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Convex and Concave Decompositions of Affine 3-Manifolds

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Abstract
A (flat) affine 3-manifold is a 3-manifold with an atlas of charts to an affine space $\mathbb{R}^3$ with transition maps in the affine transformation group $\mathbf{Aff}(\mathbb{R}^3)$. We will show that a connected closed affine 3-manifold is either an affine Hopf 3-manifold or decomposes canonically to concave affine submanifolds with incompressible boundary, toral π-submanifolds and 2-convex affine manifolds, each of which is an irreducible 3-manifold. It follows that if there is no toral π-submanifold, then M is prime. Finally, we prove that if a closed affine manifold is covered by a connected open set in $\mathbb{R}^3$, then M is irreducible or is an affine Hopf manifold.

Keywords Geometric structures · Flat affine structure · 3-manifolds

Mathematics Subject Classification Primary 57M50; Secondary 53A15 · 53A20

1 Introduction

1.1 Introduction and History

An affine manifold is a manifold with an atlas of charts to $\mathbb{R}^n$, $n \geq 2$, where the transition maps are in the affine group. Euclidean manifolds are examples. A Hopf manifold that is the quotient of $\mathbb{R}^n - \{O\}$ by a linear contraction group, i.e., a group of linear transformation generated by an element with eigenvalues of norm greater than 1 is an example. A half-Hopf manifold is the quotient of $U - \{O\}$ by a linear contraction group for a closed upper half-space U of $\mathbb{R}^n$. (See Proposition 3.)

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For the currently most extensive set of examples of affine manifolds, see the paper by Sullivan and Thurston (1983). We still have not obtained essentially different examples to theirs to this date. (See also Carrière 1989; Smillie 1977 and Benoist 2000 and Benoist 1994.) A connected compact affine 3-manifold is *radiant* if the holonomy group fixes a unique point, and has boundary tangent to a radiant vector field, which exists by the fixed point. (See Sect. 2.3 and Barbot 2000; Fried et al. 1981.) Such a manifold has a complete flow called a *radiant flow*. A generalized affine suspension is a radiant affine manifold admitting a total cross-section. (See Proposition 2.) A radiant affine n-manifold can be constructed easily from a real projective $(n-1)$-manifold using generalized affine suspension. (See Section 2.2 of Barbot 2000 or Chapter 3 of Choi 2001).

For a subset M with a manifold topology, we denote by M^o the manifold interior of M and by ∂M the manifold boundary of M.

A *tetrahedron* or 3-simplex is a convex hull of four points in a general position in an affine space $\mathbb{R}^3$. Let T be a convex simplex in an affine space $\mathbb{R}^3$ with faces F_0, F_1, F_2, and F_3. A real projective or affine 3-manifold is 2-*convex* if every projective map $f : T^o \cup F_1 \cup F_2 \cup F_3 \to M$ extends to $f : T \to M$. (Carrière 1989 first defined this concept).

A 3-manifold M is *prime* if M is a connected sum of two manifolds M_1 and M_2, then M_1 or M_2 is homeomorphic to a 3-sphere. The subject of this paper is the following: the question of Goldman in Problem 6 in the Open problems section of Apanasov et al. (1997) is whether closed affine 3-manifolds are prime.

We showed that 2-convex affine 3-manifolds are irreducible in Choi (2000). Our Theorem 3 shows that closed affine manifolds may be obtainable by gluing toral π-submanifolds which are solid tori or solid Klein bottles with special geometric properties to irreducible 3-manifolds. (See Definition 5.) This construction may result in reducible 3-manifolds as we can see from Gordon (1991). Hence, the nonexistence of solid tori or solid Klein bottles with special geometric properties in a closed affine 3-manifold M would show that M is prime. (See Corollary 1). We question whether toral π-submanifolds can occur at all. We also answer the question when M is covered by a connected open set in an affine space by Corollary 2.

For the related real projective structures on closed 3-manifolds, Cooper and Goldman (2015) showed that a connected sum $\mathbb{R}P^3\#\mathbb{R}P^3$ admits no real projective structure. For these topics, a good reference is given by Goldman (2018), originally given as lecture notes in the 1980s.

1.2 Main Results

We give some definitions which we will give more precisely later. A real projective structure on a manifold M is a maximal atlas of charts to $\mathbb{R}P^n$ with transition maps in the projective group $\mathsf{PGL}(n+1, \mathbb{R})$. M is called a *real projective manifold*. At a boundary point of M, we require that there is a chart to $\mathbb{R}P^n$ with boundary mapping to a submanifold of codimension-one.

We use the double-covering map $\mathbb{S}^n \to \mathbb{R}P^n$, and hence $\mathbb{S}^n$ has a real projective structure. The group of projective automorphisms of $\mathbb{R}P^n$ is $\mathsf{PGL}(n+1, \mathbb{R})$ and that of $\mathbb{S}^n$ is $\mathsf{SL}_{\pm}(n+1, \mathbb{R})$.

We recall the main results of Choi (1999) which we will state in Sect. 2.5 in a more detailed way. Let M be a closed real projective manifold. Let $\tilde{M}$ be the universal cover and $\pi_1(M)$ the deck transformation group. A real projective structure on M gives us an immersion $\mathbf{dev} : \tilde{M} \to \mathbb{S}^n$ equivariant with respect to a homomorphism $h : \pi_1(M) \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$. The real projective structure gives these data.

Recall a group $\mathbf{Aff}(\mathbb{R}^n)$ of affine transformations of form $x \mapsto Mx + b$ for $M \in \mathsf{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The real projective space $\mathbb{R}P^n$ contains the affine space $\mathbb{R}^n$ as a complement of a hyperspace, and affine transformation groups naturally extend to projective automorphisms. Affine geodesics also extend to projective geodesics.

Also, $\mathbb{R}^n$ identifies with an open hemisphere in $\mathbb{S}^n$ under the double covering map. The group of affine transformations is the group of projective diffeomorphisms of the open hemisphere under the identification. The open hemisphere is also called an *affine patch*.

Conversely, given a closed 3-hemisphere $\mathbb{H}$, let $\mathbf{Aut}(\mathbb{H})$ denote the group of projective automorphisms of $\mathbb{H}$. This group is isomorphic to $\mathbf{Aff}(\mathbb{H}^o)$ when $\mathbb{H}^o$ is identified with an affine space.

We will look an affine manifold as a real projective manifold, i.e., a manifold with an atlas of charts to $\mathbb{R}^n \subset \mathbb{S}^n$ with transition maps in the affine group $\mathbf{Aff}(\mathbb{R}^n) \subset \mathsf{SL}_{\pm}(n+1, \mathbb{R})$. An affine manifold has a canonical real projective structure since the charts and the transition maps are projective also. (The converse is not true).

Let K_h be the kernel of h, normal in $\pi_1(M)$. We cover M by the holonomy cover $M_h = \tilde{M}/K_h$ corresponding to K_h with

- an induced and lifted immersion $\mathbf{dev}_h : M_h \to \mathbb{S}^n$ and
- an induced holonomy homomorphism $h_h : \pi_1(M)/K_h \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ satisfying

$$\mathbf{dev}_h \circ g = h_h(g) \circ \mathbf{dev}_h \text{ for } g \in \pi_1(M)/K_h.$$

Let M_h have the path metric of the Riemannian metric pulled back from the Fubini-Study Riemannian metric of $\mathbb{S}^3$. The Cauchy completion $\check{M}_h$ of M_h is called a *Kuiper completion*. The ideal set is $M_{h,\infty} := \check{M}_h - M_h$. (See Sect. 2.5.1 for definitions.)

A 3-hemisphere is a closed 3-hemisphere in $\mathbb{S}^3$, and a 3-bihedron is the closure of a component $H - \mathbb{S}^2$ for a 3-hemisphere H with a great 2-sphere $\mathbb{S}^2$ passing H^o. These have real projective structures induced from the double-covering map $\mathbb{S}^3 \to \mathbb{R}P^3$. An *open* 3*-hemisphere* is the interior of a closed 3-hemisphere, and an *open* 3*-bihedron* is the interior of a closed 3-bihedron. An open 3-hemisphere is projectively diffeomorphic to an affine 3-space, and an open 3-bihedron is projectively diffeomorphic to a half-space in an affine 3-space.

If the universal cover $\tilde{M}$ is projectively diffeomorphic to an open hemisphere, i.e., $\mathbb{R}^n$, then M is called a *complete affine manifold*. If the universal cover $\tilde{M}$ is projectively diffeomorphic to an open 3-bihedron, we call M a *bihedral real projective manifold*.

A hemispherical 3-crescent is a 3-hemisphere in $\check{M}_h$ whose boundary 2-sphere contains a 2-hemisphere in the ideal set. A bihedral 3-crescent is a 3-bihedron B

in $\check{M}_h$ so that a boundary 2-hemisphere is the ideal set. It is *pure* if a hemispherical 3-crescent does not contain it. A concave affine 3-manifold is a codimension-zero connected compact submanifold of M defined in Choi (1999):

- A concave affine 3-manifold of type I is a compact affine manifold covered by $R \cap M_h$ for a hemispherical 3-crescent R. (See Definition 2 for the precise definition.)
- Now, we assume that there is no hemispherical 3-crescent in $\check{M}_h$. A concave affine 3-manifold of type II is a compact affine manifold covered by $U \cap M_h$ of a union U of bihedral 3-crescents in M_h extending their open ideal boundary 2-hemispheres. (See Definition 3 for the precise definition.)

We remark that when we talk about concave affine 3-manifold N of type II, then there is no hemispherical crescent in the Kuiper completion of the holonomy cover of the real projective manifold containing N. Otherwise, N is not defined.

The interior of a concave affine 3-manifold has a canonical affine structure inducing its real projective structure. The *two-faced submanifold of type I* of a real projective 3-manifold M is roughly given as the totally geodesic 2-dimensional submanifold arising from the intersection in M_h of two hemispherical 3-crescents meeting only in the boundary.

Now, we assume that there is no hemispherical 3-crescent in $\check{M}_h$. The *two-faced submanifold of type II* of a real projective 3-manifold M is roughly defined as the totally geodesic 2-dimensional submanifold arising from the intersection in M_h of two bihedral 3-crescents meeting only in the boundary. For the precise definitions, see Sect. 2.5.3.

Theorem 1 (Choi 1999) *Suppose that M is a connected compact real projective 3-manifold with empty or convex boundary that is neither complete affine nor bihedral. Suppose that M is not 2-convex. Then $\check{M}_h$ contains a hemispherical or bihedral 3-crescent.*

Now, we sketch the process of *convex-concave decomposition* in Choi (1999) which we recall in Sect. 2.5 in more details:

- Suppose that a hemispherical 3-crescent $R \subset \check{M}_h$ exists.
  - If there is the two-faced submanifold of type I, then we can split M along this submanifold to obtain M^s. If not, we let $M^s = M$. Let M_h^s denote the corresponding cover of M^s obtained by splitting M_h and taking a union of components, and let $\check{M}_h^s$ be its Kuiper completion.
  - Then hemispherical 3-crescents in $\check{M}_h^s$ are mutually disjoint, and the intersection of each hemispherical 3-crescent with M_h^s covers a compact submanifold, called a *concave affine manifold of type I*.
  - We remove all these from M^s. Then we let the resulting compact manifold be called $M^{(1)}$. The boundary is still convex.
- Let $M_h^{(1)}$ denote the cover of $M^{(1)}$ obtained by removing corresponding submanifolds from M_h^s, and let $\check{M}_h^{(1)}$ be the Kuiper completion of $M_h^{(1)}$. Suppose that there is a bihedral 3-crescent $R \subset \check{M}_h^{(1)}$.

- If there is the two-faced submanifold of type II, then we can split $M^{(1)}$ along this submanifold to obtain $M^{(1)s}$. If not, we let $M^{(1)s} = M^{(1)}$.
- Let $M_h^{(1)s}$ denote the cover of $M^{(1)s}$ obtained from $M_h^{(1)}$ by splitting and taking a union of components, and let $\check{M}_h^{(1)s}$ be the Kuiper completion. Then the intersection of $M_h^{(1)s}$ with the union of bihedral 3-crescents in $\check{M}_h^{(1)s}$ covers the union of a mutually disjoint collection of compact submanifolds, called *concave affine manifolds of type II*.
- We remove all these from $M^{(1)s}$. Then the resulting compact real projective manifold $M^{(2)}$ with convex boundary is 2-convex.

Note that given a real projective manifold M, we define

$$M^s, M_h^s, \check{M}_h^s, M^{(1)}, M_h^{(1)}, \check{M}_h^{(1)}, M^{(1)s}, M_h^{(1)s}, \text{ and } \check{M}_h^{(1)s}$$

as above, and use this terminology throughout the paper.

We will further sharpen the result in this paper. A toral π-submanifold is a compact radiant concave affine 3-manifold with the virtually infinite-cyclic fundamental group covered by a special open set in a hemisphere. We will later show that a toral π-submanifold is homeomorphic to a solid torus or a solid Klein bottle. (See Definition 5; Lemmas 16, 17). A half-Hopf manifold is an example of toral π-submanifolds; however, some toral π-submanifolds are not one. We can make examples by pasting some half-Hopf manifolds with same holonomy groups along solid tori. (This is a simple construction).

Theorem 2 *Let M be a connected compact real projective 3-manifold with empty or convex boundary that is neither complete affine nor bihedral.*

- *Let M^s be the resulting real projective 3-manifold after splitting along the two-faced totally geodesic submanifold of type I (resp. of type II).*
- *Let N be a compact concave affine 3-manifold of type I (resp. of type II) in M^s with boundary compressible into M^s.*

Then N is a toral π-submanifold of type I (resp. contains a unique maximal toral π-submanifold of type II), or M is an affine Hopf 3-manifold.

So far, our results are on real projective 3-manifolds. Now we go over to the result specific to affine 3-manifolds.

Theorem 3 *Let M be a connected compact affine 3-manifold with empty or convex boundary. Suppose that M is neither complete affine nor bihedral and is not affine Hopf 3-manifold.*

- *Let M^s be the resulting real projective 3-manifold after splitting along the two-faced totally geodesic submanifold of type I.*
- *Now, M^s decomposes into concave affine manifolds of type I with boundary incompressible in M^s and toral π-submanifolds of type I.*

Let $M^{(1)}$ be obtained by removing all concave affine manifolds of type I in M^s. Let $M^{(1)s}$ denote the $M^{(1)}$ split along the two-faced submanifold of type II. $M^{(1)s}$ decomposes into compact submanifolds as follows:

- *a* 2*-convex affine* 3*-manifold with convex boundary,*
- *toral π-submanifolds of type II in concave affine* 3*-manifolds with boundary compressible into $M^{(1)s}$ with the virtually cyclic holonomy groups, or*
- *concave affine* 3*-manifolds of type II with boundary incompressible in $M^{(1)s}$.*

The finally decomposed submanifolds from the decomposition are prime 3*-manifolds.*

However, the above decomposition is not necessarily a prime decomposition. The following gives us a criterion for the primeness. This theorem is a generalization of the 2-dimensional affine decomposition theory of Nagano and Yagi (1974).

Corollary 1 *Let M be a connected compact affine* 3*-manifold with empty or convex boundary. Suppose that there is no projectively embedded image of toral π-submanifold of type I or II in M. Then M is irreducible or is an affine Hopf* 3*-manifold and hence is prime.*

We question whether that the above concave affine manifolds are maximal in the sense of Choi (1999).

The following answers Goldman's question partially.

Corollary 2 (Choi–Wu) *Suppose that M is a connected closed affine manifold covered by an open set Ω in $\mathbb{R}^3$. Then M is either irreducible or is an affine Hopf* 3*-manifold.*

One significance of this paper is to see how far the techniques of Choi (1999, 2000) and can be applied to solve this problem. We isolated some objects here. We think that the nonexistence results of 2-convex affine 3-manifolds of certain topological type will be helpful here. We postpone this discussion to later papers.

1.3 Outline

The main tools of this paper are from three long papers of Choi (1999, 2000, 2001). We summarize the results of Choi (1999, 2001) in Sect. 2. In Sect. 2.3, we recall radiant affine n-manifolds and recall some results of Choi (2001). In Sect. 2.4, we prove various facts about affine Hopf manifolds and half-Hopf manifolds. In Sect. 2.5, we recall the convex and concave decomposition of real projective structures including 3-crescents and two-faced submanifolds in Choi (1999).

