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The Hadwiger Number of Jordan Regions Is Unbounded*

Otfried Cheong and Mira Lee

Division of Computer Science,
Korea Advanced Institute of Science and Technology,
Daejeon 305-701, South Korea
{otfried,mira}@tclab.kaist.ac.kr

Abstract. We show that for every $n > 0$ there is a planar topological disk A_0 and n translates $A_1, A_2, \dots, A_n$ of A_0 such that the interiors of $A_0, \dots, A_n$ are pairwise disjoint, but with each A_i touching A_0 for $1 \le i \le n$.

1. Introduction

For any compact body $C \subset \mathbb{R}^d$, we define $H(C)$, the Hadwiger number of C, as the maximum number of mutually non-overlapping translates of C that can be brought into contact with C (see the survey by Zong [7]). Hadwiger [5] showed that for convex sets C we have $H(C) \le 3^d - 1$ (using Minkowski's difference body method, see also [4]). The bound is tight for parallelepipeds [3], [4]. In the planar case it is known that $H(C) = 6$ for every convex C other than a parallelogram.

The arguments used in these results rely strongly on convexity. Considering the more general family of *Jordan regions*[1] in the plane, Halberg et al. [6] could show that $H(C) \ge 6$ holds for any Jordan region $C \subset \mathbb{R}^2$. More precisely, they showed that there exist six non-overlapping translates of C all touching C and whose union *encloses* C, where a set A encloses a set B if every unbounded connected set which intersects B also intersects A. It seems therefore natural to conjecture that the Hadwiger numbers of Jordan regions in the plane are bounded by an absolute constant. Some more evidence for this conjecture is a result of Bezdek et al. [1] who showed that the maximum number of pairwise touching translates of a Jordan region C in the plane is four. Since in this respect Jordan regions

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[1] A set $C \subset \mathbb{R}^2$ is a *Jordan region* or *topological disk* if it is bounded by a closed Jordan curve, or equivalently if it is homeomorphic to the unit disk.

behave in the same way as convex sets, they ask the following question:

> It seems reasonable to conjecture that $H(C) \leq 8$ for every planar Jordan region C. If this conjecture is false, is there an upper bound for $H(C)$ independent from the disk C?
>
> [1, Problem 6.1]

As a first step in settling this conjecture, Bezdek could later show that $H(C) \leq 75$ if C is a *star-shaped* Jordan region. The problem was picked up again by Brass et al. [2] (Problem 5 and Conjecture 6 in Section 2.4). We show here that the conjecture is not true in a strong sense: the Hadwiger number of Jordan regions in the plane is not bounded by *any* constant. For each $n > 0$, we construct a Jordan region that admits n mutually non-overlapping translates touching it.

The case of star-shaped Jordan regions remains open in the weaker sense of establishing the right constant: Brass et al. conjecture that this constant is 8, but the best known upper bound is 75.

2. The Proof

We consider the integer sequence $\mathcal{S} = s_1, s_2, \ldots,$ where s_i is the number of bits that must be counted from right to left to reach the first 1 in the binary representation of i:

$$\mathcal{S} = 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 6, \ldots.$$

This sequence, which is also known as the ruler function, appears as sequence $A001511$ in the on-line encyclopedia of integer sequences.[2] We need the following property of this sequence.

Lemma 1. *The prefix of length k of $\mathcal{S}$ has the smallest sum among all subsequences of length k of $\mathcal{S}$, for any $k > 0$. Formally, for any $k, r > 0$,*

$$\sum_{i=1}^{k} s_i \leq \sum_{i=r}^{r+k-1} s_i.$$

Proof. We proceed by induction. If $k = 1$, the claim is true since $s_1 = 1 \leq s_r$. Assume now that $k > 1$ and that the claim is true for all shorter prefixes. If k is odd, then $s_k = 1$, and by induction we have

$$\sum_{i=1}^{k} s_i = 1 + \sum_{i=1}^{k-1} s_i \leq 1 + \sum_{i=r}^{r+k-2} s_i \leq \sum_{i=r}^{r+k-1} s_i.$$

