



Positive feedback in coordination games: Stochastic evolutionary dynamics and the logit choice rule [☆]



Sung-Ha Hwang ^{a,*}, Luc Rey-Bellet ^b

^a College of Business, Korea Advanced Institute of Science and Technology (KAIST), Seoul, Republic of Korea

^b Department of Mathematics and Statistics, University of Massachusetts Amherst, MA, USA

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ABSTRACT

We study the problem of stochastic stability for evolutionary dynamics under the logit choice rule. We consider general classes of coordination games, symmetric or asymmetric, with an arbitrary number of strategies, which satisfies the marginal bandwagon property (i.e., there is positive feedback to coordinate). Our main result is that the most likely evolutionary escape paths from a status quo convention consist of a series of identical mistakes. As an application of our result, we show that the Nash bargaining solution arises as the long run convention for the evolutionary Nash demand game under the usual logit choice rule. We also obtain a new bargaining solution if the logit choice rule is combined with intentional idiosyncratic plays. The new bargaining solution is more egalitarian than the Nash bargaining solution, demonstrating that intentionality implies equality under the logit choice model.

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1. Introduction

Conventions and customs are sometimes determining factors for formal contracts. For example, Young and Burke (2001) show that local custom is a driving force in setting up the crop sharing contract terms in the state of Illinois. Customary patterns of behaviors such as asymmetric norms between racial groups and genders can also produce a mechanism by which inequalities persist for a long period of time (Naidu et al., 2017). Changes in informal convention sometimes induce formal institutional changes which may contribute to long-run economic growth (Hwang et al., 2016; Acemoglu et al., 2005). Thus, understanding both the disruption and emergence of conventions can shed light on problems such as economic incentives, inequality, and long-run growth. Conventions or social norms that typically form over time and last for a long period of time are frequently modeled as long run equilibria in stochastic evolutionary dynamics, in which agents play myopic best responses subject to mistakes, errors, or idiosyncratic plays (Young, 1993a; Kandori et al., 1993; Bowles, 2004). Therefore

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* Corresponding author.

E-mail addresses: sungha@kaist.ac.kr (S.-H. Hwang), luc@math.umass.edu (L. Rey-Bellet).

identifying the most likely evolutionary paths escaping from an existing convention or transitioning between conventions is a key step in studying the disruption and emergence of conventions.

Recently, one of the behavioral rules of myopic agents, the logit choice model, has become popular among researchers because of its analytic convenience (see Section 2 for existing studies using the logit choice rule). Under the logit choice rule, the probability of an agent's mistake decreases log-linearly in the payoff losses incurred by such a mistake. Recent experimental literature also supports the hypothesis that mistake probabilities decrease in payoff losses (see Section 2). In the widely used uniform mistake model, in which all possible mistakes are equally likely, the more likely path can be easily determined by comparing the number of mistakes involved and, hence, the lengths of paths (e.g., Kandori et al. (1993)). However, under the logit choice rule, determining the most likely path is far from obvious, because the probability of a path depends on the kinds of mistakes involved as well as on the length of the path. For example, Hwang and Newton (2016) provide an example in which two different kinds of mistake plays are involved in the most likely escape path from a convention under a finite population logit model (see Example 1 in Hwang and Newton (2016)).

Because of the complexity of the logit choice rule, there has been, so far, no general way to analyze the optimal evolutionary paths from one convention to another. As a concrete example, it is unknown which kind of contract conventions will emerge and persist when agents play the evolutionary version of the familiar Nash bargaining game (Nash, 1953) under the logit rules (see Section 5). The goal of this paper is to fill in this gap in the literature for general classes of games. Two recent studies address similar questions. Hwang and Newton (2016) study two-population coordination evolutionary models with *zero off-diagonal payoffs* and an arbitrary number of strategies, for both finite and infinite populations. Sandholm and Staudigl (2016) study a one-population coordination evolutionary model with *three strategies* in the infinite population limit. These two papers are discussed in more detail in Section 2 but we note here that their results are limited to specific classes of games (either games with zero off-diagonal payoffs or games with three strategies).

One of the main novelties of this paper is a new method—we call it “comparison principles”—which can be applied to one- or two-population games with an arbitrary number of strategies and which allows greatly reducing the complexity of finding the most likely evolutionary paths under the logit choice rule. Specifically, we find that under the logit choice rule, the positive feedback of agents (to coordinate) plays a key role. Kandori and Rob (1998) introduce the “marginal bandwagon property” to capture the positive feedback aspect of network externality, requiring that the advantage of strategy i over j is greater when the other player is playing strategy i . In stochastic evolutionary game theory, the (un-)likeliness of a path is measured by a quantity called the “cost”: the less costly a path is, the more likely is the transition induced by that path. We show that for *finite* population models with the logit choice rule, **(i)** positive feedback (defined by the marginal bandwagon property) implies that along the minimum cost escape paths from a status quo convention, agents always deviate first from the status quo convention strategy before deviating from other strategies (Lemma 4.1 (i)), and **(ii)** the relative strength of the positive feedback effects implies that the transitions from the status quo convention to another convention must occur consecutively in the cost optimal escape paths (Lemma 4.1 (ii)).

We then apply these results to the exit problem—the problem of finding a cost minimum path escaping from a convention—under finite population models and characterize the candidates for the cost minimizing paths, as follows. The candidates consist of (possibly) different kinds of repeated identical mistakes deviating from the status quo convention strategy (Proposition 4.1). Finally, to pin down the exact minimum cost escaping path, we consider the infinite population limit as in Sandholm and Staudigl (2016) (Proposition 4.2) and show that the most likely escape paths from the status quo convention involve only *one kind* of repeated identical mistakes of agents (Proposition 4.3, Proposition 4.4, and Theorem 4.1). This result holds for any coordination games satisfying the marginal bandwagon properties and some regularity conditions with *an arbitrary number of strategies*, regardless of symmetric or asymmetric games (hence, one- or two-population models; Theorems 4.1 and 5.1). To the best of our knowledge, this is a novel result.

As an application of our main results, we study the evolutionary bargaining convention for the Nash demand games under the logit choice rule and show that the Nash bargaining solution arises as the stochastically stable convention under the usual logit choice rule (called the unintentional logit dynamic). We also obtain a new bargaining convention when the logit choice rule is combined with intentional idiosyncratic (non-best response) plays (called the intentional logit dynamic). By *intentional idiosyncratic plays*, we mean that agents always experiment with strategies under which they would do better, should that strategy induce a convention (Naidu et al., 2010; Hwang et al., 2018). We show that the new solution under intentional logit dynamics is more egalitarian than the Nash bargaining solution, hence intentionality implies equality (Proposition 5.1). The reason for equality is as follows: under the unintentional logit rule, some transitions from the egalitarian convention (the equal division convention) to the Nash bargaining convention (the unequal division convention) are driven by a population who stands to lose by such transitions. Under the intentional logit rule, every transition is driven by the population who stands to benefit. Thus, some unfavorable transitions leading to the Nash bargaining convention are replaced by favorable transitions to the deviant population, leading to a more equal convention than the Nash bargaining convention. It can be easily seen that our comparison principle as well as our remaining arguments for the logit choice rule hold for the uniform mistake models under the assumption of the marginal bandwagon property. Thus, our comparison principles provide a unified framework for analyzing evolutionary dynamics, including the uniform mistake and logit models.

This paper is organized as follows. Section 2 discusses the related literature. Section 3 introduces the basic setup and discusses, in some detail, an example illustrating our methods. We present our main results for the exit problem for one population models in Section 4. In Section 5, we present results for two population models and analyze the Nash demand

game. In the appendix, we show that our result for the exit problem can be used to study the stochastic stability problem. The appendix also provides the technical details and proofs of the paper's results.

2. Related literature

There are many recent contributions to the analysis of stochastic evolutionary dynamics.¹ Here, we will focus on the following topics which relate most directly to our study: payoff-dependent mistake models, logit choice rules and related experimental evidence, and evolutionary bargaining.

2.1. Stochastic stability of payoff-dependent mistake models

As mentioned earlier, when mistake probabilities depend on payoffs, determining the most likely path seems *a priori* a daunting task, because the probability of a path depends on the kinds of mistakes involved as well as on the length of the path. Indeed, theoretical results for the exit and stochastic stability problems in the literature are limited to symmetric three strategy games in the one population setting or asymmetric two strategy games in the two population setting, except for only a few works.

There are two distinctive approaches in studying these questions. The first one is the so-called small noise double limit approach (Sandholm, 2010b; Staudigl, 2012; Sandholm and Staudigl, 2016; Arigapudi, 2020).² This approach takes a zero error rate limit first, as in the standard literature, and then taking an infinite population limit. Via these double limits, they obtain an optimal control problem over all escaping paths. Then, to find solutions to the optimal control problem, they solve the Hamilton-Jacobi equation associated with the obtained optimal control problem. This method, while providing a systematic approach, requires solving a (possibly challenging) Hamilton-Jacobi equation associated with the optimal control problem; hence, the results are limited to symmetric games with three strategies (see the discussion section in Sandholm and Staudigl (2016); Arigapudi (2020)) or asymmetric games with two strategies (Staudigl, 2012).

The second approach is to analyze minimum cost paths for the finite population model obtained by a zero-error limit and reduce the complex finite population problem into a lower dimension problem as in Hwang and Newton (2016). Our first step (Lemma 4.1 (i), Proposition 4.1 (i)) of comparing paths to show that agents switch first from the status quo convention strategy generalize the approach in Hwang and Newton (2016). Similarly to the current study, Hwang and Newton (2016) exploit, though implicitly, cost comparison arguments by constructing a lower bound function (see Section 5 in the cited paper). However, the arguments presented by Hwang and Newton (2016) differ from the current ones, as follows. First, in Hwang and Newton (2016), cost estimations of the constructed lower bound functions are possible only because all off-diagonal payoffs are zeros and because the cost functions are (multi-) linear with respect to the populations' states. Second, the lower bound function constructed in Hwang and Newton (2016) does not correspond to an *actual* path and, thus, the method does not provide guidance for how to construct a similar lower bound function for games other than those with zero off-diagonal payoffs. Their arguments thus cannot be applied to games with nonzero off-diagonal payoffs (e.g., the Nash demand game) or single population models in which the cost function of a path is quadratic with respect to population states. In contrast, under the condition of the *marginal bandwagon property*, our comparison methods are used to construct a lower cost path for a given arbitrary path in the state space (i.e., the simplex) and our arguments can be applied to any coordination games with an arbitrary number of strategies satisfying the marginal bandwagon property.

2.2. The logit choice rule and experimental evidence for payoff dependent mistake models

The logit choice rule, introduced by Blume (1993) to evolutionary game theory, has been widely used in stochastic evolutionary dynamics. Among them, Young and Burke (2001), mentioned in the introduction, adopted the logit choice model to study the contractual custom of cropsharing. Kreindler and Young (2013) show that fast convergence can occur when the error rate is small but non-vanishing even in a large population under logit dynamics. Belloc and Bowles (2013) also use the logit choice rule to study the effect of endogenous preferences and institutions on trade liberalization.³

The logic choice rule also receives special attention from the literature on random utility models and stochastic choices (McKelvey and Palfrey, 1995). Hofbauer and Sandholm (2002) derive the logit choice rule in two different ways: one from the random utility model and another from the optimization of perturbed expected utility. Recently, Fudenberg et al. (2015)

¹ Among them, Sawa and Wu (2018) study stochastic evolutionary dynamics of loss-averse agents who compares each strategy to a reference point (symmetric 2x2 coordination games) and show that a loss-dominant convention emerges in the long-run (see also Nax and Newton (2019)). Bilancini and Boncinelli (2020) study the stag-hunt game (the symmetric two strategy game) to study the emergence of a convention and transitions between conventions. They introduce condition-dependent mistakes in which errors converge to zero at a rate that is positively related to the payoff earned in the past and show that the payoff-dominant convention emerges when interactions are sufficiently persistent, while the maximin convention can emerge when interactions are volatile. See also Maruta (2002), Peski (2010), Sandholm (2010a). See Newton (2018) for an extensive survey.

