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Modal Analysis of Periodically Time-varying Linear Rotor Systems using Floquet Theory

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ABSTRACT

General rotor systems possess both stationary and rotating asymmetric properties, whose equation of motion is characterized by the presence of periodically time-varying parameters with the period of half the rotation. This paper employs the Floquet theory to develop the complex modal analysis method for periodically time-varying linear rotor systems. The approach is based on decomposition of state transition matrix, leading to the periodically time-varying eigensolutions. The modal analysis approach using the Floquet theory is then compared with the modulated coordinate transform method, which transforms the finite order time-varying matrix equation into an equivalent infinite order time-invariant linear equation.

KEY WORDS

general rotor system, Floquet theory, Hill's infinite order matrix, directional frequency response function

1 INTRODUCTION

According to the mechanical properties of the rotor and stator parts, rotor systems may be classified into four types [6, 10]: isotropic (symmetric) rotor system - both rotor and stator parts are axisymmetric; anisotropic rotor system - the rotor part is axisymmetric but the stator part is not; asymmetric rotor system - the stator part is axisymmetric but the rotor is not; general rotor system - neither rotor nor stator parts are axisymmetric. The general rotor system, an asymmetric rotor with anisotropic stator, thus reveals the coupled effects of anisotropic and asymmetric rotor systems.

The asymmetric (anisotropic) rotor system may look like a periodically time-varying linear system when the equation of motion is written in the stationary (rotating) coordinates, but it can be easily transformed into a time-invariant linear system by rewriting the equation of motion in the rotating (stationary) coordinates. Thus the asymmetric (anisotropic) rotor system alone essentially reduces to a time-invariant linear system, whose modal analysis scheme is well established in the literature [6, 10, 11]. On the other hand, the general rotor system, which is inherently a periodically time-varying linear system, can not be transformed into a time-invariant system by such a simple coordinate transformation. It leads to a difficulty in modal analysis of the general rotor system due to mathematical complexity associated with treatment of periodically time-varying system matrices.

Many attempts have been made for dynamic analysis of periodically time-varying linear systems by employing the periodically time-varying eigenvectors [3, 9, 13, 14]. Since these methods were strictly based on the time domain analysis, the stability analysis was of major concern. For example, Sinha, et al. [14] proposed use of the Liapunov-Floquet transformation for expanding the periodic system matrices in terms of the shifted Chebyshev polynomials of a same period, so that the original differential problem reduces to a set of linear algebraic equations. However, their method is still limited to enhancement of the stability analysis. Calico and Wiesel [2] applied the Floquet theory to develop a modal analysis method for periodically time-varying control systems, introducing the periodically time-varying eigenvectors derived from periodicity of the state transition matrix. Although their modal analysis method is mathematically sound, it requires the accurate integration of the state transition matrix over a period and it lacks the natural extension to the frequency domain analysis.

There are few investigations on complete modal analysis of periodically time-varying systems valid in both time and frequency domains. The major difficulty is because the conventional modal analysis developed for linear time-invariant systems cannot be directly applicable to linear time-varying systems, unless they can be transformed into an equivalent time-invariant system [3, 14]. Irretier [7] developed a mathematical foundation for modal testing of periodically time-varying rotor systems by expanding the periodically time-varying modal vectors in Fourier series and introducing an intuitive, but not rigorously proven, relation between modal parameters. In addition, although the resulting mathematical treatments are found to be correct, neither the computational procedure for eigensolutions nor the frequency domain analysis for modal testing was described.

For asymmetrical rotors with isotropic stators, the periodically time-varying linear differential equation expressed in the stationary coordinates can be transformed to the time-invariant linear differential equation expressed in the rotating coordinates or in the modulated stationary coordinates [15]. Then the modal analysis becomes essentially the same as the ordinary complex modal analysis method developed for anisotropic rotors, which possess asymmetric properties only in the stator part [8, 12]. On the other hand, the asymmetric rotor system with anisotropic stator cannot be transformed to a finite order equation of motion with the time-invariant parameters by coordinate transformation only.

In this paper, the complex modal analysis of periodically time-varying linear rotor systems is developed using the Floquet theory and its computational efficiency in calculation of eigensolutions is discussed, compared with the use of the reduced order Hill's matrix, which results from the modulated coordinate transform approach.

2 COMPLEX MODAL ANALYSIS OF PERIODICALLY TIME-VARYING ROTOR SYSTEMS

2.1 Equation of motion in complex form

For a rotor system with rotating and stationary asymmetry, the equation of motion can be conveniently written in the complex stationary coordinates, referring to Fig.1, as [6, 10, 4, 1]

$$\mathbf{M_f}\,\ddot{\mathbf{p}}(t)+\mathbf{C_f}\,\dot{\mathbf{p}}(t)+\mathbf{K_f}\,\mathbf{p}(t)+\{\mathbf{M_b}\,\ddot{\bar{\mathbf{p}}}(t)+\mathbf{C_b}\,\dot{\bar{\mathbf{p}}}(t)+\mathbf{K_b}\,\bar{\mathbf{p}}(t)\}+e^{j2\Omega t}\,\{\mathbf{M_r}\ddot{\bar{\mathbf{p}}}(t)+\mathbf{C_r}\dot{\bar{\mathbf{p}}}(t)+\mathbf{K_r}\bar{\mathbf{p}}(t)\}=\mathbf{g}(t) \tag{1}$$

