

Effect of External and Internal Learning Rate Differences on Manufacturing Improvement

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In this paper, we explore the effect of learning rate difference between external and internal development on manufacturing improvement. We consider a certain type of production technology that must grow, externally or internally, to maturity to be fully effective in existing production systems. For optimal operations, manufacturing firms must determine when and how much external technology in different 'maturity' stages needs to be brought into the internal production system. Employing an optimal control model, we derive two nontrivial implications. First, the optimal dynamics to bring external technology inside is sensitive to the learning rate difference between autonomous (or, external) development and induced (or, internal) development: the higher the internal learning rate, the more 'external technology' the firm should acquire. It, however, depends on the boundary condition whether the optimal path increases over time or across technology age: a significant redirection of the optimal dynamics occurs when crossing the boundary in the 'time-technology age' space, since the technology must mature to be fully effective within a limited time.

1. Introduction

Production technology development is critical in determining firms' overall performance, in particular, in terms of operations improvement (Hayes, et al. 1996). In this paper, we focus on manufacturing firms' operations strategy when the rate of *internal* learning is different from that of *external* learning. We premise that the internal learning is 'induced' by managers in the firms, whereas the external one is 'autonomous,' i.e., determined by exogenous technological factors (Adler and Clark 1991).

In the literature, an important decision making issue associated with manufacturing improvement is how to reach an optimal balancing between in-house technology development and outsourcing of the development efforts (Kim 1996). In a similar vein, Jaikumar and Bohn (1992) raised a fundamental question regarding "where" to create new technological knowledge: they suggested three such ways, (i) purchasing outside knowledge, (ii) intensive R&D outside manufacturing, and (iii) learning within existing manufacturing. This kind of decision making has been studied under several different situations. Kennedy (1993) modeled in-house versus contract maintenance, taking into account fixed costs and learning effects. Gaimon (1985 and 1989) approached a problem of automation acquisition from a binary decision making perspective.

The model we develop in this paper is an extension of the previous models in the related literature. Since the kind of decision making problem mentioned above requires dynamic approaches, most of the analytical models used optimal control models. Those models usually dealt with one dimensional space, e.g., time-related dimension. In this paper, we are concerned about two dimensions simultaneously, i.e., time-related dimension and growth-related dimension. To handle the issue appropriately, we employ a control model characterized by partial differential

equations (Butkovskiy 1969). In the next section, we elaborate on formulating the control model, and describe the solution procedure. Following the model formulation, we take on comparative analyses to examine changes in the control variable dynamics as time and growth variables vary. Based on the comparative analyses, we derive some economic and managerial implications. Finally, we discuss possible improvements for further research.

2. Model Formulation

In this paper, we adopt the definition of technology as “a set of pieces of knowledge, both directly practical and theoretical, know-how, methods, procedures, experience of successes and failures, and physical devices and equipment (Dosi 1982).” By focusing on the importance of knowledge as key to the technology development, we emphasize the ‘dynamic’ nature of production technology in that it evolves over time. Consistent with the definition, we consider such technology that in order to be fully applicable in the manufacturing process, it needs to be adapted to the manufacturing system, i.e., it has to become ‘system-specific’ to the manufacturing process. We view this as a process for production technology to grow or mature to become fully effective in the existing manufacturing system.

Let’s take an example. Suppose a shipbuilding firm wants to develop a particular welding technology to use in its cutting stage, and its level is represented with x . In order for the welding skill to be fully effective in shipbuilding, it needs to reach a certain level of advancement or sophistication, say, y . Then, the firm must nurture the welding technology until it reaches a point where $x = y$. We concentrate on two ways to accomplish it, developing it in-house and bringing in the technology that has externally matured to a certain degree.

For the modeling in this paper, let’s suppose the ‘conceptual’ age of technology is represented by ‘ ω .’ We assume that it takes ‘ $\omega=W$ ’ time for the technology to grow fully effective in the system, where we can also consider ‘ ω ’ measures the degree of technology’s being endogenous to the system. For instance, technology with $\omega=0$ is assumed completely ‘exogenous’ to the system, and need age until $\omega=W$ to be proficiently capable. A firm can bring external technology with $\omega=k$ ($k \leq W$) from outside into the internal system, and nurture it to grow for ‘ $W-k$ ’ time in order to become fully effective within the system.

The primary decision is to determine $u(\tau, \omega)$, the amount of technology ω aged which is purchased from outside at time τ . The decision time horizon is T . Thus, in the optimal control model we develop, the control variable is $u(\tau, \omega)$, $0 \leq \omega \leq W$ and $0 \leq \tau \leq T$. On the other hand, the state variable, $x(\tau, \omega)$, represents the ‘density (population)’ measure of technology ω aged at τ . For clarity of analysis, we assume, without loss of generality, that $x(0, \omega) \equiv x_{\tau}(\omega) = 0$, and $x(\tau, 0) \equiv x_{\omega}(\tau) = 0$.