² While Staudigl (2012) and Sandholm and Staudigl (2016) study the logit choice rule, Arigapudi (2020) studies the exit problem of the symmetric three strategy coordination game under the probit choice rule and identifies conditions under which the solution to the exit problem under probit choice is qualitatively similar to the logit choice. For the probit models, see Myatt and Wallace (2003), Dokumaci and Sandholm (2011).

³ Also, Alós-Ferrer and Netzer (2010), and Okada and Tercieux (2012) studied problems related to various revision rules and local potentials, respectively, under the logit choice rule.

provide two easily understood axioms under which stochastic choice corresponds to the maximization of the perturbed expected utility, and when the perturbation cost function is the commonly used one (namely, an entropy function), the optimal stochastic choice rule becomes the logit choice rule. Relatedly, in the literature on rational inattention or information acquisition, Matějka and McKay (2015) show that the decision maker's optimal information-processing strategy results in probabilistic choices that follow a logit model, where the parameter of perturbation is interpreted as the cost of information.

Experimental evidence for the class of payoff dependent mistake models to which the logit choice rule belongs is as follows. Mäs and Nax (2016) find in their experiment that a payoff decrease in the previous period would induce higher deviation rates from myopic best response behavior, which indicates that subjects' choices are sensitive to past payoff losses. Lim and Neary (2016) find that individual mistakes depend on the payoff of the myopic best-response payoff. Hwang et al. (2018) also provide experimental evidence suggesting non-best response play depends on payoffs: higher rates of non-best response play from subjects for whom the expected payoff from the best response strategy is lower (in Fig. 5 of the cited paper). Moreover, they conducted a logistic regression on payoff differences, providing further support for payoff dependence mistake models.

2.3. Evolutionary bargaining

This paper also adds to the literature on evolutionary bargaining (Young, 1993b, 1998; Binmore et al., 2003; Naidu et al., 2010; Hwang et al., 2018). In particular, Young (1993a) shows that the Nash bargaining solution emerges under the unintentional uniform deviation model of the Nash demand game and similarly, Naidu et al. (2010) find that the Nash bargaining solution emerges as well under the intentional uniform deviation model. In this paper, we find that the Nash bargaining solution is again stochastically stable under the unintentional logit choice rule, while a new bargaining convention which is more egalitarian than the Nash bargaining solution arises under the intentional logit choice rule, as explained. This shows quite well how evolutionary bargaining approaches can complement and extend the existing axiomatic bargaining approaches, hence contribute to the understanding of a certain bargaining convention in a society.

3. Stochastic evolutionary dynamics: setup and example

3.1. Basic setup: one population model

Consider a population of n agents who play a symmetric coordination game with strategy set $S = \{1, 2, \dots, |S|\}$ and payoff matrix A . The population state is described as a vector of fractions of agents using each strategy; that is, the state of the population is $x \in \Delta^{(n)}$, where $\Delta^{(n)}$ is the simplex

$$\Delta^{(n)} := \{(x_1, \dots, x_{|S|}) \in \frac{1}{n} \mathbb{Z}^{|S|} : \sum_i x(i) = 1, x(i) \geq 0 \text{ for all } i\}.$$

The expected payoff to an agent who chooses strategy i at population state x is given by $\pi(i, x) := \sum_{j \in S} A_{ij}x(j)$.

We consider a discrete time strategy updating process, defined as follows. At each period, a randomly chosen agent selects a new strategy. The new population state induced by the agent's switching from strategy i to j is denoted by $x^{i,j}$ and the state induced by two agents' transitions, first from i to j and then from k to l , is denoted by $x^{(i,j)(k,l)}$. More precisely,

$$x^{i,j} := x + \frac{1}{n}(e_j - e_i), \text{ and } x^{(i,j)(k,l)} := x + \frac{1}{n}(e_j - e_i) + \frac{1}{n}(e_l - e_k), \tag{1}$$

where e_h is the h -th element of the standard basis for $\mathbb{R}^{|S|}$. The conditional probability that an agent with strategy i chooses new strategy j given population state x is specified by the logit choice rule (Blume, 1993),

$$\text{Logit choice rule: } p_\eta^{(n)}(j|i, x) = \frac{\exp(\eta^{-1}\pi(j, x))}{\sum_l \exp(\eta^{-1}\pi(l, x))}, \tag{2}$$

where $\eta > 0$ is a positive parameter interpreted as the degree of (ir)rationality (or noise level). That is, as η decreases to 0, equation (2) converges to the so-called best-response rule, whereas as η increases to ∞ , equation (2) converges to a choice rule that assigns equal probabilities to each strategy—namely, a pure randomization rule.

The transition probabilities for the updating dynamics are

$$P_\eta^{(n)}(x, x^{i,j}) = x(i)p_\eta^{(n)}(j|i, x)$$

for $i \neq j$, where factor $x(i)$ accounts for the fact that at each period, one agent is randomly chosen to revise her strategy. The unlikeliness of a transition in stochastic evolutionary game theory is measured by the cost, $c^{(n)}(x, y)$, between two states, x, y :

$$c^{(n)}(x, y) := \begin{cases} -\lim_{\eta \rightarrow 0} \eta \ln P_\eta^{(n)}(x, y) & \text{if } y = x^{i,j} \text{ for some } i, j, i \neq j \\ 0 & \text{if } y = x \\ \infty & \text{otherwise} \end{cases}$$

which becomes

$$c^{(n)}(x, x^{i,j}) = \max\{\pi(l, x) : l \in S\} - \pi(j, x) \tag{3}$$

under the logit choice rule (2). When η is sufficiently small, we have $P_\eta^{(n)}(x, y) \asymp e^{-\eta^{-1}c^{(n)}(x,y)}$ and thus the cost between x and y , $c^{(n)}(x, y)$, is the exponential rate of decay of the probability of transition from x to y . In equation (3), the first term $\max\{\pi(l, x) : l \in S\}$ is equal to the payoff $\pi(\bar{m}, x)$ for agents playing a best response \bar{m} in the population state x , and the second term is the payoff to agents playing the new strategy, j . Thus, when a strategy-revising agent adopts the best response \bar{m} , the cost of such an action is zero. However, when she adopts a sub-optimal strategy $j \neq \bar{m}$, the cost is the payoff loss due to choosing this strategy instead of the best response. Under the uniform mistake model, $c^{(n)}(x, x^{i,j}) = 1$ if $j \neq \bar{m}$ and $c^{(n)}(x, x^{i,j}) = 0$ if $j = \bar{m}$; that is the cost in the uniform mistake model is state independent.

We consider a symmetric coordination game A in which every symmetric strategy profile (i.e. the strategy profile in which row and column players choose the same strategy) is a strict Nash equilibrium (see Condition A). A convention is defined as a state in which every agent plays the same strategy which is a strict Nash equilibrium of A . Thus, x is a convention if $x = e_i$ for some strategy i , which is a strict Nash equilibrium strategy (recall that e_i is the i -th element of the standard basis of $\mathbb{R}^{|S|}$). Focusing our attention on one such convention, we refer to convention \bar{m} as a status quo convention. A path γ is a sequence of states, $\gamma := (x_1, x_2, \dots, x_T)$, such that $x_{t+1} = (x_t)^{i,j}$ for some i, j and for all t , and we define the cost of a path as the sum of the costs of the transitions between the states in equation (3), i.e.

$$I^{(n)}(\gamma) := \sum_{t=1}^{T-1} c^{(n)}(x_t, x_{t+1}). \tag{4}$$

Next, recall the *marginal bandwagon property* (**MBP**) introduced by Kandori and Rob (1998). A symmetric game with payoff matrix A satisfies the **MBP** if

$$A_{ii} - A_{ji} > A_{ik} - A_{jk} \text{ for all distinct } i, j, k. \tag{5}$$

The condition in (5) says that the advantage of playing strategy i over strategy j is greater when the other player plays strategy i rather than another strategy k . In our population dynamic model, this implies a positive feedback effect in which the marginal advantage of switching into strategy i increases in the number of agents adopting strategy i . We also consider coordination games in which $A_{ii} > A_{ji}$ for all i, j , and assume the existence of mixed-strategy Nash equilibria supported on any arbitrary subset of the strategy set S .

Condition A: (i) A game with payoff matrix A is a coordination game (i.e., $A_{ii} > A_{ji}$ for all i, j), satisfying the **MBP** (see equation (5)),

(ii) Suppose that for any $T = \{i_1, \dots, i_K\} \subset S$, there exists a unique $q \in \Delta$ with support T such that

$$\pi(i_1, q) = \dots = \pi(i_K, q)$$

and q is a mixed strategy Nash equilibrium. \square

We define the basin of attraction of $e_{\bar{m}}$, $D^{(n)}(e_{\bar{m}})$, and its boundary, $\partial D_{\bar{m},j}^{(n)}$, as follows:

$$D^{(n)}(e_{\bar{m}}) := \{x \in \Delta^{(n)} : \pi(\bar{m}, x) \geq \pi(k, x) \text{ for all } k\}$$

$$\partial D_{\bar{m},j}^{(n)} := \{x \in D^{(n)}(e_{\bar{m}}) : x^{\bar{m},j} \in D^{(n)}(e_j)\}$$

In fact, if x belongs to $D^{(n)}(e_{\bar{m}})$, the cost of a transition from i to j is

$$c^{(n)}(x, x^{i,j}) = \pi(\bar{m}, x) - \pi(j, x) \tag{6}$$

for $i \neq j$. If $j = \bar{m}$, the cost in equation (6) is zero, and the **MBP** implies that convention \bar{m} can be reached from any $x \in D^{(n)}(e_{\bar{m}})$ at no cost; thus, $D^{(n)}(e_{\bar{m}})$ is indeed the basin of attraction of convention \bar{m} . Observe that the requirement for the existence of mixed-strategy Nash equilibria with arbitrary support implies the existence of the distinctive basins of attraction for all pure strategies. Using this setup, we next present a simple example of a three-strategy game to illustrate the main ideas and results of the paper.