Here, the $N\times 1$ complex response and force vectors, $\mathbf{p}(t)$ and $\mathbf{g}(t)$, defined by the real response vectors, $\mathbf{y}$(t) and $\mathbf{z}$(t), and the real excitation vectors, $\mathbf{f}_y(\mathrm{t})$ and $\mathbf{f}_z(\mathrm{t})$, respectively, are

$$\mathbf{p}(t)=\mathbf{y}(t)+j\,\mathbf{z}(t),\;\; \bar{\mathbf{p}}(t)=\mathbf{y}(t)-j\,\mathbf{z}(t),\;\; \mathbf{g}(t)=\mathbf{f}_y(t)+j\,\mathbf{f}_z(t),\;\; \bar{\mathbf{g}}(t)=\mathbf{f}_y(t)-j\,\mathbf{f}_z(t), \tag{2}$$

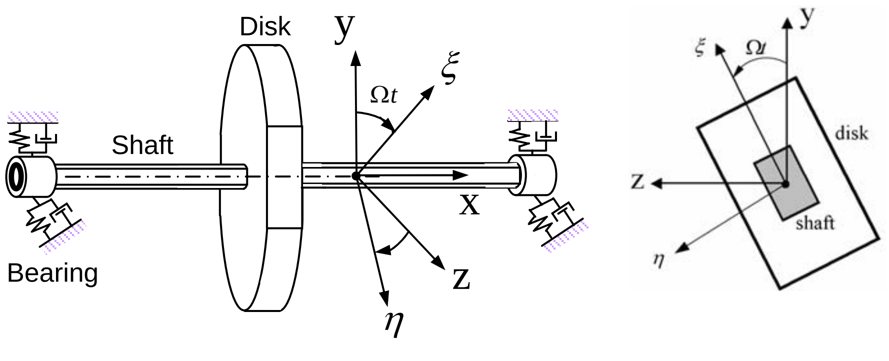


Figure 1: General rotor system: simple analysis model

where j means the imaginary number; N is the dimension of the complex coordinate vector**;** $\mathbf{g}(t)$ includes the force and moment; Ω is the rotational speed; '-' denotes the complex conjugate; $\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{K}_i$ denote the complex valued $N\times N$ generalized mass, damping and stiffness matrices, respectively; and the subscripts **f**, and, **b** and **r** refer to the mean value, and, the deviatoric values for anisotropy (stationary asymmetry) and asymmetry (rotating asymmetry), respectively. For an isotropic rotor, $\mathbf{C_b} = \mathbf{K_b} = \mathbf{M_r} = \mathbf{C_r} = \mathbf{K_r} = \mathbf{0}$; for an anisotropic rotor, $\mathbf{M_r} = \mathbf{C_r} = \mathbf{K_r} = \mathbf{0}$; and, for an asymmetric rotor, $\mathbf{C_b} = \mathbf{K_b} = \mathbf{0}$. Note here that the periodically time-varying terms, which are preceded by $e^{j2\Omega t}$ in Eq. (1), inherently appear, as both rotating and stationary asymmetries exist in the system and that Eq. (1) includes the external and internal damping, gyroscopic moment and Coriolis effect. When either rotating or stationary asymmetry does not exist, the equation of motion becomes, or it can be transformed to, a time-invariant differential equation.

In the following section, the Floquet theory is applied for complex modal analysis of the periodically time-varying linear rotor system (1). And its computational efficiency in calculating the unbalance response and the dFRFs is discussed.

2.2 Complex modal solution by Floquet theory

(1) Eigenvalue and adjoint problems

From Eq. (1) and its complex conjugate form, the complex equation of motion can be constructed as

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) + \mathbf{C}(t)\dot{\mathbf{q}}(t) + \mathbf{K}(t)\mathbf{q}(t) = \mathbf{f}(t), \tag{3}$$

where

$$\mathbf{q}(t) = \left\{\mathbf{p}^T(t) \quad \bar{\mathbf{p}}^T(t)\right\}^T , \quad \mathbf{f}(t) = \left\{\mathbf{g}^T(t) \quad \bar{\mathbf{g}}^T(t)\right\}^T ,$$

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{M_f} & \mathbf{M_r}e^{j2\Omega t} \\ \bar{\mathbf{M}}_\mathbf{r}e^{-j2\Omega t} & \bar{\mathbf{M}}_\mathbf{f} \end{bmatrix}, \; \mathbf{C}(t) = \begin{bmatrix} \mathbf{C_f} & \mathbf{C_b} + \mathbf{C_r}e^{j2\Omega t} \\ \bar{\mathbf{C}}_\mathbf{b} + \bar{\mathbf{C}}_\mathbf{r}e^{-j2\Omega t} & \bar{\mathbf{C}}_\mathbf{f} \end{bmatrix}, \; \mathbf{K}(t) = \begin{bmatrix} \mathbf{K_f} & \mathbf{K_b} + \mathbf{K_r}e^{j2\Omega t} \\ \bar{\mathbf{K}}_\mathbf{b} + \bar{\mathbf{K}}_\mathbf{r}e^{-j2\Omega t} & \bar{\mathbf{K}}_\mathbf{f} \end{bmatrix} \tag{4}$$