Now we need to model the cost structure. Let $C(\omega)$ be the total cost to manage (e.g., control and develop/grow) a unit of technology which is ω aged at τ inside the firm: in fact, we are assuming that C is independent of τ for the present decision time horizon.

Let $P(\tau, \omega)$ be the total cost to purchase a unit of *external* technology ω aged at time τ : it can be regarded as a current market value of the technology. We will show that ω relates to induced learning, and τ to autonomous learning.

The profit side of the equation is as follows. The firm can get certain ‘value’ from the technology which has aged to W , i.e., at the maturity of the technology, which is $P(\tau, W) \equiv V(\tau)$ at τ . At $\tau=T$, the technology ‘ ω ’ aged has a salvage value of $P(T, \omega) \equiv S(\omega)$. For simplicity without losing generality, we further assume $V(\tau) = V_0$ and $S(\omega) = S_0 \omega$, where V_0 and S_0 are constant.

The primary question is how much ‘technology’ the firm must acquire from the outside each time over the decision span, and how that decision is affected by the forces of induced and autonomous learning: as alluded before, induced learning is managerially nurtured inside the manufacturing system, whereas autonomous learning is largely effected by exogenous forces in the industry. We elaborate on the two learning patterns.

External Innovation (Exogenous Learning). A certain type of learning could occur without individual firms’ conscious efforts channeled to the learning activities. For instance, a technological breakthrough at an industry level can be regarded “exogenous” to an individual firm, not directly involved in the internal innovation process. Researchers have identified this type of learning as “autonomous” primarily correlated with a time-related variable (Arrow 1962, Alchian 1963, Adler and Clark 1991). Consistent with this generic perspective, we assume that the purchasing price, P , is affected by the exogenous learning. More formally, $\frac{\partial P(\tau, \omega)}{\partial \tau} \leq 0$, i.e., the purchasing price decreases over time due to the autonomous learning outside the firm.

In addition, we assume that $\frac{\partial P(\tau, \omega)}{\partial \omega} \geq 0$. That is, the more “aged” the technology, the more expensive its purchasing price. Considering that more aged technology is closer to full effectiveness, the assumption seems reasonable.

In order to concentrate on the critical factors, we use a simplified cost structure representing the external purchasing price: $P(\tau, \omega) = K_p \omega^\alpha \tau^n$, where K_p is constant, $0 \leq \alpha \leq 1$, $n = \frac{\ln \phi_n}{\ln 2}$, and ϕ_n external learning rate, indicating that the smaller ϕ_n , the faster the learning. The formulation follows a well-known learning function (Yelle 1982) and a production function, e.g., Cobb-Douglas production function (de Neufville 1990).

Internal Innovation (Induced Learning). In contrast with the exogenous learning, endogenous learning requires “intentional efforts” on the managers’ part. In effect, it is the “induced learning” planned and managed inside the firm. In the model developed in this paper, we associate the endogenous learning with $C(\omega)$, the internal cost to foster the technology, so that $\frac{\partial C(\omega)}{\partial \omega} \leq 0$. Like in modeling the exogenous

learning, we construct the internal development cost, $C(\varpi) = K_c \varpi^m$, where K_c constant, $m = \frac{\ln \phi_m}{\ln 2}$, and ϕ_m internal learning rate.

The objective is to maximize the total value, Z , contributed from the technology development:

$$Z \equiv \int_0^T \int_0^W - (r(u(\tau, \varpi) - \bar{u})^2 + P(\tau, \varpi)u(\tau, \varpi) + C(\varpi)x(\tau, \varpi)) d\varpi d\tau + \int_0^T V(\tau)x(\tau, W) d\tau + \int_0^W S(\varpi)x(T, \varpi) d\varpi,$$

where \bar{u} is a 'target' level of technology purchase and r is a positive cost attached to the squared deviation of actual purchase of technology, u , from its target, \bar{u} . The second term on the right is the total value derived from fully matured technology, and the third is the total salvage value of the technology not matured yet at T . Using the decision variables and cost parameters developed in this section, we can rewrite the optimization problem as follows:

$$Z = \int_0^T \int_0^W - (r(u(\tau, \varpi) - \bar{u})^2 + K_p \varpi^\alpha \tau^n u(\tau, \varpi) + K_c \varpi^m x(\tau, \varpi)) d\varpi d\tau + V_0 \int_0^T x(\tau, W) d\tau + S_0 \int_0^W \varpi x(T, \varpi) d\varpi. \quad (1)$$

$$\text{Subject to a state constraint, } \frac{\partial x(\tau, \varpi)}{\partial \tau} = -\frac{\partial x(\tau, \varpi)}{\partial \varpi} + u(\tau, \varpi) \quad (2)$$