3.2. Illustration of the main results

Consider a technology choice game consisting of three technologies indexed by 1, 2, and 3, respectively. For example, for PC operating systems, consider Windows, OSX, and Linux. Let b_i be the benefit of technology i that the user obtains when interacting with another user of the same technology; thus, b_i is related to the inherent quality of technology i . Suppose

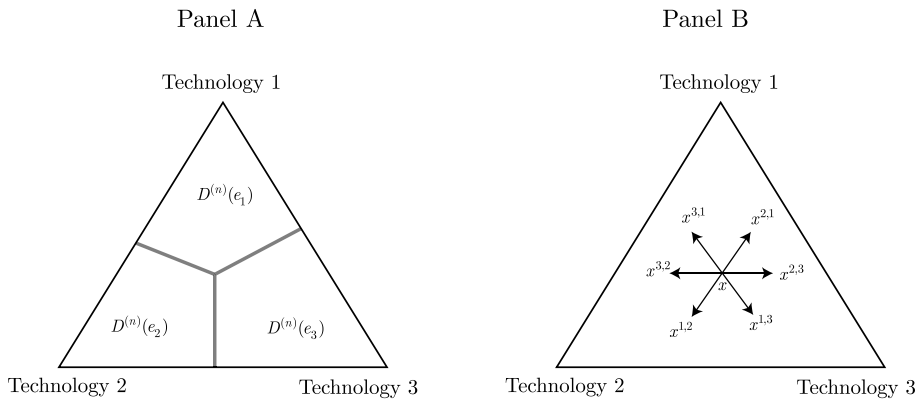


Fig. 1. Basins of attraction and paths involving a single agent's transitions. Panel A shows the basins of attractions of conventions. Panel B illustrates new states induced by a single agent's switching from state x . For example, $x^{1,2}$ is the new state induced by a single agent's switching from strategy 1 to strategy 2.

that the user of technology i experiences some utility or disutility when interacting with users of a different technology. For the sake of simplicity, the users of technologies 2, 3, and 1 derive utility d when interacting with the users of technologies 1, 2, and 3, respectively. By the same token, the users of technologies 1, 2, and 3 experience disutility d when interacting with the users of technologies 2, 3, and 1, respectively. In summary, the payoff matrix is given by

$$A = \begin{pmatrix} b_1 & -d & d \\ d & b_2 & -d \\ -d & d & b_3 \end{pmatrix}. \tag{7}$$

In the context of the technology choice game, we are interested in the positive feedback effects, where the advantages of a technology increase as the number of users of that technology increases. If

$$3d < \min_i b_i \tag{8}$$

holds, all pure strategies 1, 2, and 3 are strict Nash equilibria, and the **MBP** condition in (5) is satisfied.

Suppose that the agents' strategy revision rule is the logit choice rule. Our example is the exit problem from a convention (see Freidlin and Wentzell (1998)). Specifically, given the status quo convention of technology 1, what is the most likely way to upset this convention? Panel A of Fig. 1 shows the basins of attraction of conventions. Thus, our problem can be stated succinctly as

$$\min\{I^{(n)}(\gamma) : \gamma \text{ escapes } D^{(n)}(e_1)\}. \tag{9}$$

We now explain how the **MBP** in (5) significantly reduces the complexity of solving the minimization problem in (9).

Comparison principle 1: Lemma 4.1 (i), Proposition 4.1 (i)

Our idea is to develop systematic ways of comparing the costs of various paths and to reduce the number of candidate solutions for the minimization problem in (9) under the logit choice rule. First, consider the two paths in Panel A in Fig. 2:

$$x \rightarrow x^{2,3} \rightarrow x^{(2,3)(1,3)} \tag{10}$$

$$x \rightarrow x^{1,3} \rightarrow x^{(1,3)(2,3)}, \tag{11}$$

where $x^{(2,3)(1,3)} = x^{(1,3)(2,3)}$ (see the definitions of $x^{i,j}$ and $x^{(i,j)(k,l)}$ in equation (1)). The cost difference between these two paths is

$$\Delta_1 c := c^{(n)}(x, x^{2,3}) + c^{(n)}(x^{2,3}, x^{(2,3)(1,3)}) - [c^{(n)}(x, x^{1,3}) + c^{(n)}(x^{1,3}, x^{(1,3)(2,3)})]. \tag{12}$$

Note that under the logit choice rule, an agent switching from strategy 2 to strategy 3 compares the expected payoff of strategy 3 with the best response (strategy 1) rather than with strategy 2, implying that the costs of transitions from strategy 2 to strategy 3 and from strategy 1 to strategy 3 are the same. Indeed using equation (3) we verify that

$$c^{(n)}(x, x^{2,3}) = c^{(n)}(x, x^{1,3}), \quad c^{(n)}(x^{1,3}, x^{(1,3)(2,3)}) = c^{(n)}(x^{1,3}, x^{(1,3)(1,3)}) \tag{13}$$

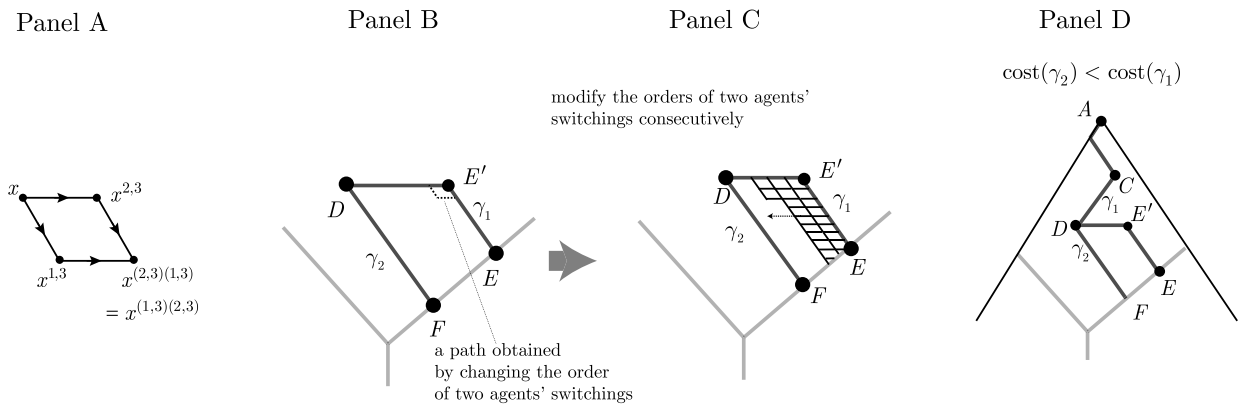


Fig. 2. Comparison Principle 1. Panel A shows two paths in equations (10) and (11). Panels B and C show how we can obtain a lower cost path by modifying the orders of two agents' switching from one strategy to another. In Panel D, we show that the cost of γ_2 in (16) is lower than that of γ_1 in (15) by applying this procedure repeatedly.

Thus, using equation (13), we can simplify (12) as follows:

$$\Delta_1 c = c^{(n)}(x^{2,3}, x^{(2,3)(1,3)}) - c^{(n)}(x^{1,3}, x^{(1,3)(1,3)})$$

Note that $x^{2,3}$ is the state where exactly one more agent than in $x^{1,3}$ plays strategy 1, because $x^{2,3} - x^{1,3} = \frac{1}{n}(e_1 - e_2)$. The positive feedback effect of strategy 1 over strategy 3, induced by the MBP, means that the payoff advantage of strategy 1 over strategy 3 is greater when the other player uses strategy 1. Thus, the marginal advantage of switching to strategy 1 is greater at state $(x^{2,3})$ where one more agent than in the other state $(x^{1,3})$ plays strategy 1. This means that the payoff loss due to the mistake of not playing strategy 1 is greater at $x^{2,3}$ than it is at $x^{1,3}$. Thus, we expect that under the logit dynamic, the transition from strategy 1 to strategy 3 will be more costly at $x^{2,3}$ than it will be at $x^{1,3}$. Indeed, we find that

$$\begin{aligned} \Delta_1 c &= c^{(n)}(x^{2,3}, x^{(2,3)(1,3)}) - c^{(n)}(x^{1,3}, x^{(1,3)(1,3)}) = \pi(1, x^{2,3}) - \pi(3, x^{2,3}) - [\pi(1, x^{1,3}) - \pi(3, x^{1,3})] \\ &= \frac{1}{n}(A_{11} - A_{31} - (A_{12} - A_{32})) > 0 \end{aligned} \tag{14}$$

which is positive from the marginal bandwagon property (condition (5)). Equation (14) also shows that when agents make the same mistake (switching from 1 to 3), the cost becomes cheaper ($c^{(n)}(x^{1,3}, x^{(1,3)(1,3)}) < c^{(n)}(x^{2,3}, x^{(2,3)(1,3)})$). In sum, under the assumption of the marginal bandwagon property, the cost of path $x \rightarrow x^{1,3} \rightarrow x^{(1,3)(2,3)}$ is cheaper than that of $x \rightarrow x^{2,3} \rightarrow x^{(2,3)(1,3)}$ (see Lemma 4.1 (i)).

Now, consider the new path (shown as a dotted line) obtained by altering a single agent's switching in Panel B of Fig. 2. The cost difference between the original and new paths in Panel B of Fig. 2 is precisely the cost difference between the two paths in Panel A. Thus, if equation (14) is positive, the cost of the new path in Panel B is strictly lower than that of the original path. Then, by successively altering a single agent's switching, we can apply the same arguments repeatedly as in Panel C of Fig. 2 (Proposition 4.1, (i)). In this way, we find that the cost of path $D \rightarrow F$ is cheaper than that of path $D \rightarrow E' \rightarrow E$ and finally the cost of path γ_2 is cheaper than that of path γ_1 (Panel D in Fig. 2), where

$$\gamma_1 : A \rightarrow C \rightarrow D \rightarrow E' \rightarrow E \tag{15}$$

$$\gamma_2 : A \rightarrow C \rightarrow D \rightarrow F. \tag{16}$$

Comparison principle 2: Lemma 4.1 (ii), Proposition 4.1 (ii)

Next, we explain the second comparison principle. Similarly, we first consider the two paths in Panel A of Fig. 3:

$$x \rightarrow x^{1,3} \rightarrow x^{(1,3)(1,2)} \tag{17}$$

$$x \rightarrow x^{1,2} \rightarrow x^{(1,2)(1,3)} \tag{18}$$

where $x^{(1,3)(1,2)} = x^{(1,2)(1,3)}$. We find

$$\begin{aligned} \Delta_2 c &:= c^{(n)}(x, x^{1,3}) + c^{(n)}(x^{1,3}, x^{(1,3)(1,2)}) - [c^{(n)}(x, x^{1,2}) + c^{(n)}(x^{1,2}, x^{(1,2)(1,3)})] \\ &= \underbrace{[c^{(n)}(x, x^{1,3}) - c^{(n)}(x^{1,2}, x^{(1,2)(1,3)})]}_{\text{(i) positive feedback of 1 over 3}} - \underbrace{[c^{(n)}(x, x^{1,2}) - c^{(n)}(x^{1,3}, x^{(1,3)(1,2)})]}_{\text{(ii) positive feedback of 1 over 2}}. \end{aligned} \tag{19}$$

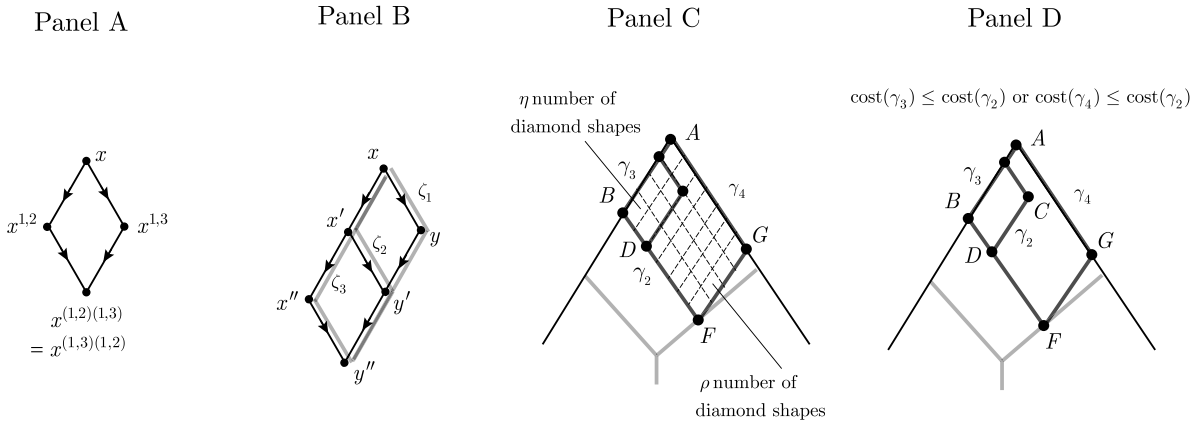


Fig. 3. Comparison Principle 2. Panel A shows two paths in (17) and (18). Panel B illustrates equations in (21), (22), and (23), showing that the cost of ζ_2 is no less than that of ζ_1 or ζ_3 . Panel C illustrates the argument in equations (26), (27), and (28). Thus, in Panel D, we obtain that the cost of γ_2 is again no less than that of γ_3 or γ_4 .