In Sect. 3, Theorem 6 claims that if the two-faced submanifold is non-π_1-injective, then the manifold is finitely covered by an affine Hopf 3-manifold. The idea for the proof is by a so-called "disk-fixed-point argument" Proposition 8. That is, we can find an attracting fixed point of a deck transformation g when a simple closed curve c bounds a disk D with the property $g(c) \subset D^o$. We prove Theorem 6 in Sect. 3.2.

The main technical core results are Theorems 7 and 8 in Sect. 3.3. We show that a cover of the concave affine 3-manifold being a union of mutually intersecting 3-crescents must be mapped to a subset in a hemisphere by $\mathbf{dev}_h$, and the boundary has a unique annulus component. Since the fundamental group of N acts on an annulus covering its boundary properly and freely, the fundamental group is virtually infinite-cyclic by Lemma 3. We complete the final part of the proof in Sect. 3.5 where we show that these concave affine 3-manifolds contain toral π-submanifolds. We also show that a toral π-submanifold is homeomorphic to a solid torus or a solid Klein bottle. We prove Theorem 2 at the end.

In Sect. 4, we discuss the decomposition of M into 2-convex real projective 3-manifolds with convex boundary and toral π-submanifolds. i.e., Theorem 10. We use the convex and concave decomposition theorem of Choi (1999) and Theorems 7 and 8 and replacing the compact concave affine 3-manifolds with compressible boundary with toral π-submanifolds. We prove Theorem 3 and Corollaries 1 and 2 lastly here.

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2 Preliminary

2.1 Some 3-Manifold Topology

Let K be a manifold. Let $\mathrm{Diff}(K)$ be the group of diffeomorphisms of K with the usual C^r-topology, $r \geq 0$, and $\mathrm{Diff}_0(K)$ the identity component of this group. We define the mapping class group $\mathrm{Mod}(K)$ of a manifold K to be the group $\mathrm{Diff}(K)/\mathrm{Diff}_0(K)$.

Since $\mathrm{Mod}(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$ is a classical work of Smale (1959), there exist only two homeomorphism types of $\mathbb{S}^2$-bundles over $\mathbb{S}^1$. If M' is orientable, then M' is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. If not, M' is a non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. The following is well known.

Lemma 1 *Let $\tilde{N} := K \times \mathbb{R}$ for a compact manifold K covers a compact manifold N as a regular cover. Suppose that* $\mathrm{Mod}(K)$ *is finite. Then N is finitely covered by $K \times \mathbb{S}^1$.*

Given an embedded surface Σ in a 3-manifold M that is either on the boundary of M or is two-sided, Σ is *incompressible* into M if $\pi_1(\Sigma)$ injects into $\pi_1(M)$. Otherwise, Σ is said to be *compressible*. A simple closed curve in Σ is *essential* if it is not null-homotopic in Σ. A compressible surface always has an essential simple closed curve that is the boundary of an embedded disk by Dehn's Lemma.

A n-manifold is *irreducible* if every embedded two-sided 2-sphere bounds a 3-ball. Also, prime 3-manifolds are either irreducible or are homeomorphic to an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. (See Lemma 3.13 of Hempel 2004).

Lemma 2 *Let L be a connected compact* 3*-manifold with the universal cover whose interior is an open cell, and $\pi_1(L)$ is virtually infinite-cyclic. Then L is homeomorphic to a solid torus or a solid Klein-bottle.*

Proof Since the interior of the universal cover of L is a cell, L is irreducible. By Theorem 5.2 of Hempel (2004), L is finitely covered by a solid torus. Hence, ∂L is homeomorphic to a torus or a Klein bottle. Since ∂L is not π-injective, ∂L is compressible by Dehn's lemma. Since ∂L is compressible, we can find a disk D with $\partial D \subset \partial L$. Since L is irreducible, $L - D$ is a cell. Therefore, the conclusion follows. □

Lemma 3 *Let G be a group G acting on an annulus A faithfully, freely, and properly discontinuously with A/G a closed surface. Then G is virtually infinite-cyclic.*

Proof Let c be an essential simple closed curve in A. Since the surface groups are locally extended residually finite by Peter Scott, there is a finite index subgroup G' preserving the ends of A of G so that c is embedded to a simple closed curve in A/G'. Then for each nontrivial $g \in G'$, we have $g(U \cup c) \subset U$ or $g^{-1}(U \cup c) \subset U$. Choosing g so that $g(c)$ is closest to c, we obtain a generator of G'. Hence, G' is infinite-cyclic. □

2.2 The Projective Geometry of the Sphere

Let V be a vector space. Define $P(V)$ as $V - \{0\}/\sim$ where $x \sim y$ if and only if $x = sy$ for $s \in \mathbb{R} - \{0\}$. $\mathsf{PGL}(V)$ acts on this space where $\mathsf{PGL}(\mathbb{R}^n) = \mathsf{PGL}(n, \mathbb{R})$.

Recall that $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$. A subspace of $\mathbb{R}P^n$ is the image $V - \{O\}$ of a subspace V of $\mathbb{R}^{n+1}$ under the projection. The group of projective automorphisms is $\mathsf{PGL}(n+1, \mathbb{R})$ acting on $\mathbb{R}P^n$ in the standard manner. A *real projective n-manifold with empty or convex boundary* is a manifold with empty or nonempty boundary with an atlas of charts to $\mathbb{R}P^n$ with transition maps in $\mathsf{PGL}(n+1, \mathbb{R})$ so that each point of the boundary has a chart to a convex domain with boundary in $\mathbb{R}P^n$. A maximal atlas is called a *real projective structure*. The boundary is *totally geodesic* if each boundary point has a neighborhood projectively diffeomorphic to an open set in a half-space of an affine space meeting the boundary.

An *affine n-manifold with empty or convex boundary* is an n-manifold with boundary and an atlas of charts to open subsets or convex domains in $\mathbb{R}^n$ and the transition maps in $\mathbf{Aff}(\mathbb{R}^n)$. Since the affine transformations are projective, an affine n-manifold has a canonical real projective structure. We consider such n-manifolds as real projective n-manifolds with special structures in this paper. A real projective manifold projectively homeomorphic to an affine manifold is called an *affine manifold* in this paper.

Definition 1 An elementary example is an *affine Hopf n-manifold* that is the quotient of $\mathbb{R}^n - \{O\}$ by an infinite-cyclic group generated by a linear map g all of whose eigenvalues have norm > 1 or by $\langle g, -\mathrm{I}\rangle$ for g as above. The quotient is a manifold by Proposition 7 in Appendix 1.

If g acts on an $(n-1)$-plane passing O, and the half-space H in $\mathbb{R}^n$ bounded by it, then $(H - \{O\})/\langle g\rangle$ is called a *half-Hopf n-manifold*. A real projective manifold projectively homeomorphic to an affine Hopf n-manifold or a half-Hopf n-manifold is called by the same name in this paper. (See Hopf 1948 for a conformally flat version and Sullivan and Thurston 1983).

Let $\mathbb{R}_+ := \{t | t \in \mathbb{R}, t > 0\}$. Define $\mathbb{S}(V)$ as $V - \{0\}/\sim$ where $x \sim y$ if and only if $x = sy$ for $s \in \mathbb{R}_+$. $\mathsf{SL}_\pm(V)$ acts on $\mathbb{S}(V)$ transitively and faithfully. There is a double cover $\mathbb{S}(V) \to P(V)$ with the deck transformation group generated by the antipodal map $\mathcal{A} : \mathbb{S}(V) \to \mathbb{S}(V)$ induced from the linear map $V \to V$ given by $v \to -v$. We denote by $(\!(v)\!)$ the equivalence class of v in $\mathbb{S}(V)$. The *homogeneous coordinate system* of $\mathbb{S}(\mathbb{R}^n)$ is given by denoting each point by $(\!(x_1, \dots, x_n)\!)$ for the vector $(x_1, \dots, x_n) \neq 0$.

We denote by $\mathbb{S}^n$ the space $\mathbb{S}(\mathbb{R}^{n+1})$. The real projective sphere $\mathbb{S}^n$ has a real projective structure given by the double covering map to $\mathbb{R}P^n$. The group of projective automorphisms of $\mathbb{S}^n$ form $\mathsf{SL}_{\pm}(n+1, \mathbb{R})$ as obtained by the standard action of $\mathsf{GL}(n+1, \mathbb{R})$ on $\mathbb{R}^{n+1}$.

We embed $\mathbb{R}^n$ as an open n-hemisphere $\mathbb{H}^o$ in $\mathbb{S}^n$ for a closed n-hemisphere $\mathbb{H}$ by sending $(x_1, x_2, \ldots, x)$ to $(\!(1, x_1, x_2, \ldots, x_n)\!)$. We identify $\mathbb{R}^n$ with $\mathbb{H}^o$. The boundary of $\mathbb{R}^n$ is a great sphere $\mathbb{S}^{n-1}_\infty$ given by $x_0 = 0$. The group of projective automorphisms acting on $\mathbb{H}$ equals the group $\mathbf{Aff}(\mathbb{R}^n)$ of affine transformations of $\mathbb{H}^o = \mathbb{R}^n$. (A good reference for all these geometric topics is the book by Berger 2009).

We take the universal cover $\tilde{M}$ of an n-manifold M. The existence of a real projective structure on M gives us

- an immersion $\mathbf{dev} : \tilde{M} \to \mathbb{R}P^n$, called a *developing map* and
- a homomorphism $h : \pi_1(M) \to \mathsf{PGL}(n+1, \mathbb{R})$, called a *holonomy homomorphism*

satisfying $\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev}$ for each $\gamma \in \pi_1(M)$.

By lifting $\mathbf{dev}$, we obtain

- a well-defined immersion $\mathbf{dev}' : \tilde{M} \to \mathbb{S}^n$ and
- a homomorphism $h' : \pi_1(M) \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$

so that $\mathbf{dev}' \circ g = h'(g) \circ \mathbf{dev}'$ for each deck transformation g of $\tilde{M}$.

Let K_h be the kernel of $h' : \pi_1(M) \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$. Let $M_h := \tilde{M}/K_h$ be a so-called holonomy cover. Then $\mathbf{dev}'$ induces an immersion $\mathbf{dev}_h : M_h \to \mathbb{S}^3$. The deck transformation group Γ_h of the covering map $p_h : M_h \to M$ is isomorphic to $\pi_1(M)/K_h$. Since the real projective structures is a real analytic structure, Γ_h acts nontrivially on every open subset of M_h. We obtain

- an immersion $\mathbf{dev}_h : M_h \to \mathbb{S}^n$, also called a *developing map* and
- a homomorphism $h_h : \pi_1(M)/K_h \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$, also called a *holonomy homomorphism*

satisfying

$$\mathbf{dev}_h \circ \gamma = h_h(\gamma) \circ \mathbf{dev}_h \text{ for } \gamma \in \Gamma_h.$$

Lemma 4 *Let M be a connected compact real projective manifold with convex boundary. Consider a cover M' of M with a covering map $p_M : M' \to M$ with a deck transformation group Γ'. Let $p_{M'} : \tilde{M} \to M'$ denote the covering map induced by the universal covering map $\tilde{M} \to M$. Then*

- *given a projective immersion $\mathbf{dev}' : M' \to \mathbb{S}^n$ satisfying $\mathbf{dev}' = \mathbf{dev} \circ p_{M'}$, there is a homomorphism $h' : \Gamma' \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ satisfying $\mathbf{dev}' \circ \gamma = h'(\gamma) \circ \mathbf{dev}'$ for every $\gamma \in \Gamma'$.*
- *$\mathbf{dev}'$ is a holonomy cover if and only if $p_M : M' \to M$ is a regular cover, and $h'|\Gamma'$ is injective.*

Proof Straightforward. □

Lemma 5 *Let M be a connected compact real projective manifold with convex boundary. For any connected submanifold N of M, let N_h denote a component of its inverse* *image in M_h. Then $p_h|N_h : N_h \to N$ is a holonomy covering map also and the deck transformation group equals the subgroup Γ_{h,N_h} of Γ_h acting on N_h. For the developing map, $\mathbf{dev}_{h,N_h} = \mathbf{dev}_h|N_h$ holds, and for the corresponding holonomy homomorphism, $h_{N_h} = h_h|\Gamma_{h,N_h}$ holds.*

Proof First, $\Gamma_{h,N_h} \to \Gamma_h$ is injective. Since $h_h|\Gamma_h$ is injective, $h_h|\Gamma_{h,N_h}$ is injective. Since Γ_{h,N_h} is the regular deck transformation group of $p_h|N_h$, we are done by Lemma 4. □

Lemma 6 *Let M be a connected compact real projective manifold with convex boundary. The deck transformation group Γ_h of M_h is residually finite. So, is Γ_{h,N_h} for any connected submanifold N of M.*

Proof Under h_h, Γ_h is mapped injectively into a linear group $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$. The Selberg-Malcev lemma implies the conclusion. □

2.3 Radiant Affine *n*-Manifolds

Given any affine coordinates $x_i, i = 1, \ldots, n$, of $\mathbb{R}^n$, a vector field $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is called a *radiant* vector field. O of the coordinate system is called the *origin* of the radiant vector field. Suppose that the holonomy group of an affine n-manifold M fixes O. Then $\mathbf{dev}_h : M_h \to \mathbb{R}^n$ is an immersion and the radiant vector field lifts to a vector field in M_h. The vector field is invariant under the deck transformations of M_h and hence induces a vector field on M. The vector field on M is also called a *radiant vector field*. (See Barbot 2000 and Chapter 3 of Choi 2001). Suppose that the vector field is tangent to ∂M. This gives us a *radiant flow* which induces an action

$$\mathbb{R} \times M \to M$$

whenever M is compact. We call an affine manifold M with a radiant flow tangent to ∂M a *radiant affine manifold*.

Let M be an affine manifold with the holonomy group fixing a point O. A *radiant line* in M_h is an arc α in M_h so that $\mathbf{dev}|\alpha$ is an embedding to a component of a complete real line l with O removed.

Proposition 1 *Let M be a connected compact affine n-manifold with empty or nonempty boundary. Suppose that the holonomy group fixes the origin of a radiant vector field, and the boundary is tangent to the radiant vector field. Then $\mathbf{dev}_h(M_h)$ misses the origin of a vector field and M_h is foliated by radiant lines.*

Proof See the proof of Proposition 2.4 of Barbot (2000). □

Let $||\cdot||$ denote the Euclidean metric of $\mathbb{R}^n$. Given a real projective $(n-1)$-manifold Σ and a projective automorphism $\phi : \Sigma \to \Sigma$, we can obtain a radiant affine n-manifold homeomorphic to the mapping torus

$$\Sigma \times I/\sim \text{ where for every } x \in \Sigma, (x,1) \sim (\phi(x), 0).$$

Let $\mathbf{dev} : \tilde{\Sigma} \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$ be a developing map with holonomy homomorphism $h : \pi_1(\Sigma) \to \mathsf{SL}_\pm(n, \mathbb{R})$. Then we extend $\mathbf{dev}$ to

$$\mathbf{dev}' : \tilde{\Sigma} \times \mathbb{R} \to \mathbb{R}^n \text{ by } (x, t) \mapsto \exp(t)\mathbf{dev}(x).$$

For each element γ of $\pi_1(\Sigma)$, we define the action of $\pi_1(\Sigma)$ on $\Sigma \times \mathbb{R}$ by

$$\gamma(x, t) = (\gamma(x), \log ||h(\gamma)(\mathbf{dev}(x))|| + t).$$

This preserves the affine structure and the radiant vector field. The automorphism ϕ lifts to $\tilde{\phi} : \tilde{\Sigma} \to \tilde{\Sigma}$ so that $\psi \circ \mathbf{dev} = \mathbf{dev} \circ \tilde{\phi}$ for $\psi \in \mathsf{SL}_\pm(n, \mathbb{R})$ where we define

$$\tilde{\phi} : \tilde{\Sigma} \times \mathbb{R} \to \tilde{\Sigma} \times \mathbb{R} \text{ by } \hat{\phi}(x, t) = (\tilde{\phi}(x), \log ||\psi(\mathbf{dev}(x))|| + t).$$

Then the result $\tilde{\Sigma} \times \mathbb{R}/\langle \hat{\phi}, \pi_1(\Sigma)\rangle$ is homeomorphic to the mapping torus. We call this construction or the manifold the *generalized affine suspension*. If ϕ is of finite order, then the manifold is called a *Benzécri suspension*.