The state constraint can be obtained as follows. Since $x(\tau, \varpi)$ represents the amount of technology ϖ aged at τ , $x(\tau + \Delta\tau, \varpi)$ consists of two parts, $x(\tau, \varpi - \Delta\tau)$, i.e., one part due to the natural growth of technology $\varpi - \Delta\tau$ old at τ , and $u(\tau, \varpi)\Delta\tau$, i.e., the other due to the sum of newly purchased technology for the instantaneous moment. Thus, we have

$$x(\tau + \Delta\tau, \varpi) = x(\tau, \varpi - \Delta\tau) + u(\tau, \varpi)\Delta\tau. \quad (3)$$

By dividing both sides of (3) with $\Delta\tau$, we obtain (2).

The Hamiltonian of the problem, (1) and (2), is:

$$H = -\left\{ r(u - \bar{u})^2 + K_p \varpi^\alpha \tau^n u + K_c \varpi^m x \right\} + \lambda \left(-\frac{\partial x}{\partial \varpi} + u \right). \quad (4)$$

By obtaining $\frac{\partial H}{\partial u} = 0$, we can show that the optimal value of u :

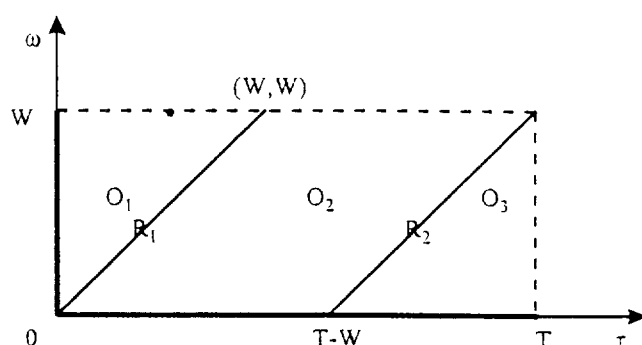
$$u^* = \bar{u} + \frac{1}{2r} (\lambda - K_p \varpi^\alpha \tau^n). \quad (5)$$

In order to have a maximum solution, we need to have $\frac{\partial^2 H}{\partial u^2} = -2r \leq 0$, which is always true as long as $r \geq 0$ as we assumed.

In order to obtain optimal values of x^* and u^* , we need to apply the maximum principle of “distributed parameter system,” which we refer to a reference (Butkovskiy 1969) for elaborate derivation. In Appendix 1, instead, we suggest intuitive ways to describe the derivation procedure. The ensuing analysis is based on the derivation in Appendix 1.

Since we deal with a control model involving two dimensions at the same time, we must resolve one more complication qualitatively different from those associated with simple control models. We are now dealing with a 2-dimensional space, i.e., (τ, ω) -space, rather than a one-dimensional, e.g., τ -space. For example, since each technology is assumed to grow for W period to be fully effective, those purchased after $T-W$, where T is the terminal time, can not have a chance to grow completely.

Figure 1. Division of (τ, ω) -space



To resolve the complication, we divide (τ, ω) -space into three regions, e.g., O_1 , O_2 , and O_3 , as in Figure 1. In O_1 , $x(\tau, \omega)$ depends on the initial condition of $x(0, \omega)$, while it depends on the initial condition of $x(\tau, 0)$ in O_2 and O_3 . Since $\lambda(\tau, \omega)$ is the shadow value of technology, it is affected by the boundary of $([0, T] \times \{W\}) \cup (\{T\} \times [0, W])$. In O_1 and O_2 , $\lambda(\tau, \omega)$ depends on $P(\tau, W) = V(\tau)$, while on $P(T, \omega) = S(\omega)$ in O_3 . This is because technology purchased in O_3 can not fully mature by T , the end of the current decision horizon: in other words, the final value of the technology purchased while in O_3 is only the salvage value at T . In order to be consistent in the boundaries between regions, e.g., R_1 and R_3 , we need to impose a constraint that on R_2 , $V_0 = S_0 W$.