If we let $x = y^{2,3}$, then $x^{1,2} = y^{1,3}$ and, as in equation (14), (i) in equation (19) becomes

$$\begin{aligned} c^{(n)}(x, x^{1,3}) - c^{(n)}(x^{1,2}, x^{(1,2)(1,3)}) &= c^{(n)}(y^{2,3}, y^{(2,3)(1,3)}) - c^{(n)}(y^{1,3}, y^{(1,3)(1,3)}) \\ &= \frac{1}{n}(A_{11} - A_{31} - (A_{12} - A_{32})). \end{aligned}$$

Furthermore, if we let $x = z^{3,2}$, then $x^{1,3} = z^{1,2}$ and (ii) in equation (19) becomes

$$\begin{aligned} c^{(n)}(x, x^{1,2}) - c^{(n)}(x^{1,3}, x^{(1,3)(1,2)}) &= c^{(n)}(z^{3,2}, z^{(3,2)(1,2)}) - c^{(n)}(z^{1,2}, z^{(1,2)(1,2)}) \\ &= \frac{1}{n}(A_{11} - A_{21} - (A_{13} - A_{23})), \end{aligned}$$

which is the positive feedback effect of strategy 1 over 2. Thus,

$$\Delta_2 c = \frac{1}{n}(-A_{12} + A_{13} + A_{21} - A_{23} - A_{31} + A_{32}) = \frac{6d}{n}. \tag{20}$$

Here, $6d$ in equation (20) can be positive, negative, or zero. When $6d = 0$, the game is a potential game—this is the well-known test for potential games by Hofbauer (1985). Sandholm and Staudigl (2016) also define (20) by “skew” and use it to compare the costs of paths in the infinite population model.

Next, using this result, we compare the three paths in Panel B of Fig. 3, defined as follows:

$$\zeta_1 : x \rightarrow y \rightarrow y' \rightarrow y'', \quad \zeta_2 : x \rightarrow x' \rightarrow y' \rightarrow y'', \quad \zeta_3 : x \rightarrow x' \rightarrow x'' \rightarrow y''$$

Then, applying equation (20), we find that

$$I^{(n)}(\zeta_1) - I^{(n)}(\zeta_2) = \frac{6d}{n} \text{ and } I^{(n)}(\zeta_2) - I^{(n)}(\zeta_3) = \frac{6d}{n} \tag{21}$$

which shows that

$$[I^{(n)}(\zeta_2) - I^{(n)}(\zeta_1)] + [I^{(n)}(\zeta_2) - I^{(n)}(\zeta_3)] = 0. \tag{22}$$

This, in turn, implies that either

$$I^{(n)}(\zeta_2) \geq I^{(n)}(\zeta_1) \text{ or } I^{(n)}(\zeta_2) \geq I^{(n)}(\zeta_3) \tag{23}$$

holds. Thus, from the inequalities in (23), either ζ_1 or ζ_3 costs less than (or is equal to) ζ_2 and we can remove ζ_2 from the candidate paths minimizing the problem in equation (9).

Next, we compare the costs of the three paths, $\gamma_2 : A \rightarrow C \rightarrow D \rightarrow F$ in (16), γ_3 , and γ_4 (Panels C and D of Fig. 3), where

$$\gamma_3 : A \rightarrow B \rightarrow F \tag{24}$$

$$\gamma_4 : A \rightarrow G \rightarrow F. \tag{25}$$

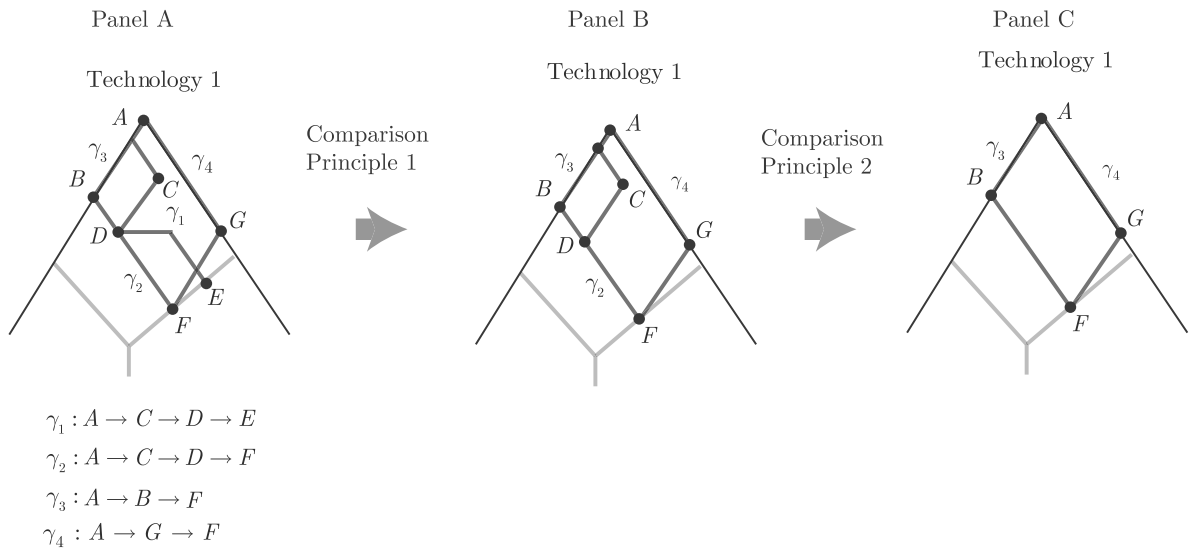


Fig. 4. Implications of Comparison Principles 1 and 2. Panel A shows four different paths, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 . By applying comparison principle 1, we eliminate γ_1 from the set of candidate solution to the exit problem in (9) as in Panel B. Then by applying comparison principle 2, we eliminate γ_2 as in Panel C.

For the purpose of exposition, assume that there are ρ diamond shapes in the area between γ_4 and γ_2 , and η diamond shapes between γ_2 and γ_3 (see Panel C of Fig. 3). Now, by applying the comparison results in equation (21) successively, we find that

$$I^{(n)}(\gamma_4) - I^{(n)}(\gamma_2) = \rho \frac{6d}{n}, \quad I^{(n)}(\gamma_2) - I^{(n)}(\gamma_3) = \eta \frac{6d}{n}. \tag{26}$$

which yields

$$\eta [I^{(n)}(\gamma_2) - I^{(n)}(\gamma_4)] + \rho [I^{(n)}(\gamma_2) - I^{(n)}(\gamma_3)] = 0. \tag{27}$$

This, in turn, implies that either

$$I^{(n)}(\gamma_2) \geq I^{(n)}(\gamma_4) \text{ or } I^{(n)}(\gamma_2) \geq I^{(n)}(\gamma_3) \tag{28}$$

holds. Using this, we can also remove γ_2 from the minimum cost candidate paths (see Panel D of Fig. 3).

Finally, we apply these two comparison principles, to obtain a class of paths comprising the candidate solutions to the cost minimization problem in equation (9) (see Fig. 4). One of these paths consists of consecutive transitions first from technology 1 to technology 2 and then from technology 1 to technology 3 (see γ_3 in Panel C in Fig. 4). Alternatively, there could be another path consisting of consecutive transitions first from technology 1 to technology 3 and then from technology 1 to technology 2 (see γ_4 in Panel C in Fig. 4). Thus, we can reduce the complicated objective function in equation (9) to a function of two variables (i.e., the number of transitions from technology 1 to technology 2, and those from technology 1 to technology 3) and easily study the minimization problem of this simple objective function using the MBP again. In general, we reduce the objective function with an arbitrary number of variables in equation (9) to an objective function with $|\mathcal{S}| - 1$ variables, where $|\mathcal{S}|$ is the number of strategies of the underlying game. We then prove that the lowest cost transition path to escape convention 1 involves the repetition of the same kinds of mistakes in the infinite population limit. That is, graphically, these paths lie on the edges of the simplex from strategy 1 to strategy 2 and from strategy 1 to strategy 3.

4. Exit from the basin of attraction of a convention: one population models

We now present our comparison principles for games with an arbitrary number of strategies for a finite population. Our first comparison principle shows that given two paths, $x \rightarrow x^{(i,k)} \rightarrow x^{(i,k)(\bar{m},l)}$ and $x \rightarrow x^{(\bar{m},k)} \rightarrow x^{(i,k)(\bar{m},l)}$, it always costs less (or the same) to first switch away from strategy \bar{m} and then to switch away from the other strategies, as already explained in Section 3 (Lemma 4.1). Our second comparison principle is based on the fact that the sizes of the two different positive feedback effects (MBP) can be compared as follows (see equation (20)):

$$[A_{ii} - A_{ji} - A_{ik} + A_{jk}] - [A_{ii} - A_{ki} - A_{ij} + A_{kj}] = [A_{ij} - A_{ji}] + [A_{jk} - A_{kj}] + [A_{ki} - A_{ik}] \tag{29}$$

Lemma 4.1. *The following statements hold.*

(i) (**Comparison principle 1**). *Suppose that the MBP holds. Consider two paths γ_1 and γ_2 (Panel A, Fig. 2) in $D^{(n)}(e_{\bar{m}})$:*

$$\gamma_1 : x \rightarrow x^{\bar{m},k} \rightarrow x^{(\bar{m},k)(i,l)}, \quad \gamma_2 : x \rightarrow x^{i,k} \rightarrow x^{(i,k)(\bar{m},l)},$$

where $i \neq \bar{m}, k \neq i, \bar{m}, l \neq i, \bar{m}$. Then,

$$I^{(n)}(\gamma_2) \geq I^{(n)}(\gamma_1).$$

(ii) (**Comparison principle 2**). *Consider three paths ζ_1, ζ_2 and ζ_3 (Panel B, Fig. 3) in $D(e_{\bar{m}})$:*

$$\zeta_1 : x \rightarrow x^{\bar{m},j} \rightarrow x^{(\bar{m},j)(\bar{m},i)} \rightarrow x^{(\bar{m},j)(\bar{m},i)(\bar{m},i)}$$

$$\zeta_2 : x \rightarrow x^{\bar{m},i} \rightarrow x^{(\bar{m},i)(\bar{m},j)} \rightarrow x^{(\bar{m},i)(\bar{m},j)(\bar{m},i)}$$

$$\zeta_3 : x \rightarrow x^{\bar{m},i} \rightarrow x^{(\bar{m},i)(\bar{m},i)} \rightarrow x^{(\bar{m},i)(\bar{m},i)(\bar{m},j)}$$

where $i \neq j$. Then, we have either

$$I^{(n)}(\zeta_2) \geq I^{(n)}(\zeta_1) \text{ or } I^{(n)}(\zeta_2) \geq I^{(n)}(\zeta_3)$$

Proof. See Appendix A. \square

Next, we present our main result on the exit problem from convention \bar{m} for finite population n . Let $\mathcal{G}_{\bar{m}}^{(n)}$ be the set of all paths escaping the basin of attraction of convention \bar{m} at some arbitrary time T ; that is,

$$\begin{aligned} \mathcal{G}_{\bar{m}}^{(n)} := \{ \gamma = (x_0, \dots, x_T) : x_0 = e_{\bar{m}}, x_t \in D^{(n)}(e_{\bar{m}}) \text{ for } 0 < t < T - 1 \\ \text{and } x_T \notin D^{(n)}(e_{\bar{m}}) \text{ for some } T > 0 \}. \end{aligned} \tag{30}$$