Proposition 2 (See Proposition 3.2 of Choi 2000) *Let M be a connected compact radiant affine n-manifold. Then M is a generalized affine suspension if and only if the radiant flow on M has a total cross section.*

Corollary 3 (Barbot-Choi Barbot and Choi 2001, Corollary A Choi 2001) *Let M be a connected compact radiant affine* 3*-manifold with empty or totally geodesic boundary. Then M admits a total cross-section to the radiant flow. As a consequence, M is affinely diffeomorphic to one of the following affine manifolds*:

- *a Benzécri suspensions over a real projective surface of negative Euler characteristic with empty or geodesic boundary.*
- *a generalized affine suspension over a real projective sphere, a real projective plane, or a hemisphere,*
- *a generalized affine suspension over a real projective torus* (*Klein bottle*)*, a real projective annulus* (*Möbius band*) *with geodesic boundary.*

There is a 6-dimensional closed radiant affine manifold giving us a counter-example to the existence of the cross-section to the radiant flow due to Fried.

Remark 1 We mention an error in Choi (2001) for Theorem A and Corollaries A and B. We state Corollary A in the corrected form above. We assume not only that the holonomy group of the affine manifold M fixes a common point but also that the boundary is tangent to the radiant vector field. Proposition 1 should fill in the gap since we just need to use the fact that radiant lines foliate the universal cover.

2.4 Affine 3-Manifolds with the Infinite-Cyclic Holonomy Groups

First, we will explore the affine Hopf manifolds (Fig. 1).

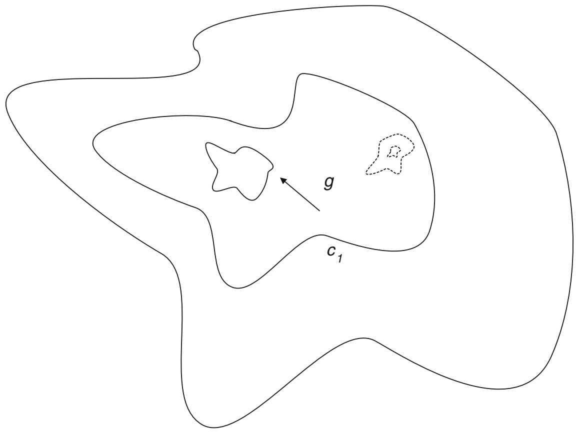


Fig. 1 There must be an image of c_1 inside the component bounded by c_1

Lemma 7 *Let X be a connected open orientable manifold with a group G acting on it properly discontinuously and cocompactly. Let c_1 be a codimension-one compact connected submanifold where $X - c_1$ has two open components, and let U be a component. Then there exist infinitely many elements $g \in G$ so that $g(c_1) \subset U$.*

Proof Let $x \in c_1$. Since the action of G on X is cocompact, there exists an infinite sequence $\{g_i\}$ of orientation-preserving $g_i \in G$ so that $g_i(x) \in U$. Since the action of G on X is properly discontinuous, $g_i(c_1) \cap c_1 = \emptyset$ except for finitely many i. We may choose g_i so that $g_i(c_1)$ is a proper subset of U. □

Proposition 3 *An affine Hopf 3-manifold M is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{R}P^2 \times \mathbb{S}^1$, or a nonorientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. A half-Hopf 3-manifold M is homeomorphic to a solid torus or a solid Klein bottle.*

Proof Let M be an affine Hopf 3-manifold. The universal cover is $\mathbb{R}^3 - \{O\}$, and hence M does not contain any fake cells. We double cover it so that it has an infinite cyclic holonomy group and call the double cover by M'. Let g be the generator of the holonomy group. Each eigenvalue of a nonidentity element $g \in h(\pi_1(M))$ has either all norms > 1 or all norm < 1 by definition. By taking g^{-1} if necessary, we assume that all the norms are < 1. Let S be a unit sphere for a norm in Lemma 18. By Lemma 18, S and $g(S)$ are disjoint. Then S and $g(S)$ bound a compact space homeomorphic to $S \times I$. We introduce an equivalence relation $\sim$ where $x \sim y$ for $x \in S$, $y \in g(S)$ if $y = g(x)$. Thus,

$$(\mathbb{R}^3 - \{O\})/\langle g \rangle$$

is an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. Since $\mathrm{Mod}(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$ is a classical work of Smale (1959), there exist only two homeomorphism types of $\mathbb{S}^2$-bundles over $\mathbb{S}^1$.

Now, M is doubly or quadruply covered by $\mathbb{S}^2 \times \mathbb{S}^1$. Since $-\mathrm{I}$ acts on S above, and $\mathrm{Mod}(\mathbb{R}P^2) = 1$, the proposition is proved.

Let M be a half-Hopf 3-manifold. Then we take a copy M' of M and glue M with M' at the boundary ∂M and $\partial M'$ by an induced map of $-\mathrm{I} : \mathbb{R}^3 \to \mathbb{R}^3$. Then the topology follows □

Theorem 4 *Let M be a connected compact affine 3-manifold with empty or totally geodesic boundary. Suppose that a virtually infinite-cyclic holonomy group of M fixes a point in the affine space. Also, suppose that the radiant flow is tangent to the boundary. Then*

- *M is finitely covered by $\mathbb{S}^2 \times \mathbb{S}^1$ or $D^2 \times \mathbb{S}^1$.*
- *M is a generalized affine suspension of a sphere, $\mathbb{R}P^2$, or a 2-hemisphere.*
- *If M is closed, then M is an affine Hopf 3-manifold and is diffeomorphic to an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ or $\mathbb{R}P^2 \times \mathbb{S}^1$. If M has totally geodesic boundary, then M is a half-Hopf manifold.*
- *Any 3-manifold covered by an affine Hopf 3-manifold or a half-Hopf 3-manifold respectively is one also.*

Proof We take a finite cover N so that N has an infinite cyclic fundamental group. By Theorem 5.2 of Hempel (2004) and Lemma 1, N has to be covered by $\mathbb{S}^2 \times \mathbb{S}^1$ or $D^2 \times \mathbb{S}^1$ finitely. Therefore, the universal cover $\tilde{M}$ is neither complete affine nor bihedral.

By taking a finite cover N of M, we may assume that $h(\pi_1(N)) = \langle g \rangle$ and g fixes a point x in the affine space. Thus the holonomy group fixes a point x. Then N is a radiant affine 3-manifold by definition in Choi (2001). (See Sect. 2.3). Since the holonomy group is virtually infinite cyclic, the classification of such affine 3-manifolds in Corollary 3 (Corollary A in Choi 2001) implies that N is a generalized affine suspension of $\mathbb{S}^2$, $\mathbb{R}P^2$, or a 2-hemisphere. To explain, N admits a total cross-section by Theorem B of Barbot (2000). This means that N and hence M are covered by $\mathbb{R}^3 - \{x\}$ or $H - \{x\}$ for the closed half-space H of $\mathbb{R}^3$ for $x \in \partial H$.

We now prove that when M is closed, the only case is the affine Hopf 3-manifold: M is a generalized affine suspension of a real projective 2-sphere or a real projective plane by the second item of the conclusion of Corollary 3. In the first case, M has an infinite cyclic group as the deck transformation group acting on $\mathbb{R}^3 - \{O\}$. By Proposition 7 in Appendix 1, M is an affine Hopf 3-manifold. In the second case, the double cover of M is an affine Hopf 3-manifold. An order-two element k centralizes the infinite cyclic group since the generator fixes a unique point in $\mathbb{R}^3$. $\pi_1(M)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. Also, k must act on a sphere in $\mathbb{R}^3 - \{O\}$ as an order two element, and hence $k = -\mathrm{I}$. Thus, M is an affine Hopf 3-manifold.

When M is a generalized affine suspension over a 2-hemisphere, similar arguments apply to show that M is a half-Hopf manifold.

Any affine 3-manifold covered by an affine Hopf 3-manifold or a half-Hopf 3-manifold satisfies the premises of the theorem. Thus, it is an affine Hopf 3-manifold or a half-Hopf 3-manifold. □

We will be using the definition of convexity in Chapter 2 of Choi (1999). A *convex line* in $\mathbb{S}^n$ is an embedded arc not containing a pair of antipodal points in its interior. A subset A of $\mathbb{S}^n$ is *convex* if any two points of A are connected by a convex segment in A. A *convex hull* of a subset B of $\mathbb{S}^n$ is the minimal convex subset containing B.

Corollary 4 *Let M be a connected closed real projective n-manifold. Suppose that a connected open subset Ω in $\mathbb{S}^n$ covers M and contains a smoothly embedded sphere S^{n-1} of codimension one with the following properties*:

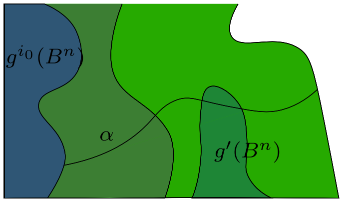


Fig. 2 Diagram for (A). We are showing the images of B^n under Γ'_h and the arc α

– *S^{n-1} bounds an n-ball B^n in $\mathbb{R}^n$ for an affine subspace $\mathbb{R}^n \subset \mathbb{S}^n$, and*
– *B^n is not contained in Ω.*

Then M is projectively diffeomorphic to an affine Hopf n-manifold.

Proof A component of $\mathbb{S}^n - S^{n-1}$ is an n-cell B^n in an affine patch. So, B^n is in a properly convex domain. Since Ω covers a compact manifold, there exists a deck transformation g so that $g(S^{n-1}) \subset B^n \cap \Omega$ by Lemma 7. Then $g(B^n \cup S^{n-1}) \subset B^n$ since the outside component of $\mathbb{S}^n - g(S^{n-1})$ is not contained in a properly convex domain. By the Brouwer fixed-point theorem, g fixes a point in the interior of B^n. Proposition 9 in Appendix 1 shows that x is an attracting fixed point in a g-invariant open hemisphere $\mathbb{H}^o$ (Fig. 2).

We now devote to showing that $\Omega = \mathbb{H}^o - \{x\}$, for this we do need a nontrivial result of Wu (2012). *The main difficulty is to show Ω contains everything between the two spheres S^{n-1} and $g(S^{n-1})$.*

(A) *We find a nonseparating sphere of dimension $n-1$ in a finite over of M*: by Lemma 6, there is a cover M' of M by taking a finite-index normal subgroup Γ'_h of Γ_h so that $p|S^{n-1}$ is an embedding to a sphere $\hat{S}$ for the covering map $p : \Omega \to M'$. Furthermore, we may assume that Γ'_h is orientation preserving.

Let g^{i_0} be the power of g in Γ'_h with least i_0, $i_0 > 0$. We obtain

$$g^{i_0}(S^{n-1}) \subset B^n \text{ and } g^{i_0}(B^n \cap \Omega) \subset B^n \cap \Omega$$

as at the beginning of the proof.

For any $g' \in \Gamma'_h$ so that $g'(S^{n-1}) \subset B^n$, we have $g'(B^n) \subset B^n$ as in the beginning of the proof: the reason is: $g'(B^n)$ is in a region bounded by $g'(S^{n-1})$ in $\mathbb{S}^n$, and the closure of the external component $\mathbb{S}^n - g'(S^{n-1})$ is not contained in an affine space while $g'(B^n)$ is in an affine space.

By the paragraph immediately above, any generic embedded open arc α in the open subset $B^n \cap \Omega - g^{i_0}(\mathrm{Cl}(B^n))$ connecting $x' \in S^{n-1}$ to $g^{i_0}(x')$ meets copies of B^n under Γ'_h other than $g^{i_0}(B^n)$

– in a compact interval disjoint from x' and $g^{i_0}(x')$ or
– in the interval containing $g^{i_0}(x')$ but not x'.

Let us call f the number of times the second case happens. We may assume that the intersection number at x' with α is $+1$ up to changing the orientation of M'. Then the oriented-intersection number of the image of α in M' meeting $\hat{S}$ is $f+1 > 0$ since the contributions of the intersection points of the first kind will cancel out. Thus, $\hat{S}$ is a nonseparating sphere (Fig. 3).

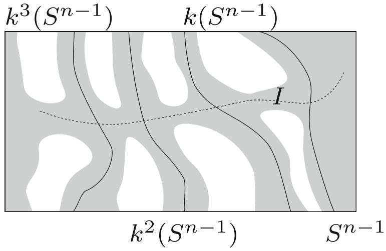


Fig. 3 Diagram for B: the action of k^i and the images of spheres. The gray region covers a neighborhood of $\hat{S} \cup I'$, and we are filling in the white regions

(B) *We will use $\hat{S}$ to obtain a sphere bounding a cell outside the neighborhood of $\hat{S}$ union an arc connecting the two sides of $\hat{S}$*: since $\hat{S}$ is nonseparating, we take a transversal embedded arc I' in M' connecting a point of $\hat{S}$ to itself and disjoint from $\hat{S}$ in the interior. We take an ϵ-neighborhood N of $I' \cup \hat{S}$ and let S'' denote the boundary sphere of the neighborhood. (See Lemma 3.8 of Hempel 2004 for the construction.) Since S'' bounds a neighborhood, it is a separating $(n-1)$-sphere. We choose sufficiently small ϵ so that S'' is homeomorphic to a sphere. Let I be the lift of I' starting from S^{n-1}. Let k be the deck transformation so that y and $k(y)$ are endpoints of I. Then S'' lifts to a sphere in Ω that is a boundary component S' of the inverse image N'' of N, which is a neighborhood of $\bigcup_{i\in\mathbb{Z}} k^i(S^{n-1} \cup I)$.

Again, k satisfies the properties of g above since $p|S^{n-1} : S^{n-1} \to \hat{S}$ is an embedding. By Proposition 9 in Appendix 1, there is an attracting fixed point of k in an affine space. By a change of notation, let x denote to the attracting fixed point of k of an affine space to be denote by $\mathbb{H}^o$

Since a sequence $\{k^j(S^{n-1})\}$ of compact sets geometrically converges to $\partial\mathbb{H}$ as $j \to -\infty$, and S^{n-1} is compact, Ω is disjoint from $\partial\mathbb{H}$ by the properness of the action of $\langle k\rangle$ on Ω. Hence, $\Omega \subset \mathbb{H}^o$. Since x is a fixed point of k, we obtain $\Omega \subset \mathbb{H}^o - \{x\}$.

Since $\Omega \subset \mathbb{H}^o$, the sphere S' is a subset of $\mathbb{H}^o$. It is obvious that M' is an affine manifold. By Theorem 1.2 of Wu (2012), S'' bounds an n-ball B'' in M', and an n-ball B' bounded by S' in $\Omega = M_h$ embeds onto B'' under p for $n \geq 3$. For $n = 2$, Nagano and Yagi (1974) shows that the universal cover of a closed affine surface is affinely diffeomorphic to a universal cover of $\mathbb{R}^2 - \{O\}$ or $\mathbb{R}^2$ or $\mathbb{R} \times (0, \infty)$. Hence, every open subset of $\mathbb{S}^2$ that covers a closed affine surface is affinely diffeomorphic to

$$\mathbb{R}^2 - \{O\}, \mathbb{R}^2, \text{ or } \mathbb{R} \times (0, \infty).$$

Separating circles in such an open subset always bound embedded disks. (See Example 1.6 of Benoist 2000).

Since $\partial B'$ is a component of $\partial N''$, we cannot have $B' \subset N''$ while N'' is not compact. Hence, the interior of B' is disjoint from that of N''.

(C) *Finally, we show that $\Omega = \mathbb{H}^o - \{x\}$*: taking a union of B' with N'', we obtain that open set D in $\mathbb{H}^o$ bounded by S^{n-1} and $k(S^{n-1})$ is in Ω. Since

$$D \subset B' \cup N \subset \Omega,$$

we obtain $\Omega \supset \bigcup_{i\in\mathbb{Z}} k^i(D)$.