As alluded before, in Appendix 1, we show intuitive processes to derive the optimal solutions of state and control variables, deferring a detailed mathematical derivation procedure to the references.

$$x(\tau, \varpi) = \begin{cases} \int_0^{\tau} u(q, \varpi - \tau + q) dq & (\tau, \varpi) \in O_1 \\ \int_0^{\varpi} u(\tau - \varpi + q, q) dq & (\tau, \varpi) \in O_2 \cup O_3 \end{cases} \quad (6)$$

$$\lambda(\tau, \varpi) = \begin{cases} V_0 - \int_{\varpi}^{\varpi} K_c \rho^m d\rho & (\tau, \varpi) \in O_1 \cup O_2 \\ (T - \tau + \varpi) S_0 - \int_{\varpi}^{\tau - \tau + \varpi} K_c \rho^m d\rho & (\tau, \varpi) \in O_3 \end{cases} \quad (7)$$

With (6) and (7), we can obtain u^* according to where the particular (τ, ω) belongs to in (τ, ω) -space. In the next section, we take on the comparative analysis with such derived u^* , deferring x^* to future study, since we are more concerned about the control variable, i.e., the primary decision variable, than the state variable in the current research.

3. Comparative Analysis

In this section, we examine changes in the control variable dynamics as other variables like time and technology age vary. First, we need to investigate two different situations separately according to the division of (τ, ω) -space.

3.1. Comparative Analysis in $O_1 \cup O_2$

By putting (7) into (5), we can obtain

$$u^* = \bar{u} + \frac{1}{2r} \left\{ V_0 - \frac{K_c}{m+1} W^{m+1} + \frac{K_c}{m+1} \varpi^{m+1} - K_p \varpi^\alpha \tau^n \right\}. \quad (8)$$

3.1.1. Changes of u^* due to ϖ

To probe changes in u^* as ϖ varies, we need to take a differentiation,

$$\frac{\partial u^*}{\partial \varpi} = \frac{1}{2r} \left\{ K_c \varpi^m - \alpha K_p \varpi^{\alpha-1} \tau^n \right\}. \quad (9)$$

In order for (9) to be positive, $K_c \varpi^m \geq \frac{\partial}{\partial \varpi} (K_p \varpi^\alpha \tau^n)$, which can be rearranged as

$$C(\varpi) \geq \frac{\partial}{\partial \varpi} P(\tau, \varpi) \quad (10)$$

Equation (10) shows the condition under which the firm, operating optimally, purchases one more unit of technology as the age of technology increases by one unit. The result is that if and only if the increment of unit purchase cost, P , as the

technology age increases by one unit, is less than the current unit cost to internally nurture the technology, C , it is profitable for the firm to acquire one more unit of technology from outside.

3.1.2. Changes of u^* due to τ

We have

$$\frac{\partial u^*}{\partial \tau} = -\frac{nK_p}{2r} \varpi^\alpha \tau^{\alpha-1} \geq 0 \text{ since } n < 0. \quad (11)$$

From (11), we can infer that it is always better to increase the purchase of outside technology, as the time passes while in $O_1 \cup O_2$, since any technology purchased in the region will fully mature by T , i.e., generate full value of the investment before the end of the decision horizon.

3.2. Comparative Analysis in O_3

From (5) and (7), we can get

$$u^* = \bar{u} + \frac{1}{2r} \left\{ (T - \tau + \varpi) S_0 - \frac{K_c}{m+1} (T - \tau + \varpi)^{m+1} + \frac{K_c}{m+1} \varpi^{m+1} - K_p \varpi^\alpha \tau^n \right\} \quad (12)$$

3.2.1. Changes of u^* due to ϖ

$$\frac{\partial u^*}{\partial \varpi} = \frac{1}{2r} \left\{ S_0 - K_c (T - \tau + \varpi)^m + K_c \varpi^m - \alpha K_p \varpi^{\alpha-1} \tau^n \right\} \quad (13)$$

In order for (13) to be positive, $S_0 - \alpha K_p \varpi^{\alpha-1} \tau^n \geq K_c \left\{ (T - \tau + \varpi)^m - \varpi^m \right\}$, which can be rearranged as $\frac{\partial}{\partial \varpi} S(\varpi) - \frac{\partial}{\partial \varpi} P(\tau, \varpi) \geq C(T - \tau + \varpi) - C(\varpi)$.