It remains to consider even k. We observe that every odd term of $\mathcal{S}$ is equal to 1, and that $\mathcal{S}$ has a nice recursive structure: removing all odd terms and subtracting 1 from all even

[2] `http://www.research.att.com/~njas/sequences/A001511.`

terms results in the same sequence $\mathcal{S}$ again. We therefore have

$$\sum_{i=1}^{k} s_i = k/2 + \sum_{i=1}^{k/2}(s_i+1) = k + \sum_{i=1}^{k/2} s_i \le k + \sum_{i=r'}^{r'+k/2-1} s_i = k/2 + \sum_{i=r'}^{r'+k/2-1}(s_i+1) = \sum_{i=r}^{r+k-1} s_i,$$

where $r' = \lceil r/2 \rceil$. □

We can now describe our planar topological disk, or, more precisely, a two-parameter family of disks. For integers $m \ge 2$ and $n \ge 1$, the disk D_n^m is the union of 2^n horizontal *bars* $B_1, \ldots, B_{2^n}$ and $2^n - 1$ vertical *connectors* $V_1, \ldots, V_{2^n-1}$. All bars are axis-parallel rectangles of width m and height 1, all connectors are axis-parallel rectangles of width 1. The height of connector V_i is s_i (the ith term of our sequence $\mathcal{S}$). Informally, connector V_i is placed above the rightmost unit square of bar B_i, while bar B_{i+1} is placed to the right of the topmost unit square of connector V_{i-1}.

Formally, bar B_i is the rectangle spanning the x-interval $[(i-1)m, im]$ and the y-interval $[y_i, y_i+1]$, where $y_i = \sum_{j=1}^{i-1} s_j$. Connector V_i spans the x-interval $[im-1, im]$ and the y-interval $[y_i+1, y_{i+1}+1]$.

Figure 1 shows D_n^m for some values of m and n.

We can give an alternative, recursive description of D_n^m, by observing that bars $B_1, \ldots, B_{2^{n-1}}$ and bars $B_{2^{n-1}+1}, \ldots, B_{2^n}$ of D_n^m form two translates of D_{n-1}^m, connected by the single connector $V_{2^{n-1}}$. We can consider D_n^m to consist of two translates of D_{n-1}^m, or four translates of D_{n-2}^m, or 2^{n-1} translates of D_1^m, or, in general, 2^k translates of D_{n-k}^m.

Lemma 2. *Let A and A' be translates of D_n^m, for $m, n \ge 2$, such that the first bar B_1' of A' is obtained from some bar B_r of A by a translation of $y^* \ge 1$ downwards and $1 \le x^* \le m-1$ to the right, where $1 \le r \le 2^n$. Then A and A' have disjoint interiors.*

Proof. Consider the vertical strip spanned by bar B_{r-1+i} of A, for $1 \le i \le 2^n - r + 1$. Since $1 \le x^* \le m-1$, this strip can intersect only bars B_{i-1}' and B_i' and connector V_{i-1}' of A'. The highest y-coordinate in $B_{i-1}' \cup V_{i-1}' \cup B_i'$ is $y_i + 1$ with respect to the

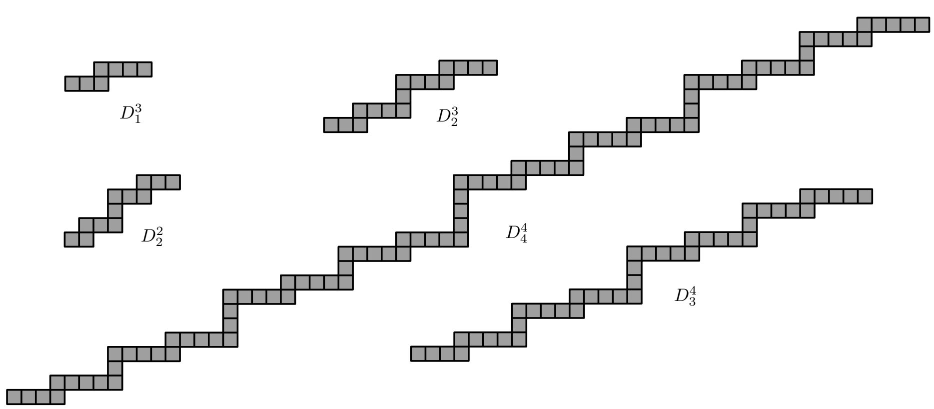

Fig. 1. Some D_n^m.