Thus, our exit problem can be written formally as

$$\min \{ I^{(n)}(\gamma) : \gamma \in \mathcal{G}_{\bar{m}}^{(n)} \}. \tag{31}$$

Using Lemma 4.1, we significantly reduce the number of candidate solutions to the problem in (31) as explained in Section 3.2 in the special case of three strategies. First, the comparison principle 1 in Lemma 4.1 implies that we can focus on the class of paths for which all transitions are from \bar{m} to some other strategy i (i.e., all paths consisting of straight lines, parallel to the edges of the simplex) (see Fig. 5)—the set of paths defined as follows:

$$\mathcal{J}_{\bar{m}}^{(n)} := \{ \gamma = (x_1, x_2, \dots, x_T) \in \mathcal{G}_{\bar{m}}^{(n)} ; x_{t+1} = (x_t)^{\bar{m},i} \text{ for some } i, \text{ for all } t \leq T - 1 \} \tag{32}$$

Then, using the comparison principle 2 in Lemma 4.1, we can further reduce the number of candidate solutions to (31) and hence consider a subset of $\mathcal{J}_{\bar{m}}^{(n)}$ in which identical transitions from \bar{m} occur consecutively:

$$\gamma : x \xrightarrow[t_1 \text{ times}]{\text{from } \bar{m} \text{ to } i_1} y \xrightarrow[t_2 \text{ times}]{\text{from } \bar{m} \text{ to } i_2} z \dots \xrightarrow[t_K \text{ times}]{\text{from } \bar{m} \text{ to } i_K} w$$

for some given (distinct) i_1, i_2, \dots, i_K . That is, γ consists of a series of consecutive transitions first from \bar{m} to i_1 , then from \bar{m} to i_2 , ..., and finally from \bar{m} to i_K . More precisely, we write γ as

$$\gamma = (x_1, x_2, \dots, x_T) = \begin{cases} x_{t+1} = (x_t)^{\bar{m},i_1} & \text{if } 1 \leq t \leq t_1 \\ x_{t+1} = (x_t)^{\bar{m},i_2} & \text{if } t_1 + 1 \leq t \leq t_1 + t_2 \\ \vdots \\ x_{t+1} = (x_t)^{\bar{m},i_K} & \text{if } \sum_{l=1}^{K-1} t_l + 1 \leq t \leq \sum_{l=1}^K t_l =: T - 1 \end{cases} \tag{33}$$

and define

$$\begin{aligned} \mathcal{K}_{\bar{m}}^{(n)} := \{ \gamma : \gamma = (x_1, x_2, \dots, x_T) \in \mathcal{J}_{\bar{m}}^{(n)}, \gamma \text{ is given by (33)} \\ \text{for some } t_1, t_2, \dots, t_K, \text{ for some distinct } i_1, i_2, \dots, i_K \}. \end{aligned} \tag{34}$$

For example, consider the following three paths:

$$\gamma_1 : x^{(1)} \xrightarrow{\text{from 1 to 2}} x^{(2)} \xrightarrow{\text{from 1 to 3}} x^{(3)} \xrightarrow{\text{from 1 to 2}} x^{(4)}$$

$$\gamma_2 : y^{(1)} \xrightarrow{\text{from 1 to 2}} y^{(2)} \xrightarrow{\text{from 1 to 3}} y^{(3)}$$

$$\gamma_3 : z^{(1)} \xrightarrow{\text{from 1 to 3}} z^{(2)} \xrightarrow{\text{from 1 to 2}} z^{(3)}$$

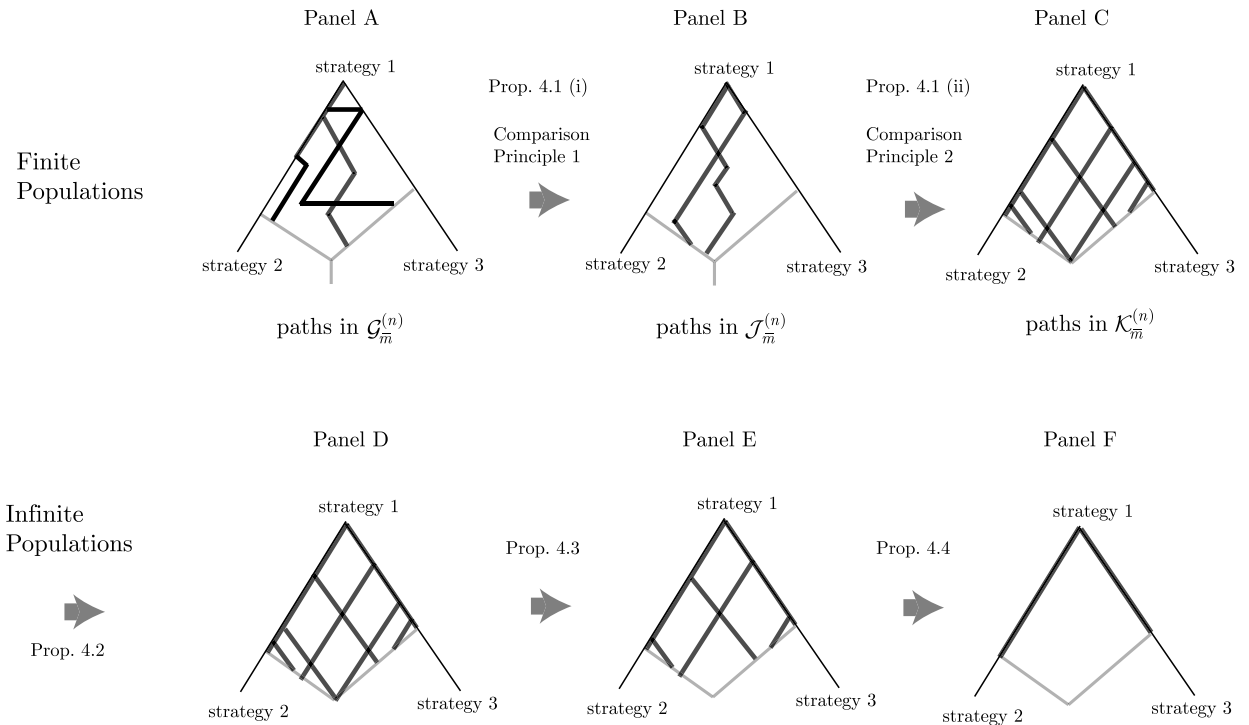


Fig. 5. Outline of the proof of Theorem 4.1. Each panel shows how each proposition reduces the number of candidate solutions to the exit problem.

Then $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{J}_1^{(n)}$ and $\gamma_2, \gamma_3 \in \mathcal{K}_1^{(n)}$, but $\gamma_1 \notin \mathcal{K}_1^{(n)}$ since $\mathcal{K}_1^{(n)}$ contains only paths in which identical transitions occur consecutively (see Panels B and C in Fig. 5). The main idea of the following Proposition 4.1 is illustrated in Section 3.2.

Proposition 4.1 (Finite populations). *Suppose that Condition A holds. Then, we have the following characterizations:*

(i) **Comparison principle I**

$$\min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_m^{(n)}\} = \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{J}_m^{(n)}\}.$$

(ii) **Comparison principle II**

$$\min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_m^{(n)}\} = \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{K}_m^{(n)}\}$$

Proof. See Appendix A. \square

Even after eliminating a substantial number of irrelevant paths from the set of candidate solutions to the problem in (31) in the setting of the finite population using Proposition 4.1 (Panels A, B, and C in Fig. 5), function $I^{(n)}$ still remains complicated, with negligible terms (in the order of n) when the population is large. Thus, we consider an infinite population limit as in Sandholm and Staudigl (2016). However, unlike Sandholm and Staudigl (2016) who solve the Hamilton Jacobi equation to find the optimal path, we directly show that the minimizing escaping paths lie on the boundaries of the simplex using the properties of MBP (Panels D, E and F in Fig. 5). This shows that the MBP is also the key condition for comparing two paths in the infinite population model. Fig. 5 illustrates the outline of the proof of the main theorem in this section, Theorem 4.1.

Specifically, to study the infinite population problem, we need to first find the continuous version of $c^{(n)}(x, x^{i,j})$ in (3). For this, we denote by $\bar{D}(e_i)$ and $\partial \bar{D}_{i,j}$ the continuous versions of $D^{(n)}(e_i)$ and $\partial D_{i,j}^{(n)}$, respectively (in an appropriate convergence sense; see equation (A.1)). Suppose that $p, q \in \Delta$ with $q = p + \alpha(e_i - e_j)$ for some $\alpha > 0$. For $p, q \in \bar{D}(e_{\bar{m}})$, define

$$\bar{c}(p, q) := \frac{1}{2}(p_j - q_j)(\pi(\bar{m}, p + q) - \pi(i, p + q)). \tag{35}$$

We show that the definition of a continuous cost function in (35) is precisely the limit of the discrete cost function at $n = \infty$ (see Lemma A.3 in Appendix A). We will denote the cost of a continuum path ζ as $\bar{I}(\zeta)$.

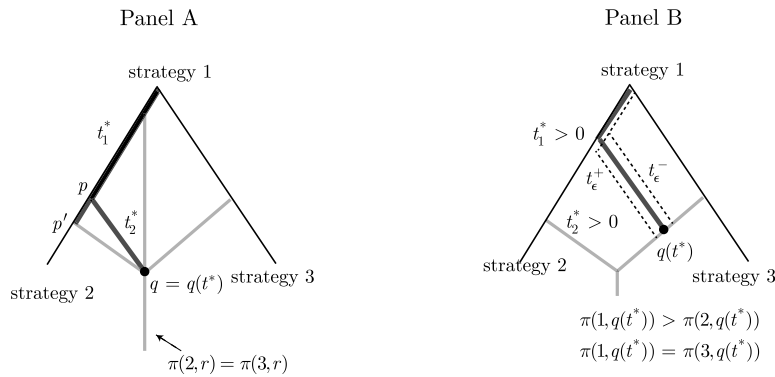


Fig. 6. Proofs of Propositions 4.3 and 4.4. Panel A illustrates Proposition 4.3; Panel B illustrates Proposition 4.4. Panel A shows that if $t_1^* > 0$, $t_2^* > 0$, $\pi(1, q(t^*)) > \pi(2, q(t^*))$, and $\pi(1, q(t^*)) = \pi(3, q(t^*))$, then either the path t_1^+ or the path t_2^- can be lower than or equal to the cost of the path (t_1^*, t_2^*) . Panel B illustrates two paths—one involving p and p' , and the other involving p and q . Proposition 4.4 shows that the path through p to p' has a lower cost than the path through p to q .