Since $C(T - \tau + \varpi) - C(\varpi) = \int_{\varpi}^{T-\tau+\varpi} \frac{\partial}{\partial \rho} C(\rho) d\rho$, we can rearrange so that

$$\frac{\partial}{\partial \varpi} S(\varpi) - \int_{\varpi}^{T-\tau+\varpi} \frac{\partial}{\partial \rho} C(\rho) d\rho \geq \frac{\partial}{\partial \varpi} P(\tau, \varpi). \quad (14)$$

The left-hand side of (14) is the net benefit increment resulting from purchasing one unit of outside technology as its age increases by one unit, while the right-hand side is the net cost increment due to purchasing the technology. The left-hand side consists of two parts: the first is the increment of salvage value because of purchasing one unit of technology aged by one more unit than the currently available technology; the second part is the total cost increment to internally nurture the technology because of its being

more aged by one unit. Since $\frac{\partial C(\varpi)}{\partial \varpi} \leq 0$ is assumed, $-\int_{\varpi}^{T-\tau+\varpi} \frac{\partial}{\partial \rho} C(\rho) d\rho$ is positive, implying there is cost saving. Therefore, adding up the salvage value increment and

cost saving, we have the net benefit increment. What (14) implies is that if and only if the cost increment due to one unit increase of ϖ is less than the net benefit increment, it is better for the firm to purchase one more unit of outside technology as ϖ increases while in O_3 . We want to highlight that since the technology purchased in O_3 will not fully mature by T , we need to take into account the salvage value, S , when determining the economics of technology purchase. Note that S did not enter the equation while in $O_1 \cup O_2$.

3.2.2. Changes of u^* due to τ

$$\frac{\partial u^*}{\partial \tau} = \frac{1}{2r} \left\{ -S_0 + K_c (T - \tau + \varpi)^m - nK_p \varpi^\alpha \tau^{n-1} \right\} \quad (15)$$

In order for (15) to be positive, $K_c (T - \tau + \varpi)^m \geq S_0 + nK_p \varpi^\alpha \tau^{n-1}$, which can be rearranged as

$$C(T - \tau + \varpi) - \frac{\partial}{\partial \tau} P(\tau, \varpi) \geq \frac{\partial}{\partial \varpi} S(\varpi) \quad (16)$$

As before, the right-hand side of (16) represents the opportunity cost increment due to purchase delay: by delaying the purchase by a unit time, the firm will lose the salvage value at T by $\frac{\partial}{\partial \varpi} S(\varpi)$. However, exactly because of delaying the purchase by a unit time, the firm also gains other benefits: due to the autonomous learning happening outside the firm, the purchasing price at $\tau + \partial\tau$ is less expensive than that at τ by $-\frac{\partial}{\partial \tau} P(\tau, \varpi)$; by delaying the purchase by a unit time, the firm will also save $C(T - \tau + \varpi)$ at T , i.e., the period during which the technology needs to be nurtured is reduced by $\partial\tau$.

Therefore, $C(T - \tau + \varpi) - \frac{\partial}{\partial \tau} P(\tau, \varpi)$ represents the total saving of opportunity cost due to delaying the technology purchase by a unit time. We can conclude that if and only if the opportunity cost increment is less than the opportunity cost saving, the firm needs to increase the purchase of outside technology as time passes while in O_3 .

4. Economic and Managerial Implications

We derive some economic and managerial implications from the comparative analysis in the previous section. In order to understand the implications more concretely, we use simulation runs with example values. Table 1 shows the actual parameter values for the ensuing sample runs.

Table 1. Parameter Values for Example Runs

T	W	V_0	S_0	\bar{u}	K_c	K_p	α	ϕ_n	r
31	10	10	1	5	1	1.5	0.8	0.95	1

Since our primary focus is on changes in the dynamics of u^* as the learning rates vary, we present four examples: (1) changes in u^* as internal learning rate changes and ω is fixed at 2 (Figure 2) representing relatively young technology, (2) changes in u^* as internal learning rate changes and ω is fixed at 8 (Figure 3) representing more mature technology, (3) changes in u^* as internal learning rate changes and τ is fixed at 15 (Figure 4), and (4) changes in u^* as internal learning rate changes and τ is fixed at 25 (Figure 5).

Based on the example dynamics in Figure 2 and 3, we can infer that the faster the internal learning occurs, an optimal strategy calls for purchasing more 'external technology' and the purchase amount increases over time until (τ, ω) reaches R_2 , i.e., the boundary after which any new technology purchased will not fully mature by the end of the current decision time horizon. Once R_2 is reached, the purchase amount of external technology drops significantly, and the higher the internal learning rate, the more significant the decrease. At T , the purchase amounts converge to a same magnitude independent of learning rate differences.

Although it seems counter-intuitive at first that more 'external technology' is purchased when 'internal' learning rate is relatively higher, the logic behind the peculiar dynamics can be reasonably explained: (1) if the internal learning occurs relatively faster, it benefits the firm to bring more 'external technology' inside the system early on, since the firm can improve the introduced external technology faster within the manufacturing system than outside, however, (2) once (τ, ω) reaches the boundary point, R_2 , the optimal solution requires the firm to decrease the purchase of external technology over time until it eventually converges to a single quantity independent of the internal learning rate, since after R_2 it makes less than full benefit to bring new 'external technology' into the system due to its inability to fully mature within the present decision horizon.