By the generalized Schoenflies theorem $k^{-i}(S^{n-1})$ and $k^i(S^{n-1})$, $i > 0$, bound a region in $\mathbb{H}^o$ homeomorphic to $S^{n-1} \times I$, and

$$k^{-i}(S^{n-1}) \to \partial\mathbb{H} \quad \text{and} \quad k^i(S^{n-1}) \to \{x\} \quad \text{as } i \to \infty$$

in the Hausdorff convergence sense. It follows that $\bigcup_{i\in\mathbb{Z}} k^i(D) = \mathbb{H}^o - \{x\}$. Hence, $\Omega = \mathbb{H}^o - \{x\}$.

Now, Theorem 4 shows that $\Omega/\langle g^{i_0}\rangle$ is an affine Hopf 3-manifold, a compact manifold. Therefore, M is finitely covered by an affine Hopf-3-manifold. Theorem 4 implies the result. □

2.5 Convex Concave Decomposition of Real Projective 3-Manifolds

2.5.1 Kuiper Completions

The immersion $\mathbf{dev}_h$ induces a Riemannian μ-metric on M_h from the standard Riemannian metric μ on $\mathbb{S}^3$. This gives us a path-metric to be denoted by $\mathbf{d}$ on M_h. (More precisely $\mathbf{d}_h$ but we omit J here). Recall from Choi (1999) the Cauchy completion $\check{M}_h$ of M_h with this path-metric is called the *Kuiper completion* of M_h. (This metric is quasi-isometrically defined by $\mathbf{dev}_h$, and hence the topology is independent of the choice of $\mathbf{dev}_h$).

- The *ideal set* is $M_{h,\infty} := \check{M}_h - M_h$, which is in general not empty.
- The immersion $\mathbf{dev}_h$ extends to a continuous map. We use $\mathbf{dev}_h$ as the notation for the extended map as well.
- If M is an affine 3-manifold, we consider M as a real projective 3-manifold since the charts and transition maps are projective. We define M_h, $\check{M}_h$, and $M_{h,\infty}$ as above.
- Γ_h acts on M_h and $M_{h,\infty}$ possibly with fixed points in $M_{h,\infty}$.

Now, we discuss subsets of $\check{M}_h$.

- For a compact convex subset K of $\check{M}$ so that $\mathbf{dev}_h|K$ is an embedding, we define ∂K to be the subset corresponding to $\partial\mathbf{dev}_h(K)$.
- If $\mathbf{dev}_h(K)$ is a compact domain in a subspace of $\mathbb{S}^3$, then we define K^o as the subset of K that is the inverse image of the manifold interior of $\mathbf{dev}_h(K)$.
- An i*-hemisphere* in $\check{M}_h$ is a compact subset H so that $\mathbf{dev}_h|H$ is an embedding to an i-hemisphere, $1 \leq i \leq 3$. For 3-hemisphere, we require $H^o \subset M_h$.
- A 3*-bihedron* in $\check{M}_h$ is a compact subset B in $\check{M}_h$ so that $B^o \subset M_h$ and $\mathbf{dev}_h|B$ is an embedding to a compact convex set K so that ∂K is the union of two 2-hemispheres with the identical boundary great circle.

2.5.2 2-Convexity and Covers

A *tetrahedron* in $\check{M}_h$ is a compact subset T so that $\mathbf{dev}_h|T$ is an embedding to a tetrahedron in an affine space in $\mathbb{S}^3$. A *face* of T is a corresponding subset of $\mathbf{dev}_h(T)$.

Proposition 4 (Proposition 4.2 of Choi 1999) *A connected compact real projective 3-manifold M is 2-convex if and only if for every tetrahedron T in $\check{M}_h$ with faces F_i, $i = 0, 1, 2, 3$, such that $T^o \cup F_1 \cup F_2 \cup F_3 \subset M_h$, T is a subset of M_h.*

2.5.3 Crescents and Two-Faced Submanifolds

A *hemispherical* 3*-crescent* is a 3-hemisphere H in $\check{M}_h$ so that $H^o \subset M_h$, and a 2-hemisphere in ∂H is a subset of the ideal set $M_{h,\infty}$. We define α_R for a hemispherical 3-crescent R to be the union of all open 2-hemispheres in $\partial R \cap M_{h,\infty}$. We define $I_R = \partial R - \alpha_R$.

Remark 2 The results of Choi (1999) are true when the manifold-boundary ∂M is convex. This is not proved there. However, it is straightforward to generalize.

By Proposition 6.2 of Choi (1999) or by its $\check{M}_h$-version, given two hemispherical 3-crescents R and S in $\check{M}_h$, the exactly one of the following holds:

- $R \cap S \cap M_h = \varnothing$,
- $R = S$, or
- $R \cap S \cap M_h$ is a union of common components of $I_R \cap M_h$ and $I_S \cap M_h$.

The components of $I_R \cap M_h$ as in the last case are called *copied components* of $I_R \cap M_h$. The union of all copied components in M_h, a *pre-two-faced submanifold of type I*, is totally geodesic and covers a compact embedded totally geodesic 2-dimensional submanifold in M_h^o by Proposition 6.4 of Choi (1999). The submanifold is called the *two-faced submanifold of type I* (arising from hemispherical 3-crescents).

Remark 3 The components of a two-faced submanifold of type I all develop to a same 2-sphere in $\mathbb{S}^n$. However, this sphere may not really be lifted in $\check{M}_h$ in a one-to-one manner.

Note it is possible that the two-faced submanifold of type I may be empty, i.e., does not exist at all.

The *splitting along* a submanifold A is given by the Cauchy completion M^s of $M - A$ of the path metric obtained by using an ordinary Riemannian metric on M and restricting to $M - A$.

A *bihedral* 3*-crescent* is a 3-bihedron B in $\check{M}_h$ so that $B^o \subset M_h$ and a 2-hemisphere in ∂B is a subset of $M_{h,\infty}$. If they are not contained in a hemispherical 3-crescent, then we say that they are *pure*. For a bihedral 3-crescent R, we define α_R as the open 2-hemisphere in $\partial R \cap M_{h,\infty}$. We define $I_R := \partial R - \alpha_R$, a 2-hemisphere. For a 3-crescent R, we define the interior of R as $R^o = R - I_R - \alpha_R$.

We say that two 3-crescents R and S *overlap* if $R^o \cap S \neq \varnothing$, or equivalently $R^o \cap S^o \neq \varnothing$. We say that $R \sim S$ if there exists a sequence of 3-crescents

$$R_1 = R, R_2, \ldots, R_n = S \quad \text{where } R_i \cap R_{i+1}^o \neq \varnothing \quad \text{for } i = 1, \ldots, n-1.$$

We say that two bihedral 3-crescents R and S intersect *transversely* if

- $I_S \cap I_R$ is a segment with endpoints in ∂I_S and ∂I_R,

- $I_S \cap R$ is the closure of a component of $I_S - I_R$, and
- $R \cap S$ is the closure of a component of $R - I_S$.

In this case, $\alpha_S \cup \alpha_R$ is a union of two open 2-hemispheres meeting at an open convex disk $\alpha_S \cap \alpha_R$. Thus, they *extend* each other. (See Chapter 5 of Choi 1999).

Proposition 5 *We assume as in Theorem 2. Suppose that two bihedral 3-crescents R and S in $\check{M}_h$ overlap. Then R and S either intersect transversely or $R \subset S$ or $S \subset R$. Moreover, $\mathbf{dev}_h | R \cup S$ is a homeomorphism to its image $\mathbf{dev}_h(R) \cup \mathbf{dev}_h(S)$ where $\mathbf{dev}_h(\alpha_R)$ and $\mathbf{dev}_h(\alpha_S)$ are 2-hemispheres in the boundary of a 3-hemisphere H.*

Proof This is a restatement of Theorem 5.4 and Corollary 5.8 of Choi (1999). □

Assuming that there is no hemispherical 3-crescent in $\check{M}_h$, we define as in Chapter 7 of Choi (1999)

$$\begin{aligned} \Lambda(R) &:= \bigcup_{S \sim R} S, \qquad \delta_\infty \Lambda(R) := \bigcup_{S \sim R} \alpha_S, \\ \Lambda_1(R) &:= \bigcup_{S \sim R} (S - I_S), \quad \delta_\infty \Lambda_1(R) := \delta_\infty \Lambda(R). \end{aligned} \tag{2.1}$$

We showed in Chapter 7 of Choi (1999) $\mathbf{dev}_h | \Lambda(R)$ maps into a 3-hemisphere H and $\mathbf{dev}_h | \delta_\infty \Lambda(R)$ is an injective local homeomorphism to ∂H (see also Corollary 5.8 of Choi 1999).

Given a subset A of $\check{M}_h$, we define $\operatorname{int} A$ to be the interior of A in $\check{M}_h$. We define $\operatorname{bd} A$ to be the topological boundary of A in $\check{M}_h$. By Lemma 7.4 of Choi (1999), there are three possibilities:

- if $\operatorname{int} \Lambda(R) \cap \Lambda(S) \cap M_h \neq \emptyset$ for two bihedral 3-crescents R and S, then $\Lambda(R) = \Lambda(S)$,
- $\Lambda(R) \cap \Lambda(S) \cap M_h = \emptyset$, or
- $\Lambda(R) \cap \Lambda(S) \cap M_h \subset \operatorname{bd} \Lambda(R) \cap \operatorname{bd} \Lambda(S) \cap M_h$.

In the third case, the intersection is a union of common components of $\operatorname{bd} \Lambda(R) \cap M_h$ and $\operatorname{bd} \Lambda(S) \cap M_h$. We call such components *copied components*. These are totally geodesic and properly embedded in M_h. The union of all copied components in M_h, a *pre-two-faced submanifold of type II*, covers a compact embedded totally geodesic 2-dimensional submanifold in M^o by Proposition 7.6 of Choi (1999). The submanifold is called the *two-faced submanifold of type II* (arising from bihedral 3-crescents).

2.5.4 Concave Affine Manifolds After Splitting

Let M^s denote the 3-manifold obtained from M by splitting along the two-faced submanifold of type I. A cover M^s_h of M^s can be obtained by splitting along the preimage of the two-faced submanifold of type I in M_h and taking a component for every component of M^s and taking the union of these. For each component A of M^s_h, let Γ_A denote the subgroup of Γ_h of elements acting on A^o. Then Γ_A extends to a deck transformation group of A. We define Γ_h^s the product group

$$\prod_{A \in \mathcal{C}} \Gamma_A \text{ for the set } \mathcal{C} \text{ of chosen components in } M_h^s.$$

Again M_h^s has a developing map $\mathbf{dev}_h^s : M_h^s \to \mathbb{S}^3$, an immersion, and $M_h^s \to M^s$ is a holonomy cover with the deck transformation group Γ_h^s. There is a map $M_h^s \to \check{M}$ by identifying along the splitting submanifolds. We can easily see that the Kuiper completion $\check{M}_h^s$ contains a hemispherical 3-crescent if and only if $\check{M}_h$ does. Also, the set of hemispherical 3-crescents of $\check{M}_h^s$ is mapped in a one-to-one manner to the set of those in $\check{M}_h$ by taking the interior of the hemispherical 3-crescent in $\check{M}_h^s$ and sending it to $\check{M}_h$ and taking the closure. Now $\check{M}_h^s$ does not have any copied components. (See Chapter 8 of Choi 1999).

Definition 2 A connected compact real projective manifold with totally geodesic boundary covered by $R \cap M_h^s$ for a hemispherical 3-crescent R is said to be a *concave affine manifold of type I* in M^s.

Let $\mathcal{H}$ be the set of all hemispherical 3-crescents in M_h^s. The union $\bigcup_{R \in \mathcal{H}} R \cap M_h^s$ covers a finite union K of mutually disjoint concave affine manifolds of type I in M^s. Then $M^s - K^o = M^{(1)}$ is a compact real projective manifold with convex boundary. The cover $M_h^{(1)}$ of $M^{(1)}$ is M_h^s with all points of hemispherical 3-crescents removed from it. Then $\check{M}_h^{(1)}$ has no hemispherical 3-crescent. (See p. 80–81 of Choi 1999.)

Now, we look at $M^{(1)}$ only. We split $M^{(1)}$ along the two-faced submanifold of type II if it exists. Let $M^{(1)s}$ denote the result of the splitting. Also, the set of bihedral 3-crescents of $\check{M}^{(1)}$ is mapped in a one-to-one manner to the set of those in $\check{M}^{(1)s}$ by taking the interior of the bihedral 3-crescent and sending it to $M^{(1)s}$ and taking the closure. (See Chapter 8 of Choi 1999). Now $\check{M}_h^{(1)s}$ does not have any copied components. For a bihedral 3-crescent R in $\check{M}_h^{(1)s}$, $\Lambda(R) \cap M_h^{(1)s}$ covers a compact 3-manifold with concave boundary in $M_h^{(1)s}$. (See p. 81–82 of Choi 1999.)

Definition 3 Suppose that $\check{M}_h$ does not contain a hemispherical 3-crescent. Let R be a bihedral 3-crescent in $\check{M}_h$. If $\Lambda(R) \cap M_h$ covers a compact real projective submanifold N, then N is called a *concave affine manifold of type II*.

A concave affine manifold of type I or II is called a *concave affine manifold.* See Chapter 8 of Choi (1999) as a reference of results stated here.

3 Concave Affine 3-Manifolds

In this section, we will prove Theorems 6, 7 and 8. The first one shows that the non-π_1-injective two-faced submanifolds of type I or type II cannot happen in general. In the second and third ones, we will show that a concave affine manifold of type I or type II with compressible boundary contains a toral π-submanifold.

3.1 A Concave Affine Manifold Has No Sphere Boundary Component

Theorem 5 *Let N be a concave affine manifold of type I or II. Then no component of ∂N is covered by a sphere.*

Proof If N is a concave affine manifold of type I, then N is covered by $\tilde{N} = R^o \cup (I_R \cap N_h)$ for a hemispherical 3-crescent R. Since $I_R \cap N_h$ is a planar open surface in I_R, the conclusion follows.

Suppose now that there is no hemispherical 3-crescent in M_h. Hence, all 3-crescents are pure bihedral 3-crescents. Then N is covered by $\Lambda(R) \cap M_h$ for a bihedral 3-crescent R. Let A be a component of $\mathrm{bd}\Lambda(R) \cap M_h$ homeomorphic to a sphere. We know that A is mapped into a convex surface in $M - N^o$ under the covering map. If A is totally geodesic, then A is tangent to $I_S \cap M_h$ for a crescent S in $\Lambda(R)$. Hence, A is a subset of $I_S \cap M_h$, each component of which is not compact. This is a contradiction.

Suppose that each point x of A has some open geodesic segment in A containing x. Since A is convex, x is on a unique maximal geodesic in A or is in a 2-dimensional totally geodesic surface in A. Since A is convex, a geodesic segment in A must end at the boundary of A. This implies that A is not compact, a contradiction.

Hence, there must be a point y where A is strictly concave. (See Appendix 1 for definition.) This contradicts Theorem 11 in Appendix 1. □

3.2 Non-π_1-Injective Two-Faced Submanifolds

Lemma 8 *Let M_h be a holonomy cover of a connected compact real projective manifold M with convex boundary. Let $\tilde{A}_1$ be a properly embedded two-sided surface in M_h covering a compact surface A_1. If $\tilde{A}_1$ is a disk, then the inclusion map $A_1 \to M$ induces an injective homomorphism $\pi_1(A_1) \to \pi_1(M)$.*

Proof The deck transformation group $\Gamma_{\tilde{A}_1}$ injects into the deck transformation group Γ_h. Since $\tilde{A}_1$ is a disk, $\Gamma_{\tilde{A}_1}$ is isomorphic to the fundamental group $\pi_1(A_1)$ of A_1. Hence, the result follows. □

Theorem 6 implies that two-faced submanifolds are π_1-injective unless M is an affine Hopf 3-manifold.

Theorem 6 *Suppose that M is a connected compact real projective 3-manifold with empty or convex boundary and M is neither complete affine nor bihedral. Let S be a component of a two-faced submanifold of type I or type II in M. Then either S is π_1-injective in M or M is an affine Hopf 3-manifold.*

Proof (I) Let A_1 be a component of the two-faced submanifold of type I. A component $\tilde{A}_1$ of $I_R \cap M_h$ for a hemispherical 3-crescent R covers A_1. If $\tilde{A}_1$ is simply connected and planar, then $\tilde{A}_1$ is a disk. By Lemma 8, A_1 is π_1-injective in M, and we are done here.