Equation (3) can be rewritten in the state space form as

$$\mathbf{A}(t)\dot{\mathbf{w}}(t) = \mathbf{B}(t)\mathbf{w}(t) + \mathbf{F}(t), \tag{5}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{M}(t) \\ \mathbf{M}(t) & \mathbf{C}(t) \end{bmatrix}, \; \mathbf{B}(t) = \begin{bmatrix} \mathbf{M}(t) & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}(t) \end{bmatrix}, \; \mathbf{w}(t) = \begin{Bmatrix} \dot{\mathbf{q}}(t) \\ \mathbf{q}(t) \end{Bmatrix}, \; \mathbf{F}(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{Bmatrix}. \tag{6}$$

Utilizing the Floquet theory for this periodically time-varying system in homogeneous part of Eq. (5) with the period $T = \pi / \Omega$, we can express the $4N\times 1$ complex state vector, $\mathbf{w}(t)$, in terms of the state transition matrix, $\mathbf{\Phi}(t,t)$, as [2, 3, 9, 13, 14]

$$\mathbf{w}(t) = \mathbf{\Phi}(t,0)\,\mathbf{w}(0), \tag{7}$$

where $\Phi(t,0)$ satisfies the differential equation, subject to the initial condition $\Phi(0,0) = \mathbf{I}_{4N\times 4N}$,

$$\dot{\mathbf{\Phi}}(t,0) = \left[\mathbf{A}^{-1}(t)\mathbf{B}(t)\right]\mathbf{\Phi}(t,0) \tag{8}$$

and the matrix decomposition relation given, subject to $\mathbf{R}(0) = \mathbf{R}(T)$, by [9, 2]

$$\mathbf{\Phi}(t,0) = \mathbf{R}(t)\, e^{\mathbf{J}t}\, \mathbf{R}^{-1}(0). \tag{9}$$

Here, $\mathbf{J}$ is the Jordan normal form of matrix, whose diagonal entries, $\mu_i, i = 1, 2, ..., 4N$, are termed Poincare exponents, equivalent to the eigenvalues for time-invariant systems.

Substituting $t = T$ into Eq. (9), we obtain

$$\mathbf{\Phi}(T,0) = \mathbf{R}(0)\, e^{\mathbf{J}T}\, \mathbf{R}^{-1}(0) \text{, or, equivalently, } \left[\mathbf{\Phi}(T,0) - \omega_i\, \mathbf{I}\right]\mathbf{R}(0) = \mathbf{0}. \tag{10}$$

It implies that $\omega_i = e^{\mu_i T}$ and $\mathbf{R}(0)$ are the eigenvalues (characteristic multipliers) and the corresponding matrix of eigenvectors, respectively, of the monodromy matrix $\mathbf{\Phi}(T,0)$. Substituting Eq. (9) into Eq. (8), we obtain

$$\dot{\mathbf{R}}(t) = \left[\mathbf{A}^{-1}(t)\mathbf{B}(t)\right]\mathbf{R}(t) - \mathbf{R}(t)\mathbf{J} \quad \text{or, equivalently, } \dot{\mathbf{r}}(t) = \left[\mathbf{A}^{-1}(t)\mathbf{B}(t) - \mu\mathbf{I}\right]\mathbf{r}(t) \tag{11}$$

where $\mathbf{r}(t)$ is a column vector of $\mathbf{R}(t)$.

Now we can construct the adjoint problem, introducing the adjoint state vector $\mathbf{z}(t)$, to the original system (5) with $\mathbf{F}(t) = \mathbf{0}$ as [2, 9]

$$\dot{\mathbf{z}}(t) = -\left[\bar{\mathbf{A}}^{-1}(t)\bar{\mathbf{B}}(t)\right]^T \mathbf{z}(t)\ , \tag{12}$$

from which we can define the adjoint matrix $\bar{\mathbf{L}}(t)$ such that

$$\dot{\bar{\mathbf{L}}}(t) = -\left[\bar{\mathbf{A}}^{-1}(t)\bar{\mathbf{B}}(t)\right]^T \bar{\mathbf{L}}(t) + \bar{\mathbf{L}}(t)\ \bar{\mathbf{J}},\ \text{ or, equivalently, }\ \dot{\bar{\mathbf{l}}}(t) = -\left[\left\{\bar{\mathbf{A}}^{-1}(t)\bar{\mathbf{B}}(t)\right\}^T - \bar{\mu}\,\mathbf{I}\right]\bar{\mathbf{l}}(t) \tag{13}$$

where $\bar{\mathbf{l}}(t)$ is a column vector of $\bar{\mathbf{L}}(t)$.