One intriguing point can be noted when no learning effect is assumed internally, i.e., $\phi_m=1$. When the internal learning is nil, u^* starts as a smaller amount, but increases gradually without being suddenly forced to decline. In other words, the purchasing dynamics shows a well-planned gradual pattern when no internal learning is present. The logic behind that the lower the internal learning, the firm purchases less external technology at any given time is understandable: since the external 'autonomous' learning occurs faster than the internal one, it makes the firm better off to let the outside mechanism do more in nurturing the technology.

Comparing Figure 2 and Figure 3, we can also conclude that for relatively young technology, i.e., technology with smaller ω , the rate of purchase decrease is much more dramatic and the decreasing point comes much earlier: these phenomena can be easily confirmed analytically. It seems reasonable since we can expect 'matured technology' to be relatively unaffected by the time progression and thus, whether (τ, ω) reaches R_2 or not. A managerial implication is that the optimal strategy can be of more dramatic shift for 'younger' technology than the older one. With the analytical forms in the previous section, we can determine the actual magnitudes of the changes in u^* as parameter values vary, provided that other parameter values are well estimated.

Figure 4 and 5 show similar, but qualitatively different dynamics of u^* . When it deals with (τ, ω) in $O_1 \cup O_2$, the firm does not have to worry about whether the new technology will eventually mature or not: thus, we can expect there would be no sudden shift in the dynamics of u^* . Figure 4 shows the situation. When the internal learning rate is reasonably close to the external one, e.g., $\phi_m = 0.85$, Figure 4 indicates that u^* is almost constant across the technology age, ω , implying that the purchase dynamics is not affected by the technology age. However, when the internal learning rate is higher, e.g., $\phi_m = 0.70$, the optimal dynamics indicates that the firm has to purchase more ‘younger technology’ than the older one since it can grow the ‘younger’ technology faster inside than outside the production system. Employing the same logic in an opposite direction, we can easily see that when the internal learning lags behind the external learning, the firm had better acquire more ‘mature technology.’ Figure 5 presents the situation when (τ, ω) is in O_3 . Although the overall dynamics resembles that in Figure 4, the most distinct difference is that now we observe a sharp redirection in the dynamics due to the presence of boundary R_2 .

Based on the examples from the comparative analysis, we can draw the first managerial implication that the differences between external and internal learning can be significant in determining the technology development dynamics. We now also know why there would be a sharp redirection in the optimal dynamics of u^* when (τ, ω) crosses the boundary of R_2 : it depends on whether the technology purchased at a specific (τ, ω) will fully mature in the end. It asks the firm to be extra careful about the shift when designing the purchasing decision. The examples help us see the particular patterns of dynamics of u^* as the internal learning rate changes in relation to the external one, and also as τ varies given ω fixed, or ω varies given τ fixed. The actual patterns as the internal learning rate changes indicate that the optimal path does not always coincide with the intuitive reasoning, underlining the value of optimal control analysis.

5. Conclusion and Discussion

We have developed a learning-induced control model based on the distributed parameter systems approach, to solve a dynamic decision problem for optimal technology development. With the control model, we have first showed the critical factors which determine the structure of comparative analysis: for instance, we have investigated the conditions under which u^* increases or decreases as other variables vary. Rooted on the comparative analysis, we have designed simulation runs to focus on the impact of learning rate differences on the optimal dynamics of u .

Two nontrivial implications can be observed from the research. First, the optimal dynamics of u is sensitive to the learning rate difference between autonomous innovation and induced innovation. From the simulation runs, we can infer that the higher the internal learning rate, the more ‘external technology’ the firm needs to purchase. It is, however, dependent on the boundary, R_2 , whether the optimal path increases or decreases over time, τ , or across technology-age, ω . Therefore, the second implication relates to the critical influence the boundary, R_2 , has on the optimal dynamics. It seems reasonable to expect a ‘sharp redirection’ in the optimal dynamics

when (τ, ω) crosses R_2 at some point, since we deal with such technology that must mature to a certain age to be fully effective within a limited time.

The control model we present in this paper is different from others in the related literature. First, although the learning effect has been often taken into account in the literature, the model in this paper incorporates two different learning effects simultaneously, autonomous and induced, into the analysis so that we can investigate the interaction between the two. Second, unlike others in the literature, our model deals with two specific dimensions related with the technology, i.e., time dimension and growth (technology-age) dimension. Thus, we have been able to study the dynamics of such technology that grows over time. We believe that by encompassing the internal evolution into the analysis, the model captures the reality better than other one-dimensional models. To properly deal with the two dimensions, we have employed a control model characterized by partial differential equations: this control model, often called "distributed parameter systems approach," is most capable of examining such dynamic relations in which we are interested in the current research (Derzko, et al. 1980).

