On the Capacity of Low-Density Parity-Check Codes

Sae-Young Chung[1]
Airvana, Inc.
25 Industrial av.
Chelmsford, MA 01824 USA.
s-chung@airvananet.com

G. David Forney, Jr.
Laboratory for Information and
Decision Systems, MIT,
Cambridge, MA 02139 USA.
forney@lids.mit.edu

Abstract — **We demonstrate that we can translate threshold values of low-density parity-check (LDPC) codes between channels accurately using a simple mapping. We develop some models of density evolution from this observation, namely erasure-channel and reciprocal-channel approximations. The reciprocal-channel approximation, based on dualizing LDPC codes, provides a very accurate model of density evolution for the additive-white Gaussian noise (AWGN) channel.**

I. Introduction

Density evolution [1, 2] is a very useful tool to calculate the capacity of low-density parity-check (LDPC) codes under various decoding algorithms. LDPC codes with a threshold value within 0.0045 dB and a simulation result within 0.04 dB of the Shannon limit were demonstrated in [4, 5] using density evolution.

As shown in [4], the channel capacity of various binary-input symmetric-output memoryless channels (binary-input channels from now on), including the binary erasure (BEC), binary symmetric (BSC), Laplace, AWGN channels, evaluated at the corresponding thresholds are very similar across these channels for various regular and irregular LDPC codes. This implies that the rate penalty for imposing an LDPC structure on optimal codes is similar across these channels.

One immediate application of this observation is that the calculation of thresholds for any binary-input channel can be carried out approximately (but reasonably accurately) by using the simplest channel, namely the BEC. We call this the "erasure-channel approximation."

This observation is also the key motivation for designing an improved model of density evolution for the AWGN channel, which we call the "reciprocal-channel approximation."

II. Reciprocal-Channel Approximation

We assume regular codes in this section. We first define a reciprocal-channel mapping $\psi_{\mathrm{X}}(\cdot) : \mathcal{X} \rightarrow \mathcal{X}$ for the channel parameter $x \in \mathcal{X} \subset \mathbb{R}$ of a binary-input channel X with its channel capacity $C_{\mathrm{X}}(x)$ (we assume this is monotonic in x) as

$$\psi_{\mathrm{X}}(x) = C_{\mathrm{X}}^{-1}(1 - C_{\mathrm{X}}(x)).$$

Note that the reciprocal-channel mapping is self-inverse. Using this definition, we define the reciprocal channel as follows.

Definition 1 *The reciprocal channel of a channel X with the parameter x is the same channel with the parameter $\psi_{\mathrm{X}}(x)$.*

We define the reciprocal-channel approximation as a one-dimensional approximation to density evolution using reciprocal-channel mappings, where all approximate messages are real numbers. To do this, we first find a mapping that maps the initial message density p_0 of a channel X to a real number a_0. We then find a symmetric computation rule $\tau(\cdot, \cdot)$ that maps $\mathbb{R}^2$ to $\mathbb{R}$. For a degree-d_v variable node, this function is applied repeatedly to compute the d_v-th output message b of the node, i.e.,

$$b = \tau(a_0, \tau(a_1, \tau(\ldots, a_{d_v-1}) \ldots),$$

where $a_1, \ldots, a_{d_v-1}$ are the approximate output messages from check nodes from the previous iteration and $b = a_0$ in the initial iteration. Abusing notation, we denote the resulting mapping by $\tau(\cdot, \ldots, \cdot)$.

For check nodes, we define the computation rule using the reciprocal channel mapping $\psi_{\mathrm{X}}(\cdot)$, i.e.,

$$a' = \psi_{\mathrm{X}}\left(\tau(\psi_{\mathrm{X}}(b_1), \psi_{\mathrm{X}}(b_2), \ldots, \psi_{\mathrm{X}}(b_{d_c-1}))\right),$$

where the b_j's are inputs and a' is the d_c-th output of the check node. Note that the reciprocal channel mapping $\psi(\cdot)$ effectively dualizes the parity-check code.

We note that this approximation becomes exact for the BEC if the message is the erasure probability and $\tau(a_1, a_2) = a_1 a_2$.

For AWGN channels, we define the mean of the LLR message as the one-dimensional message and use $\tau(a_1, a_2) = a_1 + a_2$. This is about 10 times more accurate than the erasure-channel and Gaussian [3] approximations.

Live demonstration of density evolution and full papers are available at http://truth.mit.edu/~sychung.

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[1]This work is based on S.-Y. Chung's Ph.D. thesis [4].

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