We can rewrite Eq. (11), using the identity relation $\mathbf{R}(t)\mathbf{R}^{-1}(t) = \mathbf{I}$, as

$$\dot{\mathbf{R}}^{-1}(t) = -\mathbf{R}^{-1}(t)\left[\mathbf{A}^{-1}(t)\mathbf{B}(t)\right] + \mathbf{J}\mathbf{R}^{-1}(t)\ . \tag{14}$$

From direct comparison of Eq. (13) with Eq. (14), we can obtain the biorthonormality condition as [2]

$$\mathbf{L}^T(t)\mathbf{R}(t) = \mathbf{I}_{4N\times 4N} \quad \text{or, equivalently,} \quad \mathbf{l}_i^T(t)\mathbf{r}_j(t) = \delta_{ij}\ , \quad ;\ i, j = 1 \text{ to } 4N, \tag{15}$$

where δ_{ij} is the Kronecker delta, the superscript T means the transpose, and, $\mathbf{r}_i(t)$ and $\mathbf{l}_i(t)$ are the i-th column vector of $\mathbf{R}(t)$ and $\mathbf{L}(t)$, respectively.

(2) Structure of the eigenvectors and the adjoint vectors

Substituting the relation $\mathbf{w}(t) = \left\{\dot{\mathbf{q}}^T(t) \quad \mathbf{q}^T(t)\right\}^T = \mathbf{r}(t)\eta(t)$ with $\mathbf{q}(t) = \mathbf{u_c}(t)\eta(t)$ and Eq. (11) into the homogeneous part of Eq. (5), we obtain the relation given by

$$\mathbf{r}(t) = \left\{\dot{\mathbf{u}}_\mathbf{c}^T(t) + \mu\mathbf{u}_\mathbf{c}^T(t) \quad \mathbf{u}_\mathbf{c}^T(t)\right\}^T\ . \tag{16}$$

Likewise, substituting the relation $\mathbf{z}(t) = \bar{\mathbf{l}}(t)\zeta(t)$ with $\bar{\mathbf{l}}(t) = \bar{\mathbf{A}}^T(t)\bar{\boldsymbol{l}}(t)$ and Eq. (13) into the adjoint equation (12), we obtain the relation given by

$$\boldsymbol{l}(t) = \begin{Bmatrix} -\dot{\bar{\mathbf{v}}}_\mathbf{c}(t) + \left\{\mu - \left[\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)\right]^T\right\}\bar{\mathbf{v}}_\mathbf{c}(t) \\ \bar{\mathbf{v}}_\mathbf{c}(t) \end{Bmatrix}. \tag{17}$$

Here, the modal and the adjoint vectors are composed, respectively, of

$$\mathbf{u_c}(t) = \left\{\mathbf{u}^T(t) \quad \hat{\mathbf{u}}^T(t)\right\}^T,\ \ \mathbf{v}_\mathbf{c}^T(t) = \left\{\mathbf{v}^T(t) \quad \hat{\mathbf{v}}^T(t)\right\}^T, \tag{18}$$

and, for the complex equation of motion as in Eq.(1), it holds, in general,

$$\hat{\mathbf{u}}(t) \neq \bar{\mathbf{u}}(t)\ , \quad \hat{\mathbf{v}}(t) \neq \bar{\mathbf{v}}(t)\ . \tag{19}$$

Note that the eigenvector $\mathbf{r}(t)$ (and thus $\mathbf{u_c}(t)$) and the adjoint vector $\boldsymbol{l}(t)$ (and thus $\bar{\mathbf{v}}_\mathbf{c}(t)$) are periodically time-varying vectors with the period $T = \pi / \Omega$.

For the time-invariant system, i.e. $\mathbf{r}(t) = \mathbf{r}$, $\bar{\mathbf{l}}(t) = \bar{\mathbf{A}}^T\,\bar{\boldsymbol{l}}$, $\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{B}(t) = \mathbf{B}$, Eqs. (11) and (13), and, Eqs. (16) and (17) reduce to the form of

$$\mu\mathbf{A}\mathbf{r} = \mathbf{B}\mathbf{r}\ , \qquad \bar{\mu}\,\bar{\mathbf{A}}^T\bar{\boldsymbol{l}} = \bar{\mathbf{B}}^T\bar{\boldsymbol{l}}\ , \tag{20}$$

and

$$\mathbf{r} = \left\{\mu\mathbf{u}_\mathbf{c}^T \quad \mathbf{u}_\mathbf{c}^T\right\}^T, \qquad \boldsymbol{l} = \left\{\mu\bar{\mathbf{v}}_\mathbf{c}^T \quad \bar{\mathbf{v}}_\mathbf{c}^T\right\}^T, \qquad \mathbf{u_c} = \left\{\mathbf{u}^T \quad \hat{\mathbf{u}}^T\right\}^T, \qquad \mathbf{v_c} = \left\{\mathbf{v}^T \quad \hat{\mathbf{v}}^T\right\}^T, \tag{21}$$

respectively, which are consistent with the previous results in [11].