Further improvement is, we believe, still possible from the present analysis. One can test if the economic and managerial implications can be more generalized by experimenting different cost structures. We would expect that the importance of learning rate difference remains unchanged while the shape of the optimal dynamic path might alter as different parameter and variable values are used. Although we have assumed zero values for $x(\tau, \omega)$'s initial condition without loss of generality, it might be an intriguing question how the overall dynamics would be affected as the initial condition takes a non-zero value with complicated functional forms.

Although our primary focus has been on the production technology development, the model developed in the paper can be applicable to other settings of strategic decision making as well. For instance, the type of technology we studied shares much commonality with 'human resources': firms might have to decide whether they choose to educate their own workforce early on by bring into the system more new hires in the early period, or let the market train the workers until their skill level is enhanced enough to be reasonably effective in the firms. This kind of decision making obviously depends on the difference between external and internal learning efficiency. An important question might be "How much external workforce should the firm bring into the system, and how much educated should the workforce be at the time it is employed?" We can also apply the model to other technology-related decision making situations as long as the technology has the characteristics we have dealt with in this paper: for instance, it needs to mature for a certain period of time to be fully effective in the system, and it can be nurtured either autonomously outside the system or in an induced manner inside the system. We believe most of the technology associated with business activities shares the similar characteristics, although there could be differences in terms of degree. Our research in this paper can serve as a reference point for the future research in the directions touched upon above.

Appendix 1. Intuitive Derivation of State and Control Variables

Utilizing the maximum principle developed by Pontriyagin, et al. (1962) and Butkovskiy (1969), the optimal dynamics of x and λ can be mathematically derived. Deferring the detailed mathematical derivation procedure to the references, here we provide more intuitive interpretations on the optimal dynamics.

A1.1. Intuitive Interpretation of x^*

We can confirm that the optimal dynamics of x is derived as follows.

$$x(\tau, \varpi) = \begin{cases} \int_0^{\tau} u(q, \varpi - \tau + q) dq & (\tau, \varpi) \in O_1 \\ \int_0^{\varpi} u(\tau - \varpi + q, q) dq & (\tau, \varpi) \in O_2 \cup O_3 \end{cases} \quad (A1)$$

$x(\tau, \varpi)$ with $(\tau, \varpi) \in O_1$ is affected by the boundary of $\{0\} \times [0, W]$, whereas when $(\tau, \varpi) \in O_2 \cup O_3$, it is influenced by the boundary of $[0, T] \times \{0\}$. Therefore, with $(\tau, \varpi) \in O_1$, $x(\tau, \varpi)$ is simply the sum of the entire technology purchase, $u(q, \varpi - \tau + q)$, for the period of $q \in [0, \tau]$. The reason we take an integration with regard to τ is that $x(\tau, \varpi)$ is affected by the boundary of $\{0\} \times [0, W]$, which is essentially the ω -axis. Likewise, we can make a similar interpretation for $x(\tau, \varpi)$ with $(\tau, \varpi) \in O_2 \cup O_3$. The only difference is that now we take an integration with respect to ω since the initial condition boundary coincides with the τ -axis. Since we assume $x(0, \varpi) \equiv x_{\tau}(\varpi) = 0$ and $x(\tau, 0) \equiv x_{\varpi}(\tau) = 0$, we obtain x^* as in (A1).

A1.2. Intuitive Interpretation of λ^*

From (5), we know that it is necessary to calculate λ^* in order to obtain u^* . It is easy to see that λ^* is the shadow value of the technology, that is, the net value to the objective function contributed by the newly acquired technology over its life span.

$$\lambda(\tau, \varpi) = \begin{cases} V_0 - \int_0^W K_c \rho^m d\rho & (\tau, \varpi) \in O_1 \cup O_2 \\ (T - \tau + \varpi) S_0 - \int_0^{T-\tau+\varpi} K_c \rho^m d\rho & (\tau, \varpi) \in O_3 \end{cases} \quad (A2)$$

As alluded before, λ^* with $(\tau, \varpi) \in O_1 \cup O_2$ relies on the boundary of $[0, T] \times \{W\}$, while it depends on the boundary of $\{T\} \times [0, W]$ when $(\tau, \varpi) \in O_3$.