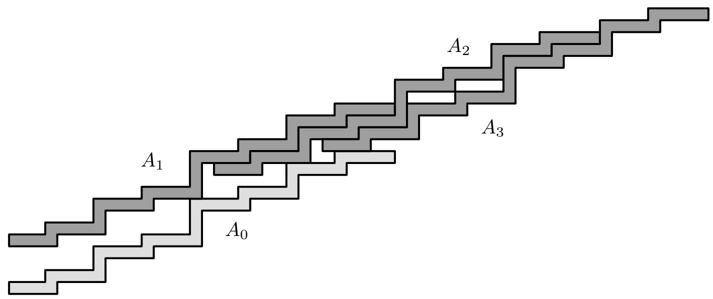

Fig. 2. The construction for $m = 4$, $n = 3$.

origin of A'. By assumption, the origin of A' is at y-coordinate $y_r - y^* \leq y_r - 1$, and so $B'_{i-1} \cup V'_{i-1} \cup B'_i$ lies below the line $y = y_r + y_i$. On the other hand, the bottom edge of B_{r-1+i} of A has y-coordinate y_{r-1+i}. We now have

$$y_{r-1+i} - (y_r + y_i) = (y_{r-1+i} - y_r) - y_i = \sum_{j=r}^{r+i-2} s_j - \sum_{j=1}^{i-1} s_j \geq 0$$

by Lemma 1. This implies that the interior of $B'_{i-1} \cup V'_{i-1} \cup B'_i$ lies stricly below B_{r-1+i}, and the lemma follows. □

We can now describe our construction of touching translates. We fix an integer $n > 1$, and pick any integer $m \geq n$. Let A_1 be D_n^m. For $2 \leq i \leq n$ we obtain A_i from A_{i-1} as follows: The *first* (leftmost) copy of D_{n+1-i}^m in A_i is a translate of the *second* copy of D_{n+1-i}^m in A_{i-1}, translated down by one and right by one.

We observe now that for any pair $1 \leq i < j \leq n$, the leftmost copy of D_{n+1-j}^m in A_j is a translate of some copy of D_{n+1-j}^m in A_i, translated down by $j - i$ and right by $j - i$. Since $1 \leq j - i < n \leq m$, Lemma 2 implies that the interiors of A_i and A_j are disjoint.

Now let A_0 be a translate of A_1, translated downwards by $n + 1$. See Fig. 2 for the entire construction for $m = 4$, $n = 3$.

It remains to show that A_i touches A_0, but that their interiors are disjoint, for $1 \leq i \leq n$.

We pick some i. Let D be the *last* (rightmost) copy of D_{n+1-i}^m in A_0. Then the first copy D' of D_{n+1-i}^m in A_i can be obtained from D by translating upwards by $n + 1$, then downwards by $i - 1$ and right by $i - 1$. In other words, D' is obtained from D by translating upwards by $n + 2 - i$ and right by $i - 1$. Now the middle vertical segment of D is a rectangle of height $n + 2 - i$, and so this translation brings D and D' into contact. On the other hand, all other vertical segments in D have length less than $n + 2 - i$, and so the interiors of D and D' are disjoint. Finally, since D is the rightmost part of A_0 and D' is the leftmost part of A_i, no other intersections between A_0 and A_i are possible, and so their interiors are disjoint.

We summarize the result in the following theorem. Figure 3 shows a larger example.

Theorem 1. *For any integer $n \geq 2$ and any integer $m \geq n$ there are $n + 1$ translates $A_0, A_1, \ldots, A_n$ of D_n^m whose interiors are pairwise disjoint, but such that A_0 touches every A_i, $1 \leq i \leq n$.*

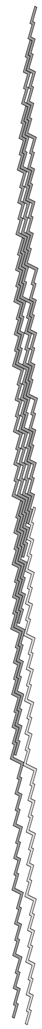

Fig. 3. The construction for $m = n = 5$.

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