(3) Modal equations and eigensolutions

The complex state and adjoint vectors, **w**(t) and **z**(t), can be expanded in terms of the eigenvectors and the adjoint vectors, respectively, for the rotor system (5), as

$$\mathbf{w}(t)=\sum_{r=1}^{4N}\left\{\mathbf{r}(t)\eta(t)\right\}_r=\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left\{\mathbf{r}(t)\eta(t)\right\}_r^i\ ,\quad \mathbf{z}(t)=\sum_{r=1}^{4N}\left\{\bar{\mathbf{l}}(t)\zeta(t)\right\}_r=\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left\{\bar{\mathbf{l}}(t)\zeta(t)\right\}_r^i\ , \tag{22}$$

where the prime notation in the summation implies exclusion of $r=0$, $\eta(t)$ and $\zeta(t)$ are the principal coordinates of the original and adjoint systems, respectively, and the superscripts B and F refer to the backward and forward modes, respectively, following the well-established convention for mode classification in rotordynamics [11]. Substituting Eq. (22) into Eq. (5), using the relation (11), premultiplying by $\bar{\boldsymbol{l}}_s^{kT}$ and using the biorthonormality condition (15), we can obtain the 4N sets of complex modal equations of motion as

$$\dot{\eta}_r^i(t)=\mu_r^i\eta_r^i(t)+\bar{\mathbf{v}}_{\mathbf{cr}}^{iT}(t)\mathbf{f}(t)=\mu_r^i\eta_r^i(t)+\bar{\mathbf{v}}_r^{iT}(t)\mathbf{g}(t)+\bar{\hat{\mathbf{v}}}_r^{iT}(t)\bar{\mathbf{g}}(t)\ ;\ r=\pm1,\pm2,\ldots,\pm N\ ;\ i=B,F. \tag{23}$$

Recalling the Floquet theory that, from the one periodic solution, the entire time response of the eigensolutions can be expressed periodically with the base of that period, we can expand the eigenvector $\mathbf{u_c}(t)$ and the adjoint vector $\mathbf{v_c}(t)$ in Eq. (18) by Fourier series as [7]

$$\mathbf{u}_r^i(t)=\sum_{m=-\infty}^{\infty}\mathbf{u}_{r(m)}^i e^{j2m\Omega t},\ \hat{\mathbf{u}}_r^i(t)=\sum_{m=-\infty}^{\infty}\hat{\mathbf{u}}_{r(m)}^i e^{j2m\Omega t},\ \mathbf{v}_r^i(t)=\sum_{m=-\infty}^{\infty}\mathbf{v}_{r(m)}^i e^{j2m\Omega t},\ \hat{\mathbf{v}}_r^i(t)=\sum_{m=-\infty}^{\infty}\hat{\mathbf{v}}_{r(m)}^i e^{j2m\Omega t} \tag{24}$$

where $\mathbf{u}_{r(m)}^i$, $\hat{\mathbf{u}}_{r(m)}^i$, $\mathbf{v}_{r(m)}^i$ and $\hat{\mathbf{v}}_{r(m)}^i$ are the complex Fourier coefficient vectors associated with the complex harmonic function of $e^{j2m\Omega t}$.

From Eqs. (23) and (24), we can obtain the forced response of the general rotor system (5) as

$$\mathbf{p}(t)=\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left\{\mathbf{u}(t)\eta(t)\right\}_r^i=\sum_{i=B,F}\sum_{r=-N}^{N}{}'\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\int_0^t e^{(\mu_r^i+j2m\Omega)(t-\tau)}\left[\mathbf{u}_{r(m)}^i\bar{\mathbf{v}}_{r(m-n)}^{iT}\mathbf{g}_{;n}(\tau)+\mathbf{u}_{r(m)}^i\bar{\hat{\mathbf{v}}}_{r(m-n)}^{iT}\bar{\mathbf{g}}_{;-n}(\tau)\right]\mathrm{d}\tau\right\} \tag{25}$$

where

$$\mathbf{g}_{;n}(t)=\mathbf{g}(t)e^{j2n\Omega t}.$$

(4) Direct numerical solution method

In this direct modal analysis approach for the periodically time-varying parameter system (1), the eigenvalues and the corresponding periodically time-varying eigenvectors can be analytically obtained from Eqs. (8) to (15) at least in theory. However, the closed form solutions are limited only to a few simple cases because of mathematical complexity. For most of practical applications, numerical approach is, instead, taken as follows. First, the monodromy matrix, $\Phi(T,0)$, is obtained by numerical integration of Eq. (8) with respect to time for given $\mathbf{A}(t)$, $\mathbf{B}(t)$ and initial condition $\Phi(0,0)=\mathbf{I}_{4N\times4N}$. Second, the characteristic multipliers ω_i and the corresponding matrix of eigenvectors **R**(0) of $\Phi(T,0)$ are calculated from Eq.(10). Then, the Jordan normal form of matrix **J** is formed with its diagonal entries $\mu_i=\log(\omega_i)/T$ and **R**(t) can be solved by numerical integration of Eq. (11) with the initial condition **R**(0). The same procedure applies to the adjoint matrix **L**(t) using **R**(t) and the biorthonormality condition (15). Note that, once the periodically time-varying modal (adjoint) vectors are obtained, calculation of the Fourier coefficient modal (adjoint) vectors, which are constant vectors, in Eq. (24) becomes straightforward.

Although the above procedure looks like a novel, analytical approach, one of its critical drawbacks is the numerical instability, since it suffers from serious accumulated error with extensive numerical integration processes [5]. For example, the complex Fourier coefficient modal (adjoint) vectors are very vulnerable to the numerical errors with **R**(t) and **L**(t).