We understand that the technology purchased while $(\tau, \varpi) \in O_1 \cup O_2$ will fully mature by T , and thus generate the value of V_0 in the end. In order to get the net value contributed by the technology acquired while $(\tau, \varpi) \in O_1 \cup O_2$, we must subtract

from the final contribution of V_0 the total cost to internally nurture the technology until it fully matures, which is $\int_{\varpi}^{\omega} K_c \rho^m d\rho$.

The technology acquired while $(\tau, \varpi) \in O_3$ will not fully mature by T , and therefore generate a salvage value of $(T - \tau + \varpi) S_0$ at T , where $(T - \tau + \varpi)$ is the final age at T of the technology ω -aged acquired at τ . The total cost to nurture the technology is $\int_{\varpi}^{T-\tau+\varpi} K_c \rho^m d\rho$. The net contribution provided by the technology acquired while $(\tau, \varpi) \in O_3$ can be obtained by subtracting the total cost from the salvage value. Therefore, we can derive λ^* as in (A2).

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Figure 2. Changes in u^* as ϕ_m changes with $\varpi = 2$

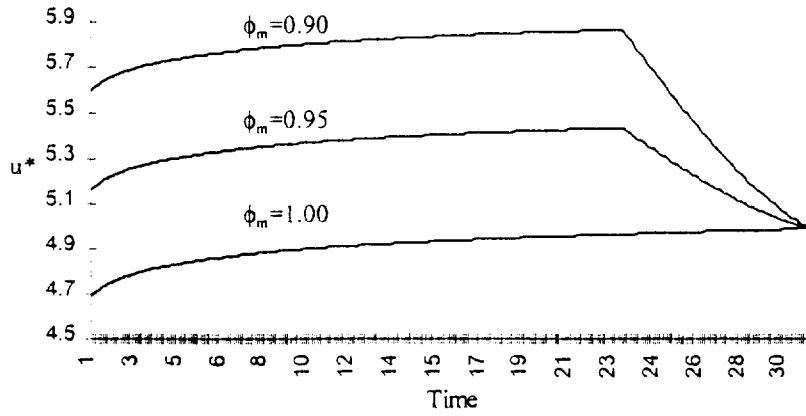


Figure 3. Changes in u^* as ϕ_m changes with $\varpi = 8$

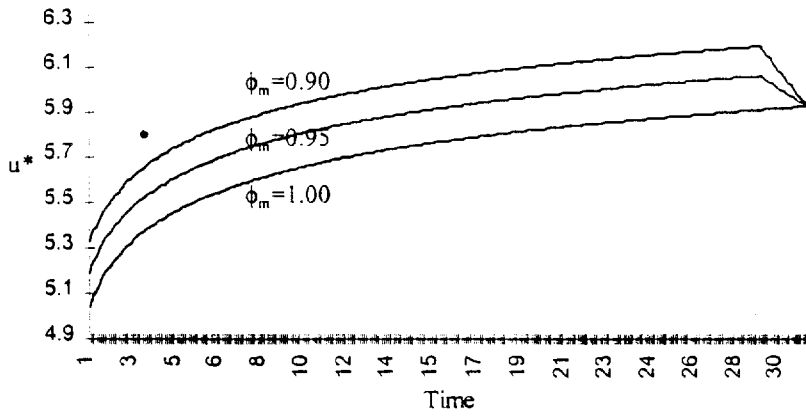


Figure 4. Changes in u^* as ϕ_m changes with $\tau = 15$

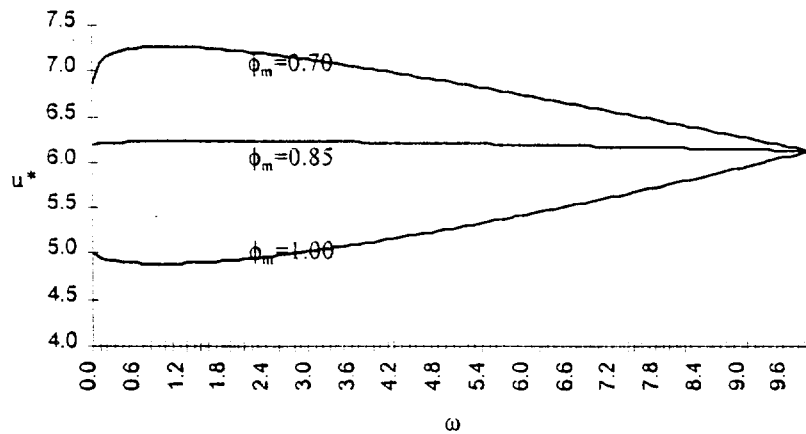


Figure 5. Changes in u^* as ϕ_m changes with $\tau = 25$

