrich), McGraw-Hill, Inc. (1992).
6. Lee, C. W., *Vibration Analysis of Rotors*, Kluwer Academic Publishers (1993).
7. Lancaster, P., *Lambda-Matrices and Vibrating Systems*, Pergamon Press (1966).
8. Meirovitch, L., *Principles and Techniques of Vibrations*, Chapter 5, Prentice-Hall, Inc. (1997).
9. Suh, J. H., Hong, S. W., Lee, C. W., Modal Analysis of Asymmetric Rotor System With Isotropic Stator Using Modulated Coordinates, J. Sound and Vibration, Vol. 284 (2005) 651-671.
10. Lee, C. W., Suh, J. H., Hong, S. W., Modal Analysis of Rotor System Using Modulated Coordinates for Asymmetric Rotor System With Anisotropic Stator, 8th International Conference on Vibrations in Rotating Machinery, Swansea (September 2004).
11. Lee, C. W., Han, D. J., Suh, J. H., Hong, S. W., Modal Analysis of Periodically Time-Varying Linear Rotor Systems, submitted to J. Sound and Vibration.

Appendix: Nomenclature, Subscripts/Superscripts, Symbols and Abbreviations

Nomenclature

$\mathbf{D}(\lambda)=\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}$: lambda matrix of degree 2.

$\mathbf{f}=\begin{bmatrix}\mathbf{g}^T & \bar{\mathbf{g}}^T\end{bmatrix}^T$: $2N\times1$ complex input vector.

$\mathbf{f}_\mathbf{y}(t)$, $\mathbf{f}_\mathbf{z}(t)$: real valued input vectors.

$\mathbf{g}(t)=\mathbf{f}_\mathbf{y}(t)+j\mathbf{f}_\mathbf{z}(t)$: $N\times1$ complex input vector.

$\mathbf{H}_{\mathbf{gp}}$, $\mathbf{H}_{\hat{\mathbf{g}}\mathbf{p}}$: normal (n-dFRM), reverse (r-dFRM), modulated (m-dFRM) directional frequency response matrices, defined in Eqs. (12b), (16) and (26).

$\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$: complex valued $2N\times2N$ generalized mass, damping and stiffness matrices.

$\mathbf{M}_i,\mathbf{C}_i,\mathbf{K}_i$: complex valued $N\times N$ generalized mass, damping and stiffness matrices.

N : dimension of the complex coordinate vector.

$\mathbf{P}(j\omega)$, $\mathbf{G}(j\omega)$, $\hat{\mathbf{G}}(j\omega)$: Fourier transforms of $\mathbf{p}(t)$, $\mathbf{g}(t)$, $\bar{\mathbf{g}}(t)$.

$\mathbf{p}(t)=\mathbf{y}(t)+j\mathbf{z}(t)$: $N\times1$ complex response vector.

$\mathbf{q}=\begin{bmatrix}\mathbf{p}^T & \bar{\mathbf{p}}^T\end{bmatrix}^T$: $2N\times1$ complex response vector.

$\mathbf{u}$, $\hat{\mathbf{u}}$: modal vectors associated with $\mathbf{p}$, $\bar{\mathbf{p}}$.

$\mathbf{u}_c=\begin{Bmatrix}\mathbf{u}^T & \hat{\mathbf{u}}^T\end{Bmatrix}^T$: right latent vector of $\mathbf{D}(\lambda)$.

$\mathbf{v}$, $\hat{\mathbf{v}}$: adjoint vectors associated with $\mathbf{p}$, $\bar{\mathbf{p}}$.

$\bar{\mathbf{v}}_c=\begin{Bmatrix}\bar{\mathbf{v}}^T & \bar{\hat{\mathbf{v}}}^T\end{Bmatrix}^T$: left latent vector of $\mathbf{D}(\lambda)$.

$\mathbf{y}(t)$, $\mathbf{z}(t)$: real valued response vectors.

α : gyroscopic moment effect.

δ : perturbation due to presence of asymmetry.

Δ : perturbation due to presence of annisotropy.

ς : damping ratio.

$\eta(t)$: principal coordinate.

$\lambda=\sigma+j\omega$: latent root, eigenvalue.

Ω : rotational speed of the shaft.

Subscripts

$\mathbf{f}$, $\mathbf{b}$: mean and deviatoric values, respectively.

$r(m)$: mode number.

Superscripts

$i = F, B$: forward and backward modes.

Symbols

$\|\mathbf{a}\|=\sqrt{\mathbf{a}^T\mathbf{a}}$: Euclidean norm of vector $\mathbf{a}$

$\|\mathbf{A}\|=\max\limits_{\forall\,\|\mathbf{a}\|\neq\mathbf{0}}\dfrac{\|\mathbf{Aa}\|}{\|\mathbf{a}\|}$: matrix norm subordinate to vector norm

$\tilde{\mathbf{a}}$: reduced order vector.

$\tilde{\mathbf{A}}$: reduced order constant matrix transformed from time varying matrix $\mathbf{A}(t)$.

$\underline{B}$, $\underline{F}$: complex conjugates of backward and forward modes.

$O(a)$: order of smallness a.

$O(a,b) = O(a) + O(b)$

$O(ab) = O(a)O(b)$

□ : approximation.

Abbreviations

dFRF (dFRM) : directional frequency response function (matrix).

HP: high pressure spool.

LP: low pressure spool.

n-, r-, m-dFRF: normal, reverse, modulated dFRFs.

NPF: nozzle passing frequency.