(5) Infinite order directional frequency response matrix (dFRM)

Fourier transforming Eq. (25), we obtain

$$\mathbf{P}(j\omega) = \sum_{n=-\infty}^{\infty} \left\{ \left[\sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \frac{\mathbf{u}_{r(m)}^{i} \bar{\mathbf{v}}_{r(m-n)}^{iT}}{j(\omega - 2m\Omega) - \mu_r^i} \right] \mathbf{G}_{;n}(j\omega) + \left[\sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \frac{\mathbf{u}_{r(m)}^{i} \bar{\hat{\mathbf{v}}}_{r(m-n)}^{iT}}{j(\omega - 2m\Omega) - \mu_r^i} \right] \hat{\mathbf{G}}_{;-n}(j\omega) \right\} \tag{26a}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \mathbf{H}_{\mathbf{g}_{;-n}\mathbf{p}}(j\omega) \mathbf{G}_{;n}(j\omega) + \mathbf{H}_{\bar{\mathbf{g}}_{;-n}\mathbf{p}}(j\omega) \hat{\mathbf{G}}_{;-n}(j\omega) \right\}$$

where $\mathbf{P}(j\omega)$, $\mathbf{G}(j\omega)$ and $\hat{\mathbf{G}}(j\omega)$ are the Fourier transforms of $\mathbf{p}(t)$, $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively, and it holds

$$\mathbf{G}_{;n}(j\omega) = \mathbf{G}\left\{ j(\omega - 2n\Omega) \right\}, \quad \hat{\mathbf{G}}_{;n}(j\omega) = \hat{\mathbf{G}}\left\{ j(\omega + 2n\Omega) \right\}. \tag{26b}$$

Although there are still an infinite number of dFRMs in Eq. (26), we introduce four dFRMs, that are important in characterizing the system asymmetry and anisotropy, as

$$\mathbf{H}_{\mathbf{gp}}(j\omega) = \mathbf{H}_{\mathbf{g}_{;0}\mathbf{p}}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}_{r(m)}^{i} \bar{\mathbf{v}}_{r(m)}^{iT}}{j(\omega - 2m\Omega) - \mu_r^i} \right], \quad \mathbf{H}_{\hat{\mathbf{g}}\mathbf{p}}(j\omega) = \mathbf{H}_{\bar{\mathbf{g}}_{;0}\mathbf{p}}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}_{r(m)}^{i} \mathbf{v}_{-r(-m)}^{iT}}{j(\omega - 2m\Omega) - \mu_r^i} \right] \tag{27}$$

$$\mathbf{H}_{\tilde{\mathbf{g}}\mathbf{p}}(j\omega) = \mathbf{H}_{\bar{\mathbf{g}}_{;-1}\mathbf{p}}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}_{r(m)}^{i} \mathbf{v}_{-r(-m+1)}^{iT}}{j(\omega - 2m\Omega) - \mu_r^i} \right], \quad \mathbf{H}_{\breve{\mathbf{g}}\mathbf{p}}(j\omega) = \mathbf{H}_{\mathbf{g}_{;-1}\mathbf{p}}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}_{r(m)}^{i} \bar{\mathbf{v}}_{r(m-1)}^{iT}}{j(\omega - 2m\Omega) - \mu_r^i} \right]$$

Here, $\mathbf{H}_{\mathbf{gp}}(j\omega)$ is referred to as the normal dFRM that represents the system symmetry, $\mathbf{H}_{\hat{\mathbf{g}}\mathbf{p}}(j\omega)$ is referred to as the reverse dFRM that represents the effect of system anisotropy, and, $\mathbf{H}_{\tilde{\mathbf{g}}\mathbf{p}}(j\omega)$ and $\mathbf{H}_{\breve{\mathbf{g}}\mathbf{p}}(j\omega)$ are the modulated dFRMs that represent the effect of system asymmetry and the coupled effect of system anisotropy and asymmetry, respectively.

2.3 Complex modal solution by coordinate transformation

It has been proven that introduction of complex modulated coordinates successfully transforms the finite order time-varying matrix equation into an equivalent infinite order time-invariant linear equation, leading to an infinite set of constant eigensolutions [6]. Then the modal analysis of the time-invariant linear system becomes straightforward, defining the Hill's infinite order matrix.

It can be easily shown that the modulated coordinate transform approach leads to the expressions for the dFRMs identical to Eqs.(26) and (27). However, the numerical procedures for the dFRM estimates are different from each other. In particular, the truncation schemes of the infinite series expansion (or, equivalently, the infinite summation) with respect to the index *m* are differenent. For example, truncation of the infinite summation is done with the complex Fourier series expansion of the periodically time-varying eigenvectors, Eq.(24), for the Floquet approach, whereas the order reduction is done with the Hill's infinite order matrix for the coordinate transform approach [6].

3 NUMERICAL EXAMPLE

This section demonstrates and compares the modal analysis methods with a simple, yet general rotor system model, which consists of a rigid rotor with asymmetric mass moments of inertia, a massless shaft with asymmetric shaft stiffnesses, and two orthotropic bearings, as shown in Fig. 1. The detailed descriptions of the rotor model are treated in [6, 12].

The eigensolutions of the analysis rotor model were calculated by using the Floquet approach with the three term approximation (the index *m* was taken to be -1, 0 and 1 in Eq. (24)) for the complex Fourier series expansion of the time-vaying eigenvectors. The calculated eigenvalues were in good agreement with those from the reduced Hill's matrix of order 24, which was found to yield fairly accurate results [6].