Let Γ_1 denote the deck transformation group of $\tilde{A}_1$ in Γ_h so that $\tilde{A}_1/\Gamma_1$ is compact and diffeomorphic to A_1. Now assume that A_1 is non-π_1-injective in M. By the above paragraph, the planar surface $\tilde{A}_1$ contains a simple closed curve c_1 not bounding a disk in $\tilde{A}_1$.

By Corollary 4 for dimension $n = 2$, $\tilde{A}_1$ is projectively diffeomorphic to $\mathbb{R}^2 - \{O\}$. Since $\tilde{A}_1 \subset I_R^o$, we obtain

$$\tilde{A}_1 = I_R - \{x\} \quad \text{for } x \in I_R^o.$$

Since $\tilde{A}_1$ covers a component of the two-faced submanifold, $\tilde{A}_1$ is a component of $I_S \cap M_h$ for a hemispherical 3-crescent S where

$$R \cap S \cap M_h = \tilde{A}_1.$$

Since $\mathrm{Cl}(\alpha_S) \cup \mathrm{Cl}(\alpha_R) \subset M_\infty$ bounds the compact domain $R \cup S$ in $\check{M}_h$, we obtain $R^o \cup S^o \cup \tilde{A}_1 = M_h$. Now, $\mathbf{dev}_h|R^o \cup I_R^o - \{x\}$ and $\mathbf{dev}_h|S^o \cup I_S^o - \{x\}$ are homeomorphisms to their images. Thus, $\mathbf{dev}_h|M_h$ is a homeomorphism to the image

$$\mathbf{dev}_h(R)^o \cup \mathbf{dev}_h(S)^o \cup \mathbf{dev}_h(I_R^o) - \mathbf{dev}_h(x).$$

Since $\mathbf{dev}_h(x)$ is an isolated boundary point, we are finished in this case by Corollary 4.

(II) Let A_1 be a component of a two-faced submanifold of type II in M that is non-π_1-injective. Now, we assume that $\check{M}_h$ has no hemispherical 3-crescent. Then as in case (I), its cover $\tilde{A}_1$ is a component of $I_R \cap M_h$ for a bihedral 3-crescent R and contains a simple closed curve not contractible in $\tilde{A}_1$. By Corollary 4, we obtain that $\tilde{A}_1 = I_R^o - \{x\}$ for a bihedral 3-crescent R.

Since $\tilde{A}_1$ is in a pre-two-sided submanifold, we obtain that $I_R \subset I_S$ for another bihedral 3-crescent S so that $S^o \cap R^o = \varnothing$. It follows that

$$I_R^o - \{x\} = I_S^o - \{x\} \quad \text{and hence } I_R = I_S.$$

Since $\mathrm{Cl}(\alpha_R) \cup \mathrm{Cl}(\alpha_S) \subset M_{h,\infty}$ forms the boundary of $R \cup S$, and M_h is disjoint from it,

$$M_h = R^o \cup S^o \cup I_R^o - \{x\}$$

holds. Hence, $\mathbf{dev}_h$ is an embedding to $\mathbf{dev}_h(R^o) \cup \mathbf{dev}_h(S^o) \cup \mathbf{dev}_h(I_R^o - \{x\})$. Since $\mathbf{dev}_h(x)$ is an isolated boundary point, Corollary 4 implies the result in this case. □

Lemma 9 *Let Ω_1 be a connected open surface in M_h with $\mathbf{dev}_h(\Omega_1)$ bounded in an affine space H^o for a 2- or 3-hemisphere H. Let G be a discrete subgroup of Γ_h acting properly discontinuously and freely on Ω_1, and $h_h|G$ is injective with $h_h(G)$ acting on H. Then Ω_1/G is noncompact.*

Proof Let H_1 be the intersection of the spanning great sphere Ω_1 with H. Since G acts on H_1, G acts as a group of affine transformations on the affine 2- or 3-space H_1^o. Let F be the compact fundamental domain of Ω_1. The closure $\mathrm{Cl}(\mathbf{dev}_h(\Omega_1))$ is a compact bounded subset of H_1^o. The convex hull C_1 of $\mathrm{Cl}(\mathbf{dev}_h(\Omega_1))$ in H_1^o is a bounded subset of H_1^o. And $h_h(G)$ acts on it and fixes the center of mass m of C_1. Since $h_h(G)$ acts on some convex domain and its interior point, $h_h(G)$ is a group of bounded affine transformations fixing m. Since $h_h(G)$ is a finite group being discrete, we choose a $h_h(G)$-invariant Euclidean metric $d_{H_1^o}$ on H_1^o. Let U be an open ϵ-$d_{H_1^o}$-neighborhood of F in Ω_1. We choose sufficiently small ϵ so that $U \subset \Omega_1$.

Since Ω_1 is open, there exists a sequence $\{y_i\}$ exiting all compact sets in Ω_1 eventually. There exists $g_i \in G$ such that $g_i(y_i) \in F$. By taking a subsequence, we may assume $\mathbf{dev}_h(y_i) \to y \in \Omega_1$ and y is in the boundary of $\mathbf{dev}_h(\Omega_1)$, i.e., $y \notin \mathbf{dev}_h(\Omega_1)$. Then $g_i^{-1}(F) \ni y_i$. Since $\mathbf{dev}_h(y_i) \to y$, $h_h(g_i^{-1})$ is an isometry group fixing m, and Ω_1, $y \in \Omega_1$, is properly embedded, it follows that

$$\mathbf{dev}_h(\Omega_1) \supset \mathbf{dev}_h(g_i^{-1}(U^o)) = h_h(g_i^{-1})(\mathbf{dev}_h(U^o)) \ni y \text{ for sufficiently large } i,$$

which is a contradiction. □

3.3 Concave Affine Manifolds and Toral π-Manifolds

Definition 4 Let S_1 and S_2 be two crescents in $\check{M}_h$ so that $I_{S_1} \cap M_h$ and $I_{S_2} \cap M_h$ intersect and are tangent but $\mathbf{dev}_h(S_1)^o \cap \mathbf{dev}_h(S_2)^o = \emptyset$. In this case S_1 and S_2 are said to be *opposite*.

Definition 5 Let M be a compact connected real projective manifold that is neither complete affine nor bihedral, and let M_h be the holonomy cover of M. Assume that M has no two-faced submanifolds. Let R be a hemispheric 3-crescent with $I_R \cap M_h = I_R^o - \{x\}$ for $x \in I_R^o$. Then a compact submanifold P covered by $R \cap M_h$ is called a *toral π-submanifold of type I*.

Suppose that $\check{M}_h$ has no hemispheric crescent. Given $\Lambda(R)$ for a bihedral 3-crescent R, we define the set $C_{R,x}$ for $x \in \mathbb{S}^3$ as follows:

$$C_{R,x} := \{R' | R' \sim R, \exists g \in \Gamma_h, g(R) = R, h_h(g)(x) = x, \mathbf{dev}_h(I_{R'}^o) \ni x\} \neq \emptyset.$$

Let $\Lambda'(R)$ be $\bigcup_{R' \in C_{R,x}} R'$ whenever $C_{R,x}$ is not empty and

$$\delta_\infty \Lambda'(R) := \bigcup_{S \in C_{R,x}} \alpha_S.$$

Then $\Lambda'(R)$ develops into a 3-hemisphere H, and $\delta_\infty \Lambda'(R)$ develops to an open disk in ∂H for a 3-hemisphere H by Chapter 7 of Choi (1999). Suppose that $\Lambda'(R) \cap M_h$ covers a compact radiant affine 3-manifold P with boundary compressible into itself. Then P is said to be a *toral π-submanifold of type II*.

Theorems 7 and 8 characterize the concave affine 3-manifolds with boundary compressible into M. One consequence is that the fundamental group is virtually infinite cyclic.

Theorem 7 *Let M be a connected compact real projective 3-manifold with empty or convex boundary. Suppose that M is neither complete affine nor bihedral. Assume that M has no two-faced submanifold of type I.*

Let N be a concave affine 3-manifold of type I in M with nonempty boundary ∂N compressible into M. Then N has a unique compressible boundary component A, and N is a toral π-submanifold P of type I.

Proof Let N be a concave affine 3-manifold of type I in M. Then $F \cap M_h$ covers N for a hemispherical 3-crescent F. Let Γ_N denote the subgroup of Γ_h acting on $F \cap M_h$ as the deck transformation group of the covering map to N.

Let $\tilde{A}_1$ denote a compressible component of $I_F \cap M_h$. By Lemma 8, $\tilde{A}_1$ is not simply connected. By Corollary 4 for dimension $n = 2$,

$$\tilde{A} = I_F^o \cap M_h = I_F^o - \{x\}$$

holds. Thus,

$$N_h = F \cap M_h = F^o \cup I_F^o - \{x\}$$

is homeomorphic to $D^2 \times \mathbb{R}$, and N is covered by $D^2 \times \mathbb{S}^1$ by Lemma 1. By Dehn's lemma, we can obtain a compressing disk with boundary in ∂N. This shows that N is finitely covered by $D^2 \times \mathbb{S}^1$. Thus, N is a toral π-submanifold. □

We will prove Theorem 8 from here to the end of this Sect. 3.3, and hence the premises of Theorem 8 are in effect to the end of Sect. 3.3:

Theorem 8 *Let M be a connected compact real projective 3-manifold with empty or convex boundary. Suppose that M is neither complete affine nor bihedral. Let M_h be the holonomy cover of M. Suppose that $\check{M}_h$ has no hemispherical 3-crescent. Assume that M has no two-faced submanifold of type II.*

Let N be a concave affine 3-manifold of type II with boundary ∂N compressible into M. Then one of the following holds:

- *M is an affine Hopf 3-manifold, or*
- *N has a unique boundary component A compressible into N, and N contains a maximal toral π-submanifold P of type II. Furthermore, the following holds*:
  - *Let $N_h \subset M_h$ be a component of the inverse image of N. The inverse image of P in N_h meets the interior of any 3-crescent in the Kuiper completion $\check{N}_h$ of N_h. The fundamental group of N is virtually infinite-cyclic.*
  - *Let R be a 3-crescent in $\mathrm{Cl}(N_h)$ in $\check{M}_h$. Then R is a bihedral 3-crescent and $\mathbf{dev}_h|\Lambda_1(R)$ is a homeomorphism to $H - K$ for a properly convex compact domain K in a 3-hemisphere H with $K \cap \partial H \neq \emptyset$.*

A toral π-submanifold is *maximal* if no toral π-submanifold of type II contains it properly.

Let N_h denote a component of the inverse image of N in M_h as in the premise.

Suppose that we obtain a bihedral 3-crescent R in $\check{N}_h$ so that a deck transformation g acts on $R^o \cup I_R^o - \{x\} \subset N_h$ properly. We call such a bihedral 3-crescent a *toral* *bihedral* 3-*crescent* and the deck transformation acting on $R^o \cup I_R^o - \{x\}$ an *associated* deck transformation.

This proof is fairly long. We give the following outline:

(II) Concave affine 3-manifolds of type II.

- (A) There exist three mutually overlapping bihedral 3-crescents in $\Lambda(R)$ for a bihedral 3-crescent R in $\check{M}_h$.
  - (i) There is a pair of opposite bihedral 3-crescents in $\Lambda(R)$. By Lemma 10, M is covered by an affine Hopf 3-manifold finitely. (See Sect. 3.3.2.)
  - (ii) Otherwise, $\mathbf{dev}_h|\Lambda_1(R)$ is a homeomorphism to $H - K$ for a properly convex domain K and a 3-hemisphere H containing K, and $\Lambda(R)$ contains a toral bihedral 3-crescent. Lemma 17 gives us a toral π-submanifold. (See Sects. 3.3.3 and 3.3.4.)
- (B) Otherwise, all bihedral 3-crescents R have $\mathbf{dev}_h(I_R)$ containing a fixed pair of points q, q_-. Then $\Lambda(R)$ is a union of segments from q to q_-. (See Sect. 3.3.5.)
  - (i) A closed curve in a component A_1 of $\mathrm{bd}\Lambda(R) \cap M_h$ bounds a disk in the union $A_{1,+}$ of lines from q to q_- passing A_1. Here, the situation is similar to (A)(i), and we use Lemma 17. (See Sect. 3.3.6.)
  - (ii) Otherwise, $A_{1,+}$ is an annulus. We show that this case does not happen. (See Sect. 3.3.7.)

3.3.1 Case (II)

Let N be a concave affine 3-manifold of type II in M. We assume that there is no hemispherical 3-crescent in $\check{M}_h$. Then N is covered by $\Lambda(R) \cap M_h$ for a bihedral 3-crescent R in $\check{M}_h$. Let Γ_N denote the subgroup of Γ_h acting on $\Lambda(R) \cap M_h$ as the deck transformation group of the covering map to N. Recall that

$$\mathbf{dev}_h(\Lambda(R)) \subset H, \mathbf{dev}_h(\delta_\infty \Lambda(R)) \subset \partial H$$

for a 3-hemisphere $H \subset \mathbb{S}^3$. (See Corollary 5.8 of Choi 1999).

(II)(A) Suppose that there exist three mutually overlapping bihedral 3-crescents R_1, R_2, and R_3 with $\{I_{R_i} | i = 1, 2, 3\}$ in general position.

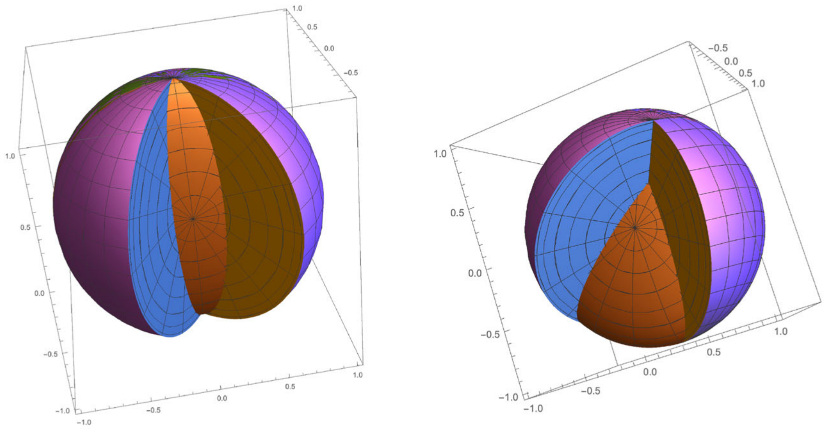
(II)(B) Suppose that there exist no such triple of bihedral 3-crescents.

See Fig. 4. We will defer (II)(B) to Sect. 3.3.5.

Now assume (II)(A). By modifying the proofs of Lemma 11.1 and Proposition 11.1 of Choi (2001) for bihedral 3-crescents, which are not necessarily radiant as in the paper, we obtain that

$$\begin{aligned} &\mathbf{dev}_h : \Lambda_1(R) \to H - K \text{ and} \\ &\mathbf{dev}_h|\Lambda_1(R) \cap M_h : \Lambda_1(R) \cap M_h \to H^o - K \end{aligned} \tag{3.1}$$

are homeomorphisms for a 3-hemisphere H and a nonempty compact properly convex set K. (See (2.1) for definition of $\Lambda_1(R)$. For Lemma 11.1 and Proposition 11.1 of



(a) Three 3-crescents R_i, $i = 1, 2, 3$, meeting transversely with I_{R_i} not in general position

(b) Three 3-crescents R_i, $i = 1, 2, 3$, meeting transversely, with I_{R_i} in general position.

Fig. 4 The image of transversal intersections of three 3-crescents under the orthogonal projection from the 3-hemisphere in $\mathbb{S}^3$ containing their images under **dev**

Choi (2001), we do not need I_R for each bihedral 3-crescent to contain the origin. There is a mistake in the third line of the proof of Lemma 11.1 of Choi (2001). We need to change $P_1 \cap L_1$ and $P_1 \cap L_2$ to $P_1 \cap L_2$ and $P_1 \cap L_3$ respectively). The general position property of I_{R_i}, $i = 1, 2, 3$, implies that K is properly convex. Also, (3.1) implies that $h_h|\Gamma_N$ is injective.