Next, we define a continuum analogue of the set of paths $\mathcal{K}_{\bar{m}}^{(n)}$ (equation (34)) in the limit of $n \rightarrow \infty$ and define an associated cost function as follows. Roughly speaking $\bar{\mathcal{K}}_{\bar{m}}$ is the set of paths consisting of a collection of piecewise straight lines. More precisely, for $t = ((t_1, \dots, t_K); (i_1, \dots, i_K))$ where $t_l \in [0, 1]$ for all l ,

$$p^{(0)} = e_{\bar{m}}, p^{(1)} = e_{\bar{m}} + t_1(e_{i_1} - e_{\bar{m}}), \dots, p^{(K)} = e_{\bar{m}} + \sum_{l=1}^K t_l(e_{i_l} - e_{\bar{m}}), \tag{36}$$

where $p^{(l)} \in \bar{D}(e_{\bar{m}})$ for all l and $p^{(K)} \in \partial \bar{D}(e_{\bar{m}}) := \bigcup_{l \neq \bar{m}} \partial \bar{D}_{\bar{m}, j}$. We also set

$$q(t) := p^{(K)}, \zeta(t) := (p^{(0)}, p^{(1)}, \dots, p^{(K)}), \text{ and } \omega(t) := \bar{I}(\zeta(t)) = \sum_{t=0}^{K-1} \bar{c}(p^{(t)}, p^{(t+1)}) \tag{37}$$

Note that path $\zeta = \zeta(t) \in \bar{\mathcal{K}}_{\bar{m}}$ is uniquely determined by the vector $t = ((t_1, \dots, t_K); (i_1, \dots, i_K))$. We then have the following infinite population limit result.

Proposition 4.2 (Infinite population limit).

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min \{I^{(n)}(\gamma) : \gamma \in \mathcal{K}_{\bar{m}}^{(n)}\} = \min \{\omega(t) : \zeta(t) \in \bar{\mathcal{K}}_{\bar{m}}\}.$$

Proof. See Appendix A. \square

Proposition 4.2 can be seen as a special case of Theorem 9 in Sandholm and Staudigl (2016), which shows the convergence over all possible paths. For completeness, we provide our own proof of Proposition 4.2 in Appendix A. Proposition 4.2 leads us to consider the following minimization problem over the set of continuous paths in $\bar{\mathcal{K}}_{\bar{m}}$:

$$\min \{\omega(t) : \zeta(t) \in \bar{\mathcal{K}}_{\bar{m}}\}. \tag{38}$$

Next, we further reduce the number of candidate solutions to (38) by comparing the costs of paths in the set of $\bar{\mathcal{K}}_{\bar{m}}$ (see Panels D, E, and F in Fig. 5). Note that at the end point q of the escape path, the following constraints must be satisfied:

$$\pi(\bar{m}, q) \geq \pi(l, q) \text{ for all } l \text{ with at least one constraint binding} \tag{39}$$

Proposition 4.3 shows that at the end point of the minimal escape path, *only one* constraint is binding (i.e., $\pi(\bar{m}, q) = \pi(k, q)$ for some k) and all other constraint are non-binding (i.e., $\pi(\bar{m}, q) > \pi(l, q)$ for some $l \neq k$). To explain why, let us consider a three-strategy game with the path $t^* = ((t_1^*, t_2^*); (2, 3))$ passing through points p and $q = q(t^*)$ in Panel A of Fig. 6, where $\pi(1, q(t^*)) = \pi(2, q(t^*)) = \pi(3, q(t^*))$. Furthermore, consider an alternative path exiting directly along the boundary between strategies 1 and 2 at p' ; hence, $\pi(1, p') = \pi(2, p')$, $\pi(1, p') > \pi(3, p')$. To compare the costs of these two paths, using the definition in (35), we find that

$$\bar{c}(p, q) = \frac{1}{2} t_2^* (\pi(1, p + q) - \pi(3, p + q)), \quad \bar{c}(p, p') = \frac{1}{2} (p_2' - p_2) (\pi(1, p + p') - \pi(2, p + p')).$$

Thus, from $t_2^* > p'_2 - p_2$ (see Panel A of Fig. 6) and the fact that q and p' are mixed-strategy Nash equilibria,

$$\bar{c}(p, q) - \bar{c}(p, p') \geq \frac{1}{2}t_2^*[\pi(1, p) - \pi(3, p) - (\pi(1, p) - \pi(2, p))] = \frac{1}{2}t_2^*(\pi(2, p) - \pi(3, p)).$$

Clearly, $\pi(2, p) > \pi(3, p)$ from Panel A of Fig. 6, because p is located to the left of the line $\pi(2, r) = \pi(3, r)$. Indeed, we confirm that

$$\begin{aligned} \pi(2, p) - \pi(3, p) &= \pi(2, q + t_2^*(e_1 - e_3)) - \pi(3, q + t_2^*(e_1 - e_3)) \\ &= t_2^*(A_{21} - A_{23} - A_{31} + A_{33}) > 0 \end{aligned}$$

using the MBP, where we again use the fact that q is the complete mixed strategy Nash equilibrium. Thus, we find that $\bar{c}(p, q) \geq \bar{c}(p, p')$. Once again, the underlying principle is that the MBP implies that a path with the same consecutive transitions is cheaper than those with different transitions.

Proposition 4.3. *Suppose that Condition A holds. Suppose that t^* such that $\omega(t^*) = \min\{\omega(t) : \zeta(t) \in \mathcal{K}_{\bar{m}}\}$. Then for some k ,*

$$\pi(\bar{m}, q(t^*)) = \pi(k, q(t^*)) \text{ and } \pi(\bar{m}, q(t^*)) > \pi(l, q(t^*)) \text{ for all } l \neq k. \tag{40}$$

Proof. See Appendix A. \square

Our final step shows that the optimal path always exit through the edge of the simplex (i.e., the mixed strategy Nash equilibrium involving only two strategies). To explain the idea behind Proposition 4.4, consider again the three-strategy game in Panel B of Fig. 6. Suppose that the optimal solution is $t^* = ((t_1^*, t_2^*); (2, 3))$ and that $t_1^*, t_2^* > 0$, $\pi(1, q(t^*)) > \pi(2, q(t^*))$ and $\pi(1, q(t^*)) = \pi(3, q(t^*))$ (see the dotted line in Panel B of Fig. 6). Then, using the linearity of the payoff functions, we consider $t_\epsilon^+ := (t_1^* + \epsilon_1, t_2^* - \epsilon_2)$ and $t_\epsilon^- := (t_1^* - \epsilon_1, t_2^* + \epsilon_2)$ which still satisfy the constraint of escaping from the basin of attraction. Then, by direct computation, we have

$$(\omega(t_\epsilon^+) - \omega(t)) + (\omega(t_\epsilon^-) - \omega(t)) = -H_{1,2;2}\epsilon_1^2 + 2H_{1,3;2}\epsilon_1\epsilon_2 - H_{1,3;3}\epsilon_2^2 \leq -(\sqrt{H_{1,2;2}}\epsilon_1 - \sqrt{H_{1,3;3}}\epsilon_2)^2 < 0$$

where $H_{i,j;k} := (A_{ii} - A_{ji}) - (A_{ik} - A_{jk})$. Thus, either $\omega(t_\epsilon^+) < \omega(t)$ or $\omega(t_\epsilon^-) < \omega(t)$ holds. Again, this shows that the MBP plays an important role in determining the minimum cost escape path under the infinite population model. The proof of Proposition 4.4 generalizes these arguments for an arbitrary n strategy game.

Proposition 4.4. *Suppose that Condition A holds. Let t^* be the solution to the minimization problem in (38). Suppose that for some k and l ,*

$$\pi(\bar{m}, q(t^*)) = \pi(k, q(t^*)) \text{ and } \pi(\bar{m}, q(t^*)) > \pi(l, q(t^*))$$

Then $t_l^* = 0$.

Proof. See Appendix A. \square

Finally, by applying Propositions 4.1, 4.2, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min\{I^{(n)}(\gamma) : \gamma \in \mathcal{G}_{\bar{m}}^{(n)}\} = \min\{\omega(t) : \zeta(t) \in \bar{\mathcal{K}}_{\bar{m}}\}. \tag{41}$$

Now let t^* be the solution to $\min\{\omega(t) : \zeta(t) \in \bar{\mathcal{K}}_{\bar{m}}\}$. Then, from Proposition 4.3, there exists k such that equations in (40) hold. By applying Proposition 4.4, we conclude that t^* involves only transitions from \bar{m} to k and this path exits at the mixed strategy involving only \bar{m} and k . Then, we find the cost of t^* as follows (see Lemma A.3)

$$\omega(t^*) = \frac{1}{2} \left(1 - \frac{A_{kk} - A_{\bar{m}k}}{(A_{\bar{m}\bar{m}} - A_{k\bar{m}}) + (A_{kk} - A_{\bar{m}k})}\right) (\pi(\bar{m}, e_{\bar{m}}) - \pi(k, e_{\bar{m}})) = \frac{1}{2} \frac{(A_{\bar{m}\bar{m}} - A_{k\bar{m}})^2}{(A_{\bar{m}\bar{m}} - A_{k\bar{m}}) + (A_{kk} - A_{\bar{m}k})}.$$

Thus, we obtain

$$\min\{\omega(t) : \zeta(t) \in \bar{\mathcal{K}}_{\bar{m}}\} = \omega(t^*) \geq \min_{j \neq \bar{m}} \left\{ \frac{1}{2} \frac{(A_{\bar{m}\bar{m}} - A_{j\bar{m}})^2}{(A_{\bar{m}\bar{m}} - A_{j\bar{m}}) + (A_{jj} - A_{\bar{m}j})} \right\}. \tag{42}$$

Since the expression in curly braces of the right hand side of (42) is the cost of the straight line escape path from \bar{m} to j and these paths belong to $\bar{\mathcal{K}}_{\bar{m}}$, (42) holds as an equality. Thus, (41) and (42) (as an equality) yield the following theorem.

Theorem 4.1 (Exit problem: one population model). Assume that Condition A holds. Then, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min \{ I^{(n)}(\gamma) : \gamma \in \mathcal{G}_{\bar{m}}^{(n)} \} = \min_{j \neq \bar{m}} \left\{ \frac{1}{2} \frac{(A_{\bar{m}\bar{m}} - A_{j\bar{m}})^2}{(A_{\bar{m}\bar{m}} - A_{j\bar{m}}) + (A_{jj} - A_{\bar{m}j})} \right\} \tag{43}$$

Proof. See Appendix A. \square

In the much-studied uniform mistake model, the probabilities of mistakes are identical for all states and, hence, are independent of the states (see, e.g. Binmore et al. (2003)). Therefore, the threshold number of deviant agents who induce other agents to change their best responses is the only determinant of the expected escape time and stochastic stability. The number $n \frac{(A_{\bar{m}\bar{m}} - A_{j\bar{m}})}{(A_{\bar{m}\bar{m}} - A_{j\bar{m}}) + (A_{jj} - A_{\bar{m}j})}$ is the threshold number of agents deviating from strategy \bar{m} to strategy i and inducing others to best respond with strategy i . Obviously, our arguments for Theorem 4.1 can be applied to the uniform mistake model, as explained in the introduction. In particular, comparison principle 1 holds as an equality, which can still be used to find a minimum cost path and comparison principle 2 holds without any modification. Thus, when the MBP holds,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma \in \mathcal{G}_{\bar{m}}^{(n)}} I_{uniform}^{(n)}(\gamma) = \min_{j \neq \bar{m}} \left\{ \frac{(A_{\bar{m}\bar{m}} - A_{j\bar{m}})}{(A_{\bar{m}\bar{m}} - A_{j\bar{m}}) + (A_{jj} - A_{\bar{m}j})} \right\} \tag{44}$$

where $I_{uniform}$ is the cost function under the uniform mistake model (see also Binmore et al. (2003); Kandori and Rob (1998)).

Comparing (43) and (44) reveals that the logit rule cost also accounts for the opportunity cost of mistakes ($A_{\bar{m}\bar{m}} - A_{j\bar{m}}$) adopting sub-optimal strategy j instead of the best response strategy \bar{m} as well as the threshold fraction of the population inducing a new best response. Specifically, under the logit model, the threshold fraction of agents inducing a new best response is weighted by individuals' average cost of mistakes, $\frac{1}{2}(A_{\bar{m}\bar{m}} - A_{j\bar{m}})$ (equation (43)). This (plausible) modification of costs creates a different prediction for the exit path and time, as well as a stochastically stable state from the uniform model (see Section 5.2).