Figure 2 compares the typical unbalance responses of the rotor calculated by four different methods. Note that both the reduced Hill's matrix of order 24 and the Floquet's method with three term approximation yielded non-distinguishable results from the exact numerical one. Both methods are found to be superior in calculation of unbalance response to the conventional harmonic balance method [4], as shown in Fig. 2.

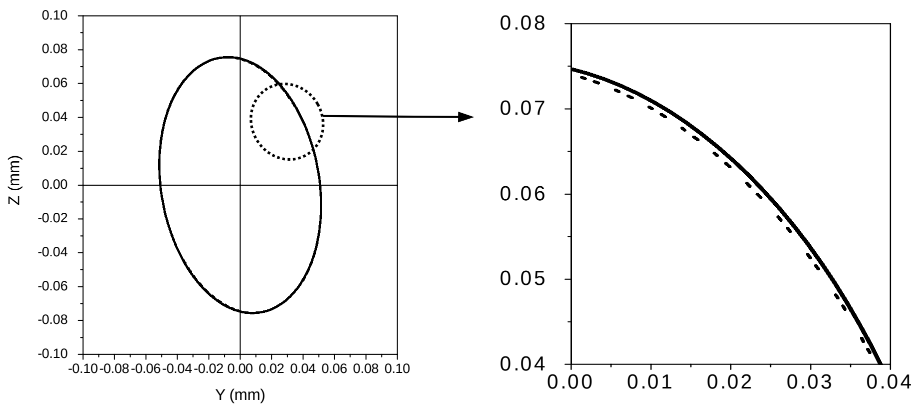


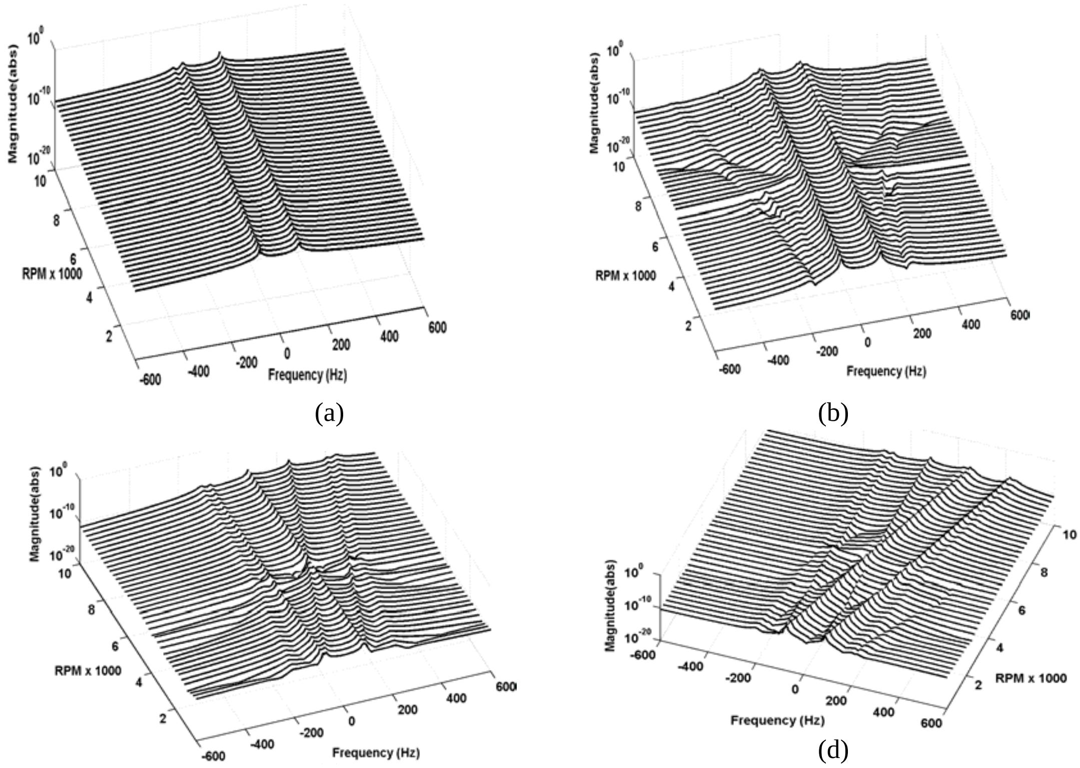
Figure 2: Unbalance response at 4,000 rpm ($\delta = \Delta = 0.3$): ——— Numerical integration; ——— Hill's matrix of order 24; —— Floquet's method; -- -- Harmonic balance method

Figures 3 and 4 compare the waterfall plots for the magnitude of the four important dFRFs (given in Eq.(27)) obtained by the Floquet method using the three term approximation and the reduced Hill's matrix of order 24, respectively, as the rotational speed is varied up to 10,000 rpm for the analysis rotor model with $\delta = \Delta = 0.3$. The number of assumed modes used for calculation of dFRFs is kept unchanged for the latter method, but it varies for the former method, depending upon the type of dFRFs, due to the inherent nature of approximation. It often leads to relatively large discrepancies in the logarithmically scaled dFRF estimates obtained by two methods. The dFRFs shown in Figs. 3(a) and 4(a) are almost identical, but other types of dFRFs show some discrepancies with the order of magnitude less by 5 to 6 than the dominant peak values in the normal dFRFs, which corresponds perhaps to the measurement noise level in practice. The overall trends of the dFRFs, particularly the resonant peaks, are not much different for both methods. Some discrepancies are, however, noticable, at the anti-resonant regions and near the unstable regions.