We collect some facts from the above paragraph:

- $\mathbf{dev}_h(\alpha_{R'}) \subset \partial H$ for $R' \sim R$.
- $\mathrm{bd}\Lambda_1(R) \cap M_h$ is mapped into $\mathrm{bd}K$ under $\mathbf{dev}_h$.
- $h_h(\Gamma_N)$ is an affine transformation group of H^o since it acts on an affine space H^o as a projective automorphism group.

We can have two possibilities:

(II)(A)(i): Suppose that there exist two opposite bihedral 3-crescents $S_1, S_2 \sim R$.
(II)(A)(ii): There are no such bihedral 3-crescents.

3.3.2 Case(II)(A)(i)

Here, we will show that M is an affine Hopf 3-manifold. The following finishes the proof of Theorem 8 for the case (A)(i).

Lemma 10 *Suppose that there exist two bihedral 3-crescents S_1, S_2 in $\check{M}_h$ so that $I_{S_1} \cap M_h$ and $I_{S_2} \cap M_h$ intersect and are tangent but $\mathbf{dev}_h(S_1)^o \cap \mathbf{dev}_h(S_2)^o = \varnothing$. Assume $S_1, S_2 \sim R$, and (II)(A)(i). Then there exists a unique component of $I_{S_i} \cap M_h$ equal to $I^o_{S_i} - \{x\}$ for a point x of $I^o_{S_i}$, $i = 1, 2$, and M is an affine Hopf 3-manifold.*

Proof First, $I_{S_1} \cap M_h$ and $I_{S_2} \cap M_h$ meet at the union of their common components since such a component is totally geodesic and complete in M_h and they are tangent.

Let K' denote the inverse image in $\mathrm{bd}\Lambda_1(R)$ of K. At least one component A_1 of $I^o_{S_1} \cap M_h$ contains $I^o_{S_1} - K'$ for a properly convex compact set K' by (3.1). Since A_1 is also in $I^o_{S_2} \cap M_h$, it follows that A_1 is a common component of $I^o_{S_1} \cap M_h$ and $I^o_{S_2} \cap M_h$. By the classification of the affine 2-manifolds (see Nagano and Yagi 1974 and Benoist 2000), the only possibility is

$$A_1 = \begin{cases} I^o_{S_1} \text{ or} \\ I^o_{S_1} - \{x\}, x \in I^o_{S_1}. \end{cases}$$

In the first case, we obtain that

$$\mathbf{dev}_h(\Lambda_1(R) \cap M_h) = H^o \text{ and}$$
$$\partial H = \mathrm{Cl}(\alpha_{S_1}) \cup \mathrm{Cl}(\alpha_{S_2}) \subset \check{M}_{h,\infty}.$$

Hence, M_h is projectively diffeomorphic to the complete affine space which contradicts a premise of Theorem 8, which we are proving here.

Now suppose that $A_1 = I^o_{S_1} - \{x\}$. Since

$$\mathrm{Cl}(\alpha_{S_1}) \cup \mathrm{Cl}(\alpha_{S_2}) \subset M_{h,\infty},$$

$S^o_1 \cup A_1 \cup S^o_2$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and $M_h = S^o_1 \cup A_1 \cup S^o_2$. Since S_1 and S_2 are mapped into the closures of two different components of $\mathbb{S}^3 - \mathbb{S}^2$ respectively,

$$\mathbf{dev}_h | S^o_1 \cup A_1 \cup S^o_2$$

is an embedding onto its image by geometry. Since $\mathbf{dev}_h(x)$ is an isolated boundary point, Corollary 4 implies the result that M is an affine Hopf manifold. □

3.3.3 Case (II)(A)(ii)

From now on, we assume that M is not an affine Hopf 3-manifold. In this case, K^o is a nonempty properly convex open domain, and $\mathrm{bd}\Lambda(R) \cap M_h$ is mapped into $\mathrm{bd}K$: otherwise, $\dim K \leq n - 1$ and K is a subset of a hyperspace V. Then the two components of $H^o - V$ lift to open 3-cells in $\Lambda(R)^o$ by (3.1). The closures of two cells in $\Lambda(R)$ are bihedral 3-crescents again by (3.1). The two crescents are opposite. Thus, we are in case (i), a contradiction.

By Lemma 11 and (3.1), we have $\mathrm{bd}\Lambda_1(R) \cap M_h = \mathrm{bd}\Lambda(R) \cap M_h$. For following Lemma 11, we do not need to assume (II)(A) and K needs not to be properly convex.

Lemma 11 *Let M be as in Theorem 8 with a concave affine 3-manifold N with compressible boundary in M. Assume that M is not an affine Hopf 3-manifold. Suppose that $\mathbf{dev}_h|\Lambda_1(R)$ is a homeomorphism to $H - K$ for a compact convex domain K. Then $\mathrm{bd}\Lambda_1(R) \cap M_h = \mathrm{bd}\Lambda(R) \cap M_h$, and the interior of K is an open domain in H^o.*

Proof Since for each crescent S, S^o is dense in S, we obtain

$$\mathrm{bd}\Lambda(R) \cap M_h \subset \mathrm{bd}\Lambda_1(R) \cap M_h.$$

Given a point $x \in \mathrm{bd}\Lambda_1(R)$, choose a convex open neighborhood $B(x) \subset M_h$ with $\mathbf{dev}_h|B(x)$ is an embedding. $B(x) \cap S^o$ for a crescent S, $S \sim R$, is the closure of a component of $B(x) - I_S \cap B(x)$ for a totally geodesic disk $I_S \cap B(x)$ with boundary in $B(x)$ since α_S is disjoint from $B(x)$. (See Choi 1999.) The set $B(x) - \Lambda_1(R)$ is a convex set K'' in $B(x)$. Since $\mathbf{dev}_h(\Lambda_1(R))$ is a homeomorphism to $H - K$ by a premise,

- $\mathbf{dev}_h(x) \in K$,
- $\mathbf{dev}_h|B(x) \cap \Lambda_1(R)$ is an embedding to $\mathbf{dev}_h(B(x)) - K$, and hence
- $\mathbf{dev}_h|B(x) - \Lambda_1(R)(= K'')$ is an embedding to $\mathbf{dev}_h(B(x)) \cap K$.

Suppose that K'' has the empty interior. So does $\mathbf{dev}_h(B(x)) \cap K$. Since $\mathbf{dev}_h(B(x))$ is a convex open ball,

$$\mathbf{dev}_h(K'') = \mathbf{dev}_h(B(x)) \cap K$$

has the empty interior. Thus, K has the empty interior since K^o is dense in K. Since $\mathbf{dev}_h(B(x)) \cap K \neq \emptyset$, there is a proper subspace P such that $K \subset P \cap H$ and $H^o - P$ is in a union of two bihedrons disjoint from K. The inverse image of these in $\Lambda_1(R)$ are also bihedrons by a premise. We take closures. Then we have an opposite pair of bihedral 3-crescents with the interiors of the images disjoint from P. This is a contradiction by Lemma 10.

Hence, K'' and K have nonempty interiors. The interior of K is disjoint from $\mathbf{dev}_h(T)$ for any crescent T, $T \sim R$ since otherwise

$$\mathbf{dev}_h(T^o) \cap K^o \neq \emptyset \text{ while } K^o \cap \mathbf{dev}_h(\Lambda_1(R)) = \emptyset.$$

Thus, $K''^o \cap \Lambda(R) = \emptyset$, and $x \in \mathrm{bd}\Lambda(R)$. □

Lemma 12 *Assume as in Lemma 11. Then K is an unbounded subset of an affine space H^o. Moreover, $K \cap \partial H$ is a nonempty compact convex set, and* $\mathrm{bd}K \cap H^o$ *is homeomorphic to a disk.*

Proof Suppose that K is a bounded subset of H^o. Then $\mathbf{dev}_h|\Lambda_1(R)$ is a homeomorphism to $H - K$ by (3.1). By Theorem 5, components of $\mathrm{bd}\Lambda(R) \cap M_h = \mathrm{bd}\Lambda_1 \cap M_h$ by Lemma 11 are not homeomorphic to the 2-sphere or the projective plane. $\mathrm{bd}\Lambda_1(R) \cap M_h$ is mapped into a surface in ∂K. Since the map cannot be onto the sphere ∂K, there exists a noncompact component A_1 of $\mathrm{bd}\Lambda(R) \cap M_h = \mathrm{bd}\Lambda_1 \cap M_h$ covering a closed surface B_1. By Lemma 9, this is a contradiction as $h_h|\Gamma_N$ is injective by (3.1). Hence, K is unbounded in H^o. Since $\mathrm{bd}K \cap H^o$ is the complement of a compact cell $K \cap \partial H$, the final statement follows (Fig. 5). □

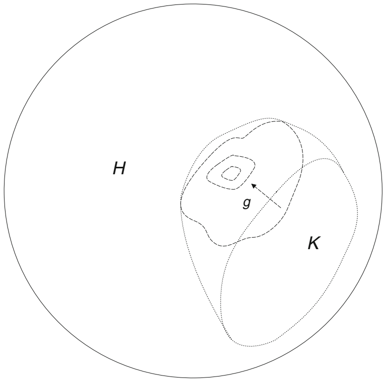


Fig. 5 The diagram for H, K and $c_1 \subset \mathrm{bd}K$ for the case (II)(A)(ii)

3.3.4 Case (II)(A)(ii) Continued: Obtaining a Toral Bihedral 3-Crescent

The next step is to show that $\mathbf{dev}_h|\Lambda(R) : \Lambda(R) \to H$ is injective.

If each component of $\mathrm{bd}\Lambda(R) \cap M_h$ is simply connected, then each component is a disk by Theorem 5. By Lemma 8, these components are π_1-injective. Hence, a component A_1 of $\mathrm{bd}\Lambda(R) \cap M_h$ is not simply connected. $\mathrm{bd}K \cap H^o$ is homeomorphic to a disk by Lemma 12. Since A_1 is not simply connected, there exists a simple closed curve $c_1 \subset A_1$ so that $\mathbf{dev}_h(c_1)$ bounds a disk D_1 in $\mathrm{bd}K \cap H^o$. By Lemma 11, the premises of Lemma 13 hold. By Lemmas 13 and 17, N contains a toral π-submanifold. This finishes the proof of Theorem 8 for the case (II)(A)(ii).

Lemma 13 *Assume as in Lemma 11. Let S be a properly embedded disk in H^o that is the boundary of a convex domain K as in Lemma 11. Suppose that*

$$\mathrm{bd}\Lambda(R) \cap M_h = \mathrm{bd}\Lambda_1(R) \cap M_h$$

is mapped into S. Let A_1 be a non-simply connected relatively open subset in $\mathrm{bd}\Lambda(R) \cap M_h$ *containing a curve c_1 so that $\mathbf{dev}_h(c_1)$ bounds an open disk D_1 in S. Then the following hold*:

- $\mathbf{dev}_h|\Lambda(R) : \Lambda(R) \to H$ *is injective.*
- $\mathrm{bd}\Lambda(R) \cap M_h$ *has a unique component.*
- $\Lambda(R)$ *contains a toral bihedral* 3*-crescent R_P.*
- *The fundamental group of N is virtually infinite cyclic.*

Proof Let Γ_1 be the subgroup of Γ_N acting on A_1 cocompactly. There exists an element $g \in \Gamma_1$ such that $h_h(g)(\mathbf{dev}_h(c_1)) \subset D_1 \cap \mathbf{dev}_h(A_1)$ by Lemma 7. Also, $h_h(g)(D_1) \subset D_1$ since the external component of $S - \mathbf{dev}_h(c_1)$ is not homeomorphic to a disk. By (3.1), $\Lambda_1(R)$ is homeomorphic to $H^o - K$, a cell. We find an open disk D'_1 in $\Lambda_1(R)$ that forms the interior of a compact disk D''_1 with boundary c_1 in $\mathrm{bd}\Lambda(R) \cap N_h$. Then a component of $\Lambda(R) \cap N_h - D_1$ is a bounded domain B_1 where B^o_1 is a cell. Since the group action is proper, and D''_1 in N_h is compact, we obtain $g \in \Gamma_1$ where $g(B_1 \cup D''_1) \subset B_1$ by Lemma 7. Thus, we can find a fixed point x in $\mathrm{Cl}(K)$ for g by the Brouwer fixed-point theorem. We can verify the premises of Proposition 8 in

Appendix 1 for $\mathbf{dev}_h(D_1'')$ by using a supporting hyperplane at x since K is convex and the boundary of $\mathbf{dev}_h(D_1'')$ is in K. Proposition 8 in Appendix 1 implies that x is the fixed point of the largest norm eigenvalue of $h_h(g)$ and the global attracting fixed point of $h_h(g)|H^o$.

Now, we prove the injectivity of $\mathbf{dev}_h|\Lambda(R)$: let x_j, $j = 1, 2$ be points of $\Lambda(R)$. Let $R_1, R_2 \sim R$ be two bihedral 3-crescents where $x_j \in R_j$, $j = 1, 2$. We may assume that $\mathbf{dev}_h(R_j)$ meets $\mathbf{dev}_h(\mathrm{bd}\Lambda_1(R)) - \partial H$ by taking the maximal bihedral 3-crescents. Then $h_h(g)^i(\mathbf{dev}_h(R_j))$ meets a neighborhood of x for sufficiently large i by (3.1). Since $\mathbf{dev}_h(g^i(I_{R_1}))$ and $\mathbf{dev}_h(g^i(I_{R_2}))$ are very close containing nearby points for sufficiently large i and supporting a properly convex domain K^o, we obtain that $\mathbf{dev}_h(g^i(R_1))$ and $\mathbf{dev}_h(g^i(R_2))$ meet in the interior. By (3.1), we obtain (3.1),

$$g^i(R_1)^o \cap g^i(R_2) \neq \varnothing$$

for sufficiently large i and hence

$$R_1^o \cap R_2^o \neq \varnothing.$$

By Theorem 5.4 and Proposition 3.9 of Choi (1999), $\mathbf{dev}_h|R_1 \cup R_2$ is injective. Therefore, $\mathbf{dev}_h|\Lambda(R)$ is injective. This proves the first item.

Since $\mathbf{dev}_h|\Lambda(R) \cap M_h$ is injective, the restriction of an immersion

$$\mathbf{dev}_h|\mathrm{Cl}(K) \cap \mathrm{bd}\Lambda(R) \cap M_h$$

is a homeomorphism to its image Y in $\mathrm{bd}K$. The set Y is an open surface. Then $Y/h_h(\Gamma_N)$ is a union of closed surfaces. Let Y_1 be the image of A_1. $Y_1/h_h(\Gamma_1)$ is a connected closed surface homeomorphic to A_1/Γ_1.

Since $\mathbf{dev}_h(x)$ is a unique attracting fixed point of $h_h(g)$ in H^o, $h_h(g)^i(c_1)$ goes into an arbitrary neighborhood of $\mathbf{dev}_h(x)$ in $\mathrm{bd}K$ for sufficiently large i. $h_h(g)^i(c_1)$ goes into an arbitrary tubular neighborhood of $\mathrm{bd}K \cap \partial H$ in $\mathrm{bd}K$ for sufficiently small negative number i. Using i and $-i$ for large integer i, $h_h(g)^i(c_1)$ and $h_h(g)^{-i}(c_1)$ bound a compact annulus in $\mathrm{bd}K \cap H^o$. If there is a component $\tilde{Y}_j$ of $Y \subset \mathrm{bd}K$ other than $\mathbf{dev}_h(A_1)$, then it lies in one of the annuli, a bounded subset of H^o, and $\tilde{Y}_j$ covers a compact surface Y_j for some j. By Lemma 9, this is a contradiction. Thus, $\mathrm{bd}\Lambda(R) \cap M_h$ has a unique component.