5. Two population models and the application: the logit solutions for the Nash demand game

5.1. Two population models

For the application of our results to the Nash demand game, we briefly introduce the two population setup and state the result for two population models (Theorem 5.1), which is proved in a similar way to the one population model result (see Appendix B). Consider two populations denoted by α and β , consisting of the same number of agents n and a bimatrix game (A^α, A^β) , where A^κ is an $|S| \times |S|$ matrix for $\kappa = \alpha, \beta$. An α -agent playing i against j obtains a payoff A_{ij}^α , while a β -agent playing j against i obtains A_{ij}^β . We introduce the following definition.

Definition 5.1. We say that (A^α, A^β) is a coordination game if $A_{ii}^\alpha > A_{ji}^\alpha$ and $A_{ii}^\beta > A_{ij}^\beta$ for all i, j . We also say that (A^α, A^β) satisfies the weak marginal bandwagon property (WBP) if

$$A_{\bar{m}\bar{m}}^\alpha - A_{i\bar{m}}^\alpha \geq A_{\bar{m}j}^\alpha - A_{ij}^\alpha \text{ and } A_{\bar{m}\bar{m}}^\beta - A_{\bar{m}i}^\beta \geq A_{j\bar{m}}^\beta - A_{ji}^\beta \tag{45}$$

for all distinct i, j, \bar{m} .

Note that condition (45) is weaker than the MBP which requires strict inequalities. We relax the MBP, because we would like to study a broader class of games including Nash demand games (Nash (1953)).⁴ Let $x = (x_\alpha, x_\beta) \in \Delta_\alpha^{(n)} \times \Delta_\beta^{(n)}$, where $\Delta_\kappa^{(n)} := \{x_\kappa : \sum x_\kappa(i) = 1, \text{ and } x_\kappa(i) \geq 0 \text{ for all } i\}$ for $\kappa = \alpha, \beta$. Then, the expected payoffs are similarly given by

$$\pi_\alpha(i, x) = \pi_\alpha(i, x_\beta) = \sum_{j=1}^n x_\beta(j) A_{ij}^\alpha, \quad \pi_\beta(j, x) = \pi_\beta(j, x_\alpha) = \sum_{i=1}^n x_\alpha(i) A_{ij}^\beta$$

Similarly to Condition A, we make the following assumptions:

Condition B

(i) A game (A^α, A^β) is a coordination game, satisfying the WBP.

(ii) Suppose that

for any $T = \{i_1, \dots, i_K\} \subset S$, there exists a unique $q \in \Delta_\beta$ with support T such that

⁴ It is straightforward to check that a discrete Nash demand game satisfies (45). See Appendix D.

$$\pi_\alpha(i_1, q) = \dots = \pi_\alpha(i_K, q),$$

for any $T = \{i_1, \dots, i_K\} \subset S$, there exists a unique $p \in \Delta_\alpha$ with support T such that

$$\pi_\beta(i_1, p) = \dots = \pi_\beta(i_K, p). \quad \square$$

We denote by $x^{\alpha,i,j}$ (or $x^{\beta,i,j}$) the state induced by an α -agent's (or β -agent's) switching from i to j from x . We also denote by $x^{\alpha,i,j,\rho}$ (or $x^{\beta,i,j,\rho}$) the state induced from x by α -agents' ρ -times consecutive switchings (or β -agents' ρ -times consecutive transitions) from i to j . We write e_i^α and e_j^β as the i -th and j -th standard basis for $\mathbb{R}^{|S|}$ and thus $(e_i^\alpha, e_i^\beta) \in \Delta_\alpha^{(n)} \times \Delta_\beta^{(n)}$ is a convention. We similarly define a basin of attraction of convention i as follows:

$$D^{(n)}(e_{\bar{m}}) := \{x = (x_\alpha, x_\beta) \in \Delta_\alpha^{(n)} \times \Delta_\beta^{(n)} : \pi_\alpha(\bar{m}, x_\beta) \geq \pi_\alpha(k, x_\beta), \pi_\beta(\bar{m}, x_\alpha) \geq \pi_\beta(k, x_\alpha) \text{ for all } k\}$$

and compute the cost functions between states:

$$c^{(n)}(x, x^{\alpha,i,j}) := \pi_\alpha(\bar{m}, x) - \pi_\alpha(j, x), \quad c^{(n)}(x, x^{\beta,i,j}) := \pi_\beta(\bar{m}, x) - \pi_\beta(j, x) \tag{46}$$

for $x \in D^{(n)}(e_{\bar{m}})$. We similarly define the cost function of a path, $I^{(n)}$, as in equation (4) and the set of all paths escaping the basin of attraction of convention \bar{m} as $\mathcal{G}_{\bar{m}}^{(n)}$, as in equation (30).

For two population models, two kinds of deviant behaviors naturally arise depending on the intentionality of deviant plays (Naidu et al., 2010; Hwang et al., 2018), namely an unintentional or intentional logit choice rule. The unintentional logit choice rule refers to the standard logit rule in the context of the two population model, defined as follows:

Unintentional logit choice rule: $p_\eta^{U,(\kappa)}(l|x) := \frac{\exp(\eta^{-1}\pi_\kappa(l, x))}{\sum_{l'} \exp(\eta^{-1}\pi_\kappa(l', x))}$ for $\kappa = \alpha, \beta$. (47)

By the intentional choice rule, we mean that agents choose a non-best-response strategy from the set of strategies that would give a higher payoff than the payoff at the status quo convention when adopted as a convention. Thus, the intentional logit choice rule means that agents play a non-best-response strategy among the set restricted by “intentional” behaviors, with probabilities specified by the logit choice rule. Experimental evidence for intentional as well as payoff-dependent behaviors captured by the logit rule is provided in Mäs and Nax (2016), Lim and Neary (2016), and Hwang et al. (2018) (see Section 2).⁵

More precisely, to define the intentional logit choice rule, we introduce the sets of the permissible suboptimal strategies under the intentional dynamic:

$$\tilde{S}_m(\kappa) := \{l : A_{ll}^\kappa \geq A_{mm}^\kappa\}, \quad \hat{S}_x(\kappa) := \{l : A_{ll}^\kappa \geq A_{mm}^\kappa \text{ for } m \in \arg \max_i \{\pi_\kappa(i, x)\}\} \tag{48}$$

That is, $\tilde{S}_m(\kappa)$ is the set of all strategies which yield payoffs higher than (or equal to) strategy m if adopted as a convention, while $\hat{S}_x(\kappa)$ is the set of all strategies which yield payoffs higher than (or equal to) the current convention at state x if adopted as a convention. We make the following assumption for conflict of interests between the two populations, α and β :

$$\tilde{S}_m(\alpha) \cup \tilde{S}_m(\beta) = S, \quad \tilde{S}_m(\alpha) \cap \tilde{S}_m(\beta) = m \tag{49}$$

Thus, except for the current convention strategy m , the set of strategies that α population prefers is different from the set of strategies that β population prefers. Then under the intentional logit choice rule, the conditional probability that a $\kappa = \alpha, \beta$ agent chooses strategy l at a given state x is given by

Intentional logit choice rule: $p_\eta^{I,(\kappa)}(l|x) := \begin{cases} \frac{\exp(\eta^{-1}\pi_\kappa(l, x))}{\sum_{l' \in \hat{S}_\kappa(x)} \exp(\eta^{-1}\pi_\kappa(l', x))} & \text{if } l \in \hat{S}_\kappa(x) \\ 0 & \text{if } l \notin \hat{S}_\kappa(x). \end{cases}$ (50)

To state our result for the two population model, we let ζ_α (or ζ_β) the threshold fractions of β -population (or α -population) inducing a new best response in α -population (or β -population) be

$$\zeta_{m,j}^\alpha := \frac{(A_{mm}^\alpha - A_{jm}^\alpha)}{(A_{mm}^\alpha - A_{jm}^\alpha) + (A_{jj}^\alpha - A_{mj}^\alpha)}, \quad \zeta_{m,j}^\beta := \frac{(A_{mm}^\beta - A_{mj}^\beta)}{(A_{mm}^\beta - A_{mj}^\beta) + (A_{jj}^\beta - A_{jm}^\beta)}.$$

⁵ Lim and Neary (2016) find that individual mistakes are directed in the sense that they are group-dependent. The directed mistakes in their paper are intentional behaviors of deviant agents; for example, they find that 2.25% of subjects play mistakes when the best response is the preferred strategy, whereas 20.85% of subjects play mistakes when the best response is the less preferred strategy (Figure 5 (a) on p. 19).

Table 1
Bargaining solutions for the Nash demand game. The bargaining solutions for the Nash demand game under the logit choice rule are new.

	Unintentional	Intentional
Uniform	$\frac{f(s^{NB})}{s^{NB}} = -f'(s^{NB})$	$\frac{f(s^{NB})}{s^{NB}} = -f'(s^{NB})$
Logit	$\frac{f(s^{NB})}{s^{NB}} = -f'(s^{NB})$	$\left(\frac{f(s^I)}{s^I}\right)^2 = -f'(s^I)$

In Theorem 5.1 below, we show that the minimum cost escaping path from convention m is similarly given by the threshold fraction weighted by the individual’s opportunity cost ($A_{mm}^\beta - A_{mj}^\beta$ for β -population agents and $A_{mm}^\alpha - A_{jm}^\alpha$ for α -population agents). To state this, we let

$$R_{mj}^U := (A_{mm}^\beta - A_{mj}^\beta)\zeta_{mj}^\alpha \wedge (A_{mm}^\alpha - A_{jm}^\alpha)\zeta_{mj}^\beta, \quad R_{mj}^I := \begin{cases} (A_{mm}^\alpha - A_{jm}^\alpha)\zeta_{mj}^\beta, & \text{if } j \in \tilde{S}_m(\alpha) \\ (A_{mm}^\beta - A_{mj}^\beta)\zeta_{mj}^\alpha, & \text{if } j \in \tilde{S}_m(\beta). \end{cases} \tag{51}$$

Note that under the intentional logit choice rule (R_{mj}^I in (51)), the sets of strategies for which minimum cost transitions occur are precisely the sets of strategies that the transition driving population prefers. This is because under the intentional dynamic, the deviant plays always involve strategies that the deviant population prefers. For the two population models, the minimum cost escape path from convention m again occurs at the boundary of the simplex (the state space), as is the case for the one population model:

Theorem 5.1. *Suppose that Condition B and equations (49) hold. Then*

$$\textbf{Unintentional:} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma \in \mathcal{G}_m^{(n)}} I^{(n)}(\gamma) = \min_{j \neq m} R_{mj}^U$$

$$\textbf{Intentional:} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\gamma \in \mathcal{G}_m^{(n)}} I^{(n)}(\gamma) = \min_{j \neq m} R_{mj}^I$$

Proof. See Appendix B. \square

5.2. The logit solutions for the Nash demand game

In this section, we present the application of our results to the Nash demand games (Nash, 1953), using Theorem 5.1. Consider a bargaining set, $S \subset \mathbb{R}^2$, consisting of payoffs to two populations when they agree to split. We normalize the “disagree” point to $(0, 0)$. We describe the bargaining frontier by the function f that is decreasing, differentiable and strictly concave. That is,

$$S = \{(x, y) : y \leq f(x)\}$$

where $f(x) \geq 0$, $f'(x) < 0$, and $f''(x) < 0$ for all x . We let \bar{s} be the maximum payoff to populations α : i.e., $\bar{s} := \max\{x : (x, y) \in S\}$. Bargaining solutions dictate how to divide the surpluses (defined by the bargaining set) between two populations.⁶

Which is the bargaining norm arising through decentralized evolutionary bargaining processes under the logit choice rule? To answer this question, following the standard literature on evolutionary bargaining (Young, 1993b, 1998; Binmore et al., 2003; Hwang et al., 2018), we discretize the bargaining set as follows and consider a Nash demand game. More specifically, let L be a positive integer and $\delta := \frac{\bar{s}}{L}$. Define a Nash demand game:

$$(A_{ij}^\alpha, A_{ij}^\beta) := \begin{cases} (\delta i, f(\delta j)), & \text{if } i \leq j \\ (0, 0), & \text{if } i > j, \end{cases} \tag{52}$$

where $i \in \{1, 2, \dots, L - 1\}$ and we will let $\delta \rightarrow 0$ (or $L \rightarrow \infty$) eventually. Then, we can easily check that the game in (52) satisfies Condition B (See Appendix D).