In general, the number of assumed modes used in the Floquet method is less than, or equal at best to, the coordinate transform method. Thus, it can be concluded that the coordinate transform method is superior in estimation of dFRFs than the Floquet method. Note that the coordinate transform method, which is essentially a frequency domain approach, succeeds in approximating the dFRFs with a limited number of assumed modes (Ritz vectors), whereas the Floquet method, which is essentially a time domain approach, fails in using an effective set of base harmonics required to better estimate the dFRFs in the frequency domain. Theoretically speaking, as the number of assumed modes increases indefinitely, both methods will eventually lead to the identical results. Although there exist some discrepancies in the logarithmic magnitudes of dFRFs, the typical response calculations in the time domain by both methods yield little difference, as demonstrated previously in Fig.2.

4 CONCLUSION

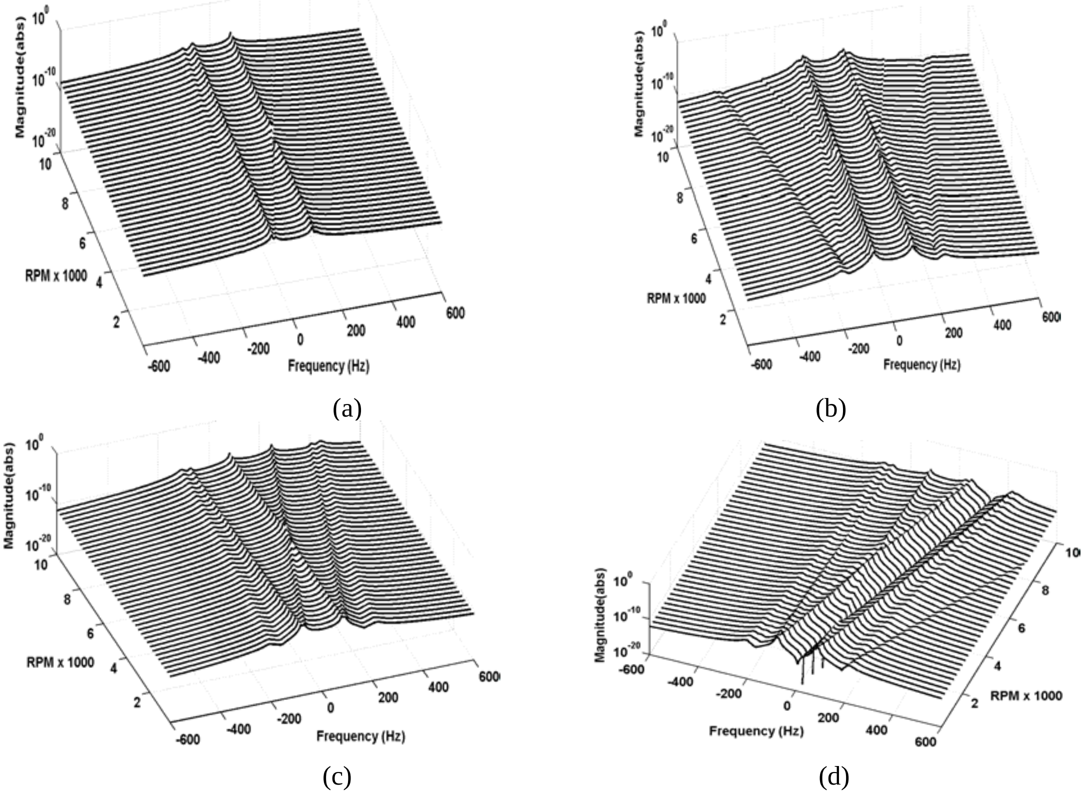
The complex modal analysis by the Floquet theory is developed for periodically time-varying linear rotor systems and compared with the coordinate transform approach. The Floquet method provides clear physical understanding of the eigenvalues and the corresponding eigenvectors. It is found that the Floquet method with a few approximation terms estimates the eigenvalues and the eigenvectors of lower order with fair accuracy, leading to satisfactory response calculations in the time domain. However, it is not efficient in calculating the eigenvectors of higher order and thus the frequency domain characteristics, including the directional frequency response functions, compared with the coordinate transform approach.



(a) (b) (c) (d)

Figure 3: Waterfall plots of dFRFs by the Floquet method: $\delta = \Delta = 0.3$

(a) $H_{g_0\,p}(j\omega)$ (b) $H_{\hat{g};0\,p}(j\omega)$ (c) $H_{\hat{g};-1\,p}(j\omega)$ (d) $H_{g;-1\,p}(j\omega)$



(a) (b) (c) (d)

Figure 4: Waterfall plots of dFRFs from the Hill's matrix of order 24: $\delta = \Delta = 0.3$

(a) $H_{g_0\,p}(j\omega)$ (b) $H_{\hat{g};0\,p}(j\omega)$ (c) $H_{\hat{g};-1\,p}(j\omega)$ (d) $H_{g;-1\,p}(j\omega)$

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