Since $h_h(g)^i(c_1)$ are disjoint from Q_K and $\langle h_h(g)\rangle$ acts on $Q_K := S - \mathbf{dev}_h(\tilde{A}_1)$, the set Q_K is either $\{\mathbf{dev}_h(x)\}$ or a closed set with infinitely many components. We obtain $Q_K = \{\mathbf{dev}_h(x)\}$ by Lemma 14 and

$$(\mathrm{bd}K - \{\mathbf{dev}_h(x)\}) \cap H^o = \mathbf{dev}_h(A_1). \tag{3.2}$$

Since Γ_N acts faithfully, properly discontinuously, and freely on an annulus A_1, Γ_N is virtually infinite-cyclic. The existence of g shows that the $h_h(\Gamma_N)$ fixes the unique point $\mathbf{dev}_h(x)$ corresponding to one of the ends. This proves the fourth item.

Let $K_x \subset \mathbb{S}^2_x$ denote the subspace of directions of the segments with endpoints in $\mathbf{dev}_h(x)$ and K^o. Obviously, K_x is a convex open domain in an open half-space of $\mathbb{S}^2_x$.

Our $h_h(g)$ acts on K_x, and $\mathbb{S}^2_x$ has a $h_h(g)$-invariant great circle $\mathbb{S}^1$ outside K_x as we can deduce from the existence of K_x.

We take a union of maximal segments in $\mathbf{dev}_h(\Lambda(R))$ from $\mathbf{dev}_h(x)$ in directions in $\mathbb{S}^1$. Their union is a 2-hemisphere P with boundary in ∂H, and $\mathbf{dev}_h(x) \in P$.

We find an open bihedron $B \subset H - K$ whose boundary contains an open 2-hemisphere in ∂H and P. By taking the inverse $(\mathbf{dev}_h|\Lambda_1(R))^{-1}(B)$ and the closure, we obtain a bihedral 3-crescent $R_P \subset \Lambda(R)$ with $x \in I_{R_P}$. By the first item, g acts on R_P, I_P, and x.

The last step is to show that R_P has the desired property. By our choice of K_x and P, we obtain

$$\mathbf{dev}_h(I_{R_P})^o - \mathbf{dev}_h(x) \subset H - K^o.$$

By (3.2) and the first item, we obtain

$$I^o_{R_P} - \{x\} \subset A_1 \cup \Lambda_1(R). \tag{3.3}$$

Hence, $I^o_{R_P} - \{x\} \subset N_h$ for our bihedral 3-crescent R_P above. There is an element $g \in \Gamma_N$ acting on $R^o_P \cup I^o_{R_P} - \{x\}$. □

Lemma 14 *Let S_0 be a properly embedded disk or cylinder in $\mathbb{H}^o$. Let $\tilde{A}_0 \subset S_0$ be a connected open set covering a closed surface A_0 with the deck transformation group G_1 also acting on S_0 for $G_1 \subset$* $\mathbf{Aut}(\mathbb{H}^o)$. *Suppose that there exists a collection of simple closed curves $c_i \in A_0$, $i \in \mathbb{Z}$, so that for any end neighborhood of S_0 there is a component of $S_0 - c_i$ in it. Then $S_0 - \tilde{A}_0$ cannot have infinitely many components.*

Proof Suppose not. Then $\tilde{A}_0$ is an open planar surface with infinitely many ends. Giving an arbitrary complex structure on A_0, the cover $\tilde{A}_0$ admits a Koebe general uniformization as $\mathbb{C}P^1 - \Lambda$ for a Cantor set Λ. (See Simha 1989.) That is $A_0 = \tilde{A}_0/G_1$ is homeomorphic to a closed Schottky Riemann surface $(\mathbb{C}P^1 - \Lambda)/\Gamma_1$ where Γ_1 is a group in $\mathsf{PSL}(2, \mathbb{C})$ isomorphic to G_1. (See p. 77 of Marden (2007) for the proof.) The set of the pairs of fixed points of elements of G_1 are dense in $\Lambda \times \Lambda - \Delta(\Lambda)$ for the diagonal $\Delta(\Lambda)$ of Λ. (See Kulkarni 1978 or Theorem 2.14 of Apanasov 2000). We can find a closed curve c in the surface $\tilde{A}_0/\Gamma_1$ so that c lifts to a curve $\tilde{c}$ ending at two points k_1 and k_2 in Λ fixed by an element of Γ_1.

On $\tilde{A}_0$, simple closed curves bound end neighborhoods. We may assume that k_1 corresponds to an end of $\tilde{A}_0$ whose end neighborhood is bounded by a simple closed curve d_1, and k_2 corresponds to an end of $\tilde{A}_0$ whose end neighborhood is bounded by a simple closed curve d_2. We can choose k_i for $i = 1, 2$ so that $\mathbf{dev}_h(d_i)$ bounds an open disk D_i whose closure is compact in S_0 since we can choose k_1 and k_2 arbitrarily. We can choose disjoint disks D_1 and D_2.

Choose an orientation-preserving element $g_c \in \Gamma_1$ acting on $\tilde{c}$. Then

$$h_h(g_c^n)(\mathbf{dev}_h(d_1)) \subset D_2 \quad \text{for some } n.$$

By orientation considerations of how $\tilde{c}$ meets d_1 and $g_c^n(d_1)$, we obtain

$$h_h(g_c^n)(D - D_1) \subset D_2.$$

Since g_c acts on ∂H, $D - D_1$ has limit points in ∂H, and D_2 has no limit points in ∂H, this is a contradiction. □

3.3.5 Case (B)

Again, we keep assuming that M is not an affine Hopf 3-manifold. Now suppose that $\Lambda(R)$ contains no triple of mutually overlapping bihedral 3-crescents S_i, $i = 1, 2, 3$, with $\mathbf{dev}_h(I_{S_i})$ in general position.

By induction on overlapping pairs of bihedral 3-crescents, we obtain that $\mathbf{dev}_h(I_S)$ for a bihedral 3-crescent S, $S \sim R$, share a common point $q \in \partial H$ and hence its antipode $q_- \in \partial H$. Then $\Lambda(R)$ is a union of segments whose developing images end at q, q_-. The interior of such segments in $\Lambda(R)$ is called a *complete q-line*. Also, *q-lines* are subarcs of complete q-lines. $\mathrm{bd}\Lambda(R) \cap M_h$ is foliated by subsets of q-lines.

Suppose that $\mathbf{dev}_h(I_S)$ for S, $S \in R$, always have the same boundary circle. Then $\Lambda(R)$ is a union of bihedral 3-crescents S so that $S \supset R$ or $S \subset R$ by Proposition 5. It is straightforward to show $\mathbf{dev}|\Lambda(S)$ is a diffeomorphism to a 3-bihedron, and hence $\Lambda(S)$ is a bihedral 3-crescent since we do not have hemispherical 3-crescent by the premise of Theorem 8.

The premises of Lemma 11 are satisfied. By Lemmas 11, 12, and 13, we obtain a toral bihedral 3-crescent. By Lemma 17, we obtain a toral π-submanifold from the bihedral 3-crescent T.

Now assume otherwise. Then q is determined by $\Lambda(R)$ uniquely, and hence the group of deck transformations acting on $\Lambda(R) \cap M_h$ acts on $\{q, q_-\}$. Let $\mathbb{S}^2_q$ denote the sphere of directions of complete affine lines from q, and let $\mathbb{S}^2_q$ have a standard Riemannian metric of curvature 1. The space of q-lines in $\Lambda_1(R) \cap M_h$ whose developing image go from q to its antipode q_- is an open surface S_R with an affine structure. We have a fibration

$$l \to \Lambda_1(R) \cap M_h \stackrel{\Pi_R}{\to} S_R \tag{3.4}$$

where fibers are q-lines. The developing map $\mathbf{dev}_h$ induces an immersion $\mathbf{dev}_{h,q}$: $S_R \to \mathbb{S}^2_q$. The surface S_R develops into a 2-hemisphere $H_q \subset \mathbb{S}^2_q$ whose interior H_q^o is identifiable with an affine 2-space. Denote by $\Pi_q : H^o \to H_q^o$ the projection.

We give $\mathbb{S}^2_q$ a Fubini-Study metric. The Kuiper completion $\check{S}_R$ of S_R has an ideal subset c' that is the image of $\mathrm{bd}\Lambda(R) \cap M_h$ and a geodesic boundary subset corresponding to $\delta_\infty \Lambda(R)$ and is mapped to ∂H_q. We denote the extension by the same symbol $\mathbf{dev}_{h,q} : \check{S}_R \to \mathbb{S}^2_q$.

The subspace $N_h := \Lambda(R) \cap M_h$ covers a concave affine manifold in M_h. If each component of $\partial N_h = \mathrm{bd}\Lambda(R) \cap M_h$ is simply connected, then it is incompressible by Theorem 5. Thus, there is a component A_1 of $\mathrm{bd}\Lambda(R) \cap M_h$ containing a simple closed curve c that is not null-homotopic in A_1.

We will use the same notation $\mathbf{dev}_h$ for the extension of $\mathbf{dev}_h|N_h$ to $\check{N}_h$. Let $\mathcal{L}_q$ denote the set of complete q-lines l such that

$$l \subset R''' \text{ for } R''' \sim R,\, R''' \subset \check{N}_h, \text{ and } l \cap A_1 \neq \varnothing.$$

We define

$$A_{1+} := \bigcup_{l \in \mathcal{L}_q} l.$$

We claim that A_{1+} is homeomorphic to the injective image of a topologically open surface: recalling the surface S_R above, we obtain a fibration $\Pi_R : \Lambda_1(R) \cap M_h \to S_R$ extending to $\check{N}_h \to \check{S}_R$, to be denoted by Π_R again. Π_R maps A_{1+} to a set a_{1+} in the ideal boundary of $\check{S}_R$ of S_R. Since

- q-complete lines pass the open surface A_1 foliated by q-arcs,
- $\mathbf{dev}_{h,q}|a_{1+}$ maps locally injectively to an embedded arc in H_q^o, and
- A_1 is a surface,

it follows that a_{1+} is a locally injective open arc.

Suppose that two leaves l_1 and l_2 of A_{1+} go to the same point of an open arc α in a_{1+} where $\mathbf{dev}_{h,q}|\alpha$ is an embedding. Since l_1 and l_2 are fibers, there is a point $\Pi_R(l)$ in S_R of $\mathbf{d}$-distance $< \epsilon$ from the images $\Pi_R(l_1)$, $\Pi_R(l_2)$ of these lines in $\check{S}_R$. Inside $\Lambda_1(R)$, there exist paths of $\mathbf{d}$-length $< \epsilon$ from l_1 and l_2 to any point of a common line l in $\Lambda_1(R)$ corresponding to $\Pi_R(l)$ by spherical geometry. Taking $\epsilon \to 0$ and l closer to l_i, we obtain $l_1 = l_2$. Hence, we showed that A_{1+} fibers over a_{1+} locally.

This implies that A_{1+} is the image of an open surface. We give a new topology on A_{1+} by giving a basis of A_{1+} as the set of components of the inverse images of open sets in $\check{M}_h$. Then A_{1+} is homeomorphic to a surface with this topology.

As above, A_1 contains a simple closed curve c not homotopic to a point in A_1.

$$c''' := \Pi_R(c) \subset a_{1+}$$

is either a compact arc, i.e., homeomorphic to an interval or a circle (Fig. 6).

We divide into two cases:

(i) c''' is homeomorphic to an interval.
(ii) c''' is homeomorphic to a circle.

3.3.6 Case (B)(i)

Then c bounds an open disk D' in the fibered space A_{1+}. Let Γ_1 be the subgroup of Γ_N acting on A_1. We can use a similar argument to (II)(A)(ii): first, there exists $g \in \Gamma_1$ so that $g(c)$ is in $D' \cap A_1$ by Lemma 7. Hence g fixes a point x in D'^o that is a fixed point on A_{1+} by the Brouwer fixed-point theorem. A_{1+} is either homeomorphic to an annulus or a disk since A_{1+} is foliated by q-lines. We have $g(D' \cup c) \subset D'$ since exactly one component of $A_{1+} - g(c)$ is homeomorphic to a disk.

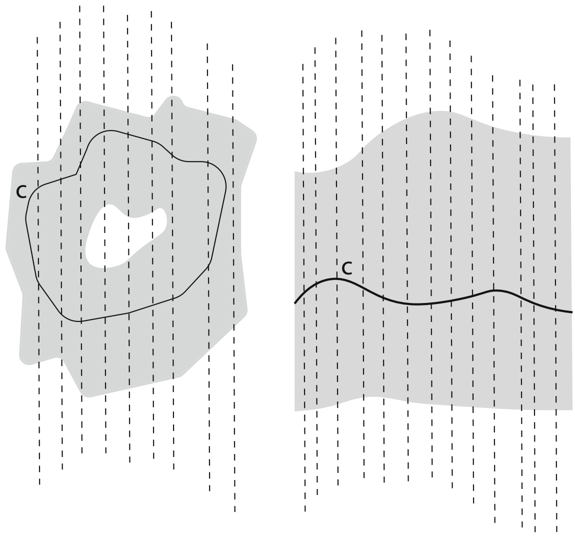


Fig. 6 The diagram for two cases (B)(i) and (B)(ii) where the gray area indicates A_1, and the dotted lines the fibers of A_{1+}. In (B)(ii) we drew c partially but it should be a closed curve and A_{1+} should be an annulus

Let x_q denote the complete q-line containing x in a_{1+}. Let $g_q : \check{S}_R \rightarrow \check{S}_R$ be the induced map of $g : \Lambda(R) \rightarrow \Lambda(R)$. Recall the affine plane H^o_q. Consider x_q as the origin. Since the induced affine transformation

$$h(g)_q | H^o_q : H^o_q \rightarrow H^o_q$$

is not trivial,

- $h(g)_q$ has an isolated fixed point or
- has a line l of fixed points in H^o_q.

We consider the first case. Since $g(c)$ is in the open disk in A_{1+} bounded by c, a compact arc neighborhood of x_q in a_{1+} goes into itself strictly under g_q. It must be that $\mathbf{dev}_{h,q}(x_q)$ is the attracting fixed point under $h(g)_q$. Thus, the local arc $\mathbf{dev}_{h,q}(a_{1+})$ is the union $\bigcup_{i \geq 0} h(g)_q^{-i}(I)$ for a small embedded open arc I in α_{1+} containing $\mathbf{dev}_{h,q}(x_q)$. Since I is embedded, $\mathbf{dev}_{h,q}(a_{1+})$ is also an embedded arc. By the classification of the infinite cyclic linear automorphism groups of H^o_q, we can show that $\mathbf{dev}_{h,q}(a_{1+})$ is a properly embedded convex arc in H^o_q.

In the second case, $h(g)_q$ acts on lines parallel to l, or acts on a parallel set of lines transversal to the line l as can be deduced from elementary linear algebra. The action on $\mathbf{dev}_{h,q}(a_{1+})$ of $h(g)_q$, its fixed point x_q in H^o_q is locally isolated, or there is a geodesic subarc of fixed points forming a neighborhood or a one-sided neighborhood of x_q in the local arc $\mathbf{dev}_{h,q}(a_{1+})$. A similar argument to the above paragraph will show $\mathbf{dev}_{h,q}(a_{1+})$ is a properly embedded convex arc in H^o_q. We obtain the same fact where we replace I with an ϵ-$\mathbf{d}$-neighborhood of in the above paragraph.

Consider the commutative diagram

$$\begin{array}{ccc} A_{1+} & \xrightarrow{\Pi_R} & a_{1+} \\ \downarrow \mathbf{dev}_h & & \mathbf{dev}_{h,q} \downarrow \\ H^o & \xrightarrow{\Pi_q} & H^o_q. \end{array}$$

Since the left arrows of the above commutative diagrams are fibrations,

$$\mathbf{dev}_h | A_{1+} : A_{1+} \to H^o$$

is a proper embedding to H^o. Since $\mathbf{dev}_{h,q}(a_{1+})$ is a properly embedded convex arc, A_{1+} is a properly embedded surface bounding a convex domain K in H^o.

We claim that $\mathbf{dev}_h : \Lambda_1(R) \to H$ is an embedding: $\mathbf{dev}_h | R^o \cup \alpha_R$ is an embedding. We can choose a crescent S so that

- S overlaps with R and $\mathbf{dev}_h(I_S)^o$,
- S contains a generically chosen $y \in A_1$, and
- the closure of the arc $\mathbf{dev}_{h,q}(\alpha_S)$ does not contain the endpoint of a_{1+}.