The assumption for conflict in interests in (49) also holds for the game defined in (52). Thus, while the α -population prefers strategy m to strategy m' (for $m > m'$), the β -population prefers strategy m' to strategy m (for $m > m'$). This is because at a higher (or lower) index convention, an α -population agent (or a β -population agent) claims a large share.

⁶ Three axiomatic bargaining solutions are most commonly used: the Nash bargaining solution (Nash, 1950), the Kalai-Smorodinsky bargaining solution (Kalai and Smorodinsky, 1975), and the egalitarian bargaining solution (Kalai, 1977).

Thus, under the intentional logic dynamic, an α population agent idiosyncratically plays a suboptimal strategy from the set of strategies with higher indices than the current strategy, whereas a β population agent does the opposite.

Next, we find a stochastic stable state using Theorem 5.1 (see Appendix C and Theorem C.1). To explain this, first consider the intentional logit choice rule for its simplicity. We would like to find the minimum cost of transitions from convention m (R_{mj}^I in (51)); the costs of transition driven by each population are given as follows:

$$\alpha\text{-pop.} \quad \min_{j \in \hat{S}_m(\alpha)} \left\{ (A_{mm}^\alpha - A_{jm}^\alpha) \frac{(A_{mm}^\beta - A_{mj}^\beta)}{(A_{mm}^\beta - A_{mj}^\beta) + (A_{jj}^\beta - A_{jm}^\beta)} \right\} = \min_{m < j} m\delta \frac{f(\delta m) - f(\delta j)}{f(\delta m)} \tag{53}$$

$$\beta\text{-pop.} \quad \min_{j \in \hat{S}_m(\beta)} \left\{ (A_{mm}^\beta - A_{mj}^\beta) \frac{(A_{mm}^\alpha - A_{jm}^\alpha)}{(A_{mm}^\alpha - A_{jm}^\alpha) + (A_{jj}^\alpha - A_{mj}^\alpha)} \right\} = \min_{m > j} f(m\delta) \frac{(m - j)\delta}{m\delta} \tag{54}$$

for a given m . In the second equation in (53), the minimum is taken over $\{j : m < j\}$ since the α -population prefers the strategy with a higher index, while the opposite holds true for the β -population in (54). We then find the minimum cost of transitions, also called the radius for convention m , as follows:

$$\min_j R_{mj}^I = \min \left\{ f(m\delta) \frac{\delta}{m\delta}, m\delta \frac{f(\delta m) - f(\delta(m+1))}{f(\delta m)} \right\} \tag{55}$$

It can be shown that a convention m which maximizes $\min_j R_{mj}^I$ is a stochastically stable state for the game in (52) under the unintentional or intentional logit choice rule (see Appendix D). Note that the first term in the minimum of equation (55) is decreasing in m and the second term in the minimum of equation (55) is increasing in m . Thus, the maximum of (55) is achieved where the gap between the two terms in the minimum of equation (55) is smallest. Heuristically, we find that

$$f(\delta m) \frac{\delta}{\delta m} = \delta m \frac{f(\delta m) - f(\delta(m+1))}{f(\delta m)} \iff \left(\frac{f(\delta m)}{\delta m} \right)^2 = \frac{f(\delta m) - f(\delta(m+1))}{\delta} \rightarrow \left(\frac{f(x)}{x} \right)^2 = -f'(x)$$

if $\delta m \rightarrow x$ and $\delta \rightarrow 0$

More precisely, we define

$$\left(\frac{f(s^I)}{s^I} \right)^2 = -f'(s^I) \text{ and } \frac{f(s^{NB})}{s^{NB}} = -f'(s^{NB}). \tag{56}$$

and s^{NB} is the familiar Nash bargaining solution and s^I is a new solution under the intentional logit choice rule. We also let s^E be the egalitarian solution:

$$f(s^E) = s^E.$$

We then find

$$\left(\min_j R_{mj}^I, \arg \min_j R_{mj}^I \right) = \begin{cases} \left(m\delta \frac{f(\delta m) - f(\delta(m+1))}{f(\delta m)}, m + 1 \right) & \text{if } \delta m < s^I \\ \left(\frac{f(m\delta)}{m\delta} \delta, m - 1 \right) & \text{if } \delta m > s^I \end{cases} \tag{57}$$

Let $m^I \in \arg \max_m \min_j R_{mj}^I$. Then it is straightforward to show that m^I is stochastically stable and

$$\delta m^I \rightarrow s^I \text{ as } \delta \rightarrow 0$$

(see Appendix D); thus, the solution s^I defined in (56) is the stochastically stable bargaining convention under the intentional logit dynamic. In addition, the following theorem shows that the Nash bargaining solution defined in (56) is the stochastically stable bargaining norm under the unintentional logit rule (See Table 1).

Theorem 5.2. *Suppose that $f(x) \geq 0$, $f'(x) < 0$, and $f''(x) < 0$ for all x . Let m^* and m^I be the stochastically stable states under the unintentional and intentional logit models, respectively. Then we have*

$$\delta m^* \rightarrow s^{NB} \text{ and } \delta m^I \rightarrow s^I$$

as $\delta \rightarrow 0$.

Proof. See Appendix D. \square

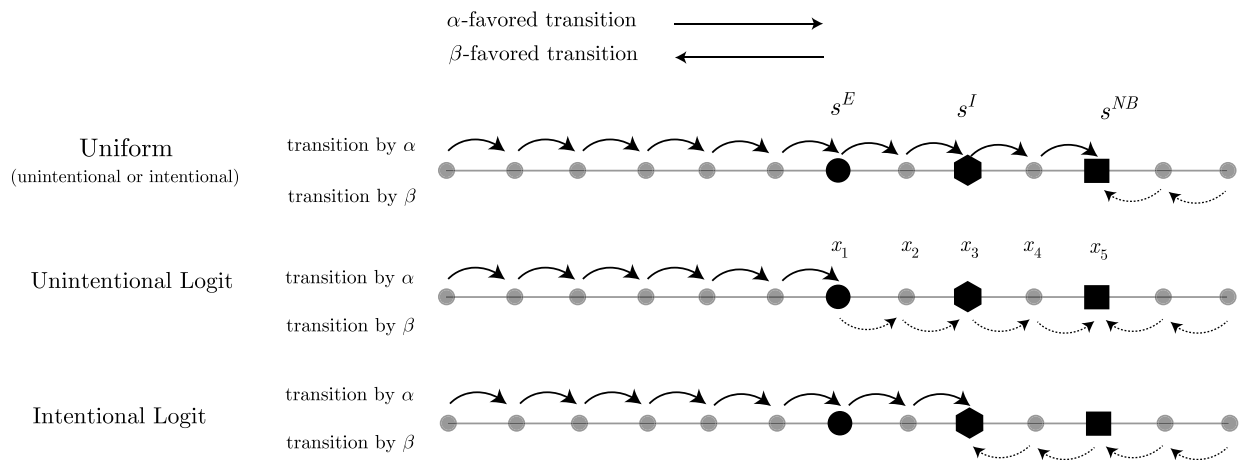


Fig. 7. Minimal trees. The closed circle, hexagon, and square correspond to the egalitarian solution (s^E), the intentional logit solution (s^I), and the Nash bargaining solution (s^{NB}), respectively. Each small dot represents a convention. The regular arrows show transitions induced by the α -population, while the dotted arrows show transitions induced by the β -population. Under the unintentional logit choice rule, when α 's share of Nash bargaining solution s^{NB} is greater than that of the egalitarian solution s^E , the transitions from the egalitarian solution to the Nash bargaining solution are driven by β -population.

Fig. 7 compares cost minimizing paths under the various mistake models for the Nash demand game (see also Table 1). Under the unintentional logit choice rule, the Nash bargaining solution arises as a bargaining norm, as in the uniform model (Theorem 5.2 and see Binmore et al. (2003)). However, the underlying mechanism is dramatically different from the uniform model as Fig. 7 shows. Under the unintentional logit choice rule, unlike the uniform mistake model, when α population's share at the Nash bargaining solution (s^{NB}) is greater than the equal division (s^E), transitions from the egalitarian solution to the Nash bargaining solution are induced by β -population agents even if these transitions are α -population's favorite transitions (transitions to a higher index convention). This is because under the logit dynamic, this opportunity cost of the deviant play (as well as the threshold fraction) matters and when the α -population claims a larger share than the β -population, the opportunity cost can make transitions induced by the α -population more costly than those induced by the β -population. Thus, under the unintentional logit choice model, when the Nash bargaining solution favors the α -population, transitions from the egalitarian solution are driven by the β -population (see x_1, x_2, x_3, x_4 in Fig. 7), leading to the Nash bargaining solution in which the α -population claims a larger share than the β -population.

Under the intentional logit choice rule, transitions always occur by those who stand to benefit from the transitions. That is, every transition to a higher index convention is driven by the α -population, while every transition to a lower index convention is driven by the β -population. Thus, some transitions from the egalitarian solution driven by the β -population under the unintentional model are now replaced by transitions to a lower index convention by the β -population and this pushes the stochastically stable state toward a more equal convention (see x_4, x_5 in Fig. 7). In this way, intentionality equalizes the division of the surplus.

Proposition 5.1. *The logit intentional bargaining solution is more equal than the logit unintentional bargaining solution: i.e., either*

$$s^{NB} > s^I > s^E \text{ or } s^E > s^I > s^{NB}$$

holds.

Proof. See Lemma D.6. \square

Hwang et al. (2018) also find that for the contract game (the coordination game in which all off-diagonal payoffs are zeroes), intentionality makes the stochastically stable division of surpluses between the two populations more equal under the logit choice rule. However, the mechanism under which equality is achieved is again different. Under the contract game, intentionality equalizes the threshold fractions of one population inducing a new best response for the other population (at 1/2), and equality is achieved. For the Nash demand game, intentionality replaces unfavorable transitions for the deviant population by the favorable transitions and a more equal convention arises.

6. Summary

Relying on positive feedback conditions and the relative strengths of these effects we developed methods to identify the most likely paths for evolutionary population dynamics under the logit rule. We identified two main factors determining the minimum cost path escaping from a convention: (1) the existence of positive feedback effects, and (2) the relative

strengths of positive feedback effects. This leads us to simple but powerful comparison principles that drastically reduce the number of candidate paths for minimizing the escaping cost from a convention. To summarize, we showed that the path with minimal cost involves only the repeated identical mistakes of the agents. We also applied our finding to the bargaining problem to find the stochastically stable states and obtain a new bargaining convention.

Appendix. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.geb.2021.01.003>.

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