Then for any crescent T overlapping with S, $\mathbf{dev}|S^o \cup \alpha_S \cup T^o \cup \alpha_T$ is an embedding by Proposition 5. For any crescent T_1 overlapping with T, since $\mathbf{dev}_h(\alpha_{T_1})$ is not antipodal to $\mathbf{dev}_h(\alpha_S)$ by our choice,

$$\mathbf{dev}_h | T^o \cup \alpha_T \cup T_1^o \cup \alpha_{T_1}$$

is injective, T_1 overlaps with S also. This implies that

$$\mathbf{dev}_h | S^o \cup \alpha_S \cup T^o \cup \alpha_T \cup T_1^o \cup \alpha_{T_1}$$

is injective. By induction, we obtain that $\mathbf{dev}_h | \Lambda_1(R)$ is an embedding into H.

We obtain $\mathrm{bd}\Lambda_1(R) \cap M_h = \mathrm{bd}\Lambda(R) \cap M_h$ by Lemma 11. By Lemmas 13 and 17, we obtain a toral π-submanifold from the bihedral 3-crescent T. This completes the proof of Theorem 8 for case (B)(i).

3.3.7 Case (B)(ii)

This case does not occur; we show that $\Lambda(R)$ is not maximal here:

The open surface A_{1+} is homeomorphic to an annulus foliated by complete affine lines. Here, c is an essential simple closed curve. There exists an element $g \in \Gamma_1$ sending c into a component U_1 of $A_1 - c$ by Lemma 7. Replacing U_1 by $g^i(U_1)$ if necessary, we may assume that $g(U_1 \cup c) \subset U_1$. Then g is of infinite order.

We will use the many results of Sect. 3.3.5 in Sect. 3.3.7. We outline this Sect. 3.3.7 since it is complicated.

(a) First, we will show that $\check{S}_R$ is essentially $\Lambda_1(S)$ for a bihedral 2-crescent S, and $\mathbf{dev}|\Lambda_1(R) \cap M_h$ is finite-to-one. (See Lemma 15.)

(b) We will now determine the affine form of g to be a "translation with a transverse orthogonal part".
(c) We show $A_{1+} = A_1$. Then we show that there is a $\langle g\rangle$-invariant neighborhood in M_h using the property of g from (b). We show that $\Lambda(R)$ is not maximal showing us a contradiction.

(a) We can embed c''' into $\check{S}_R$. Recall a fibration (3.4):

$$l \to \Lambda_1(R) \cap M_h \overset{\Pi_R}{\to} S_R \tag{3.5}$$

where fibers are q-lines. Π_R extends to $\Lambda_1(R) \to \check{S}_R$.

Step (a) is the following:

Lemma 15 *We have $S_R = \Lambda(S) \cap S_R$ for a bihedral 2-crescent S in $\check{S}_R$. Also, $\check{S}_R$ is homeomorphic to a compact annulus with a boundary component c''' and a closed curve in an arc $\delta_\infty \Lambda(S)$. Furthermore, $\mathbf{dev}_h|\Lambda_1(R) \cap M_h$ is finite-to-one to its image.*

Proof The map Π_R sends the interior of a bihedral 3-crescent R' to an open lune B' developing into a 2-hemisphere $H_q \subset \mathbb{S}^2_q$ as we discussed in Sect. 3.3.5 after (3.4). Let S' be the closure of B' in S_R, which is a bigon with the union of two segments as its boundary. (See Choi 1994.) Since S' develops into H_q and one edge goes to ∂H_q, it follows that an edge of S' is in the ideal boundary of the Kuiper completion. Hence, S' is a bihedral 2-crescent.

Let us choose one which we denote by S. We define $\Lambda(S)$ in $\check{S}_R$ as usual in Chapter 7 of Choi (1999) for the dimension two. Since $\Pi_R(\Lambda_1(R) \cap M_h) = S_R$, every point of S_R is in a bihedral 2-crescent equivalent to S, we obtain the first equality.

The next step is showing that $\check{S}_R$ is compact:

We take for each point z of c''' a 2-dimensional crescent S_z so that $z \in I_{S_z}$, which exists since we can go back to $\Lambda(R)$ and corresponding points of c. Then $\bigcup_{z\in c'''} S_z \cap S_R$ is a closed subset of $\Lambda(S) \cap S_R$ since we can use a sequence argument and c''' is compact. Using the fact that $\mathbf{dev}_{h,q}|c'''$ is a closed convex curve, we perturb the crescent S_z by choosing different $z, z \in c'''$. Consider developing images of crescents of form $S_{z'}$, $z' \in c'''$ and that of open disks in S_R meeting $\Lambda(S)$. We can show that $\Lambda(S) \cap S_R$ is also open in S_R Hence, we have

$$\bigcup_{z\in c'''} S_z \cap S_R = \Lambda(S) \cap S_R \text{ and hence } \bigcup_{z\in c'''} S_z = \Lambda(S) = \check{S}_R.$$

Recall that $\mathbf{dev}_{h,q}$ sends $\check{S}_R$ into a 2-hemisphere H_q. Since c is a compact arc, $\mathbf{dev}_{h,q}|c'''$ is a map to a compact arc in H^o_q. For each S_z, we choose a segment s_z in S_z connecting z to a point of α_{S_z}. The complement $\check{S}_R - \bigcup_{z\in c'''} s_z$ is a disjoint union of properly convex triangles each of which has a vertex in c''' and two edges from the vertex to points of the closure of $\delta_\infty \Lambda(S_R)$ and an edge in $\delta_\infty \Lambda(S_R)$.

We cover each triangle by maximal segments from the vertex in c'''. This corresponds to blowing up c''' for the vertices of the triangles so that the segments are now disjoint. The blown-up c''' is still homeomorphic to a compact circle since the set of directions of the segments from c''' form a compact set. We obtain a surface S' foliated by segments mapped to segments or segments for the triangles. Since c'' is compact, S' is a compact surface. Hence, the image $\check{S}_R = \Lambda(S)$ of S' is a compact surface.

The final step is as follows: since $\check{S}_R = \Lambda(S)$, it follows that $\check{S}_R$ is a compact surface with two boundary components c''' and another simple closed curve in $\delta_\infty \Lambda(S)$. For each point $t \in \check{S}_R$, there is a neighborhood where $\mathbf{dev}_{h,q} : \check{S}_R \to H$ restricts to a homeomorphism to an open disk with possibly an embedded arc as the boundary. Since $\check{S}_R$ is compact with a finite covering by such charts, $\mathbf{dev}_{h,q}$ is finite-to-one.

Hence $\mathbf{dev}_{h,q}|S_R$ is a finite-to-one map to its image in H^o_q by above. Therefore, $\mathbf{dev}_h : \Lambda_1(R) \cap M_h \to H^o$ is finite-to-one to its image. □

(b) Now, $\langle h_h(g)\rangle$ acts on a nontrivial closed curve $\mathbf{dev}_{h,q}(c''')$ bounded in an affine space H^o_q of $\mathbb{S}^2_q$. Thus, $\langle h_h(g)\rangle$ acts as an isometry group on $\mathbb{S}^2_q$ with respect to a standard metric corresponding to a choice of coordinates on $\mathbb{S}^2_q$. Let $L(g)$ denote the linear part of $h_h(g)$ considered as an affine transformation of the affine space H^o. We obtain an affine transformation form

$$h_h(g)(x) = L(g)x + v(g)$$

where $v(g)$ is a 3-vector. We identify it in the matrix form with

$$h_h(g) = \begin{pmatrix} L(g) & v(g) \\ 0 & 1 \end{pmatrix}.$$

Let $v_q \in \mathbb{R}^3$ denote the vector in the direction of q in the boundary of $\mathbb{R}^3$. By the classification of elements of $\mathsf{SL}_\pm(4, \mathbb{R})$, the following hold:

- $h_h(g)$ induces an orthogonal linear map on $H^o_q := \mathbb{R}^3/\langle v_q\rangle$ up to a choice of coordinates of H^o_q since $h_h(g)$ acts as an isometry on $\mathbb{S}^2_q$ preserving H^o_q.
- There is a fixed point of $h_h(g)$ in H^o_q. We replace the coordinate system if necessary so that the fixed point is the origin.
- Since $h_h(g)$ acts on a complete affine line in $\mathbb{R}^3$ passing the origin and in the direction of v_q, the linear map $L(g)$ has v_q as an eigenvector corresponding to a positive eigenvalue by taking a power of g if necessary,
- Hence, the affine form of g is obtained by post-composing $L(g)$ with a translation in the direction of v_q. Thus, $v(g)$ is parallel to v_q.

Suppose that $h_h(g)$ is unipotent with eigenvalues all equal to 1. By the second property above, $h_h(g)$ acts as the identity on H^o_q, and g is a translation on each q-line in A_{1+}. Since $\langle g\rangle$ acts on A_{1+}, and as the identity on H^o_q, $h_h(g)$ is of form

$$\begin{pmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ for } \alpha, \beta, \gamma \in \mathbb{R}. \tag{3.6}$$

If α, β are not all zero, then we can find a plane P_g of fixed points given by $\alpha y + \beta z + \gamma = 0$ in $\mathbb{R}^3$. The inverse image P'_g of P_g in $\Lambda_1(R) \cap M_h$ is not empty. Also,

$\mathbf{dev}_h : P'_g \to P_g$ is finite-to-one onto its image by Lemma 15. Then g acts P'_g with a finite order since $h_h(g)|P_g$ is the identity map. This contradicts our choice of g in the beginning of (B)(ii). Thus,

$$\alpha = 0, \beta = 0. \tag{3.7}$$

Finally, $\gamma \neq 0$ since $g \neq \mathrm{I}$. Hence, g is a translation in the direction of v_g as we desired.

Otherwise, the only possibility is that $h_h(g)$ acts on a one-dimensional subspace parallel to v_q and a complementary subspace, and $h_h(g)$ has the form

$$\begin{pmatrix} \lambda & 0 & \gamma \\ 0 & \mu O_g & 0 \\ 0 & 0 & \mu \end{pmatrix}, \lambda\mu^3 = 1, \lambda, \mu > 0$$

for an orthogonal 2×2-matrix O_g.

Recall that $A_{1+} \cap M_h$ has a component A_1 containing c. Suppose that $\mu \neq \lambda$. Then $g^i(c)$ geometrically converges to a compact closed curve in the interior of A_{1+} as $i \to \infty$ or $i \to -\infty$. The limit of $\mathbf{dev}_h(g^i(c))$ must be on a totally geodesic subspace P by the classification of elements of $\mathsf{SL}_{\pm}(4, \mathbb{R})$ passing $\mathbf{dev}(A_{1+})$. Recall (3.4). By Lemma 15, the annulus $\check{S}_R$ has c''' as a boundary component.

- S_R contains an annulus $A_{1,R}$ with boundary c'''.
- The inverse image P' of P under $\mathbf{dev}_h$ contains an annulus $A'_{1,R}$ embedding to $A_{1,R}$ under Π_R.
- Then $\mathbf{dev}_h|A'_{1,R}$ is a finite-to-one map by Step (a). We may assume that g acts on $A_{1,R}$, and hence g acts on $A'_{1,R}$.

Since g^i is represented as a sequence of uniformly bounded matrices on $A'_{1,R}$ for every $i \in \mathbb{Z}$, and g is of infinite order, this is a contradiction to the properness of the action of $\langle g \rangle$. Therefore, we obtain

$$\mu = \lambda = 1. \tag{3.8}$$

Since $h_h(g)(q) = q$, and h_h acts on H^o, it follows that $h_h(g)$ restricts to an affine transformation in H^o acting on the set of a parallel collection of lines. $h_h(g)$ acts as a translation composed with a rotation on H^o with respect to a Euclidean metric since the 3×3-matrix of $L(g)$ decomposes into an orthogonal 2×2-submatrix and the third diagonal element equal to 1.

Thus, in all cases as indicated by Eqs. (3.7) or (3.8), g is of form

$$\begin{pmatrix} 1 & 0 & \gamma \\ 0 & O_g & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for a trivial or nontrivial orthogonal 2×2-matrix O_g under a coordinate system. Since $g(c) \subset U_1$ from the beginning of Sect. 3.3.7, g does not act on a parallel collection of circles in A_{1+}. Hence, $\gamma \neq 0$. Thus, g is either a nontrivial translation in the direction of $v(g)$ or of above form with nonzero γ, which is the desired conclusion for the step (b).

(c) Let L be an annulus bounded by c and $g(c)$ in A_{1+}. Since g acts on A_1, we have $g^i(c) \subset A_1$ for every $i \in \mathbb{Z}$. Now $g^i(c)$ and $g^{-i}(c)$ bound an annulus A_i. For any bounded subset K of A_{1+}, the set K is contained in A_i for sufficiently large i since g acts as above. Since there is no bounded component of $L \cap M_h$ by Lemma 9, A_1 is the unique component of M_h. By Lemma 14, $A_{1+} - A_1$ has finitely many components. By the action of $\langle g \rangle$, the planar surface A_1 has only one or two ends or infinitely many ends. Hence, $A_{1+} = A_1$. Thus, we conclude $A_{1+} \subset M_h$.

There exists an open neighborhood N of L in M_h, and $\bigcup_{i \in \mathbb{Z}} g^i(N) \subset M_h$ covers A_{1+}. By restricting a Euclidean metric H^o, we obtain a Euclidean metric on an open set in M_h containing $\Lambda(R) \cap M_h$ and $\bigcup_{i \in \mathbb{Z}} g^i(L)$. We obtain a closed set

$$\Lambda' \subset \bigcup_{i \in \mathbb{Z}} g^i(N) \cup \Lambda(R) \cap M_h,$$

that is foliated by complete q-lines and

$$\Lambda(R) \cap M_h \subset \Lambda'$$

properly. Also, Λ' contains an ϵ-neighborhood of $\Lambda(R) \cap M_h$ in the Euclidean metric.

The subspace Λ' fibers over the surface Σ of complete q-lines in Λ' as before in the beginning of (B). Then the Kuiper completion $\check{\Sigma}$ has an affine structure. We extend the above fibration

$$\Pi_R : \Lambda_1(R) \cap M_h \rightarrow S_R \text{ to } \Pi_R : \Lambda' \rightarrow \Sigma$$

to be denoted by Π_R again.

Recall from Lemma 15 that $S_R \subset \Sigma$ denote the image of $\Lambda_1(R) \cap M_h$ under Π_R. Let $\check{\Sigma}$ denote the Kuiper completion of Σ. Each point of $\alpha_{R'}$ for $R' \sim R$ maps to an ideal point of Σ since we can consider path in $\Lambda_1(R)$ converging to this point which maps to finite length path in Σ under Π_R and cannot end at a point of Σ under Π_R. Hence, Π_R extends to $\Lambda_1(R) \rightarrow \mathrm{Cl}(S_R) \subset \check{\Sigma}$. Also, for each crescent R', $R' \sim R$, $\Pi_R(R'^o)$ has the closure that is a bihedral 2-crescent which we showed in the proof of Lemma 15. We denote this closure by $S(R')$. The image of $\Lambda_1(R)$ under π_R is the union of $S(R') - I_{S(R')}$ for a bihedral 2-crescent $S(R')$ in $\check{\Sigma}$. We denote this by $\Lambda_1(S(R))$.

Since c''' is closed and not geodesic, we can take a short geodesic k in Σ connecting the endpoints of the short subarc α_1 in c''' so that they bound a disk in Σ. k can be extended in $\Lambda_1(S(R))$ until it ends in the ideal set of $\check{\Sigma}$ corresponding to complete q-lines in $\delta_\infty \Lambda(R)$. We choose a 2-dimensional crescent S'' in $\check{\Sigma}$ bounded by k containing a 2-dimensional crescent S_2 in $\check{S}_R$, and containing α_1. (See the maximum property in Section 6.2 of Choi 1994). The inverse image $\Pi_R^{-1}(S_2^o) \subset \Lambda'$ has the closure S' in $\check{M}_h$. Since $\Lambda_1(R) \cap M_h$ has the ideal boundary in the open hemisphere in the boundary of S' considering the developing image in ∂H, we may verify that S' is a bihedral 3-crescent. This contradicts the maximality of $\Lambda(R)$, which is a contradiction to how we defined $\Lambda(R)$ in (2.1). □

3.4 The Irreducibility of Concave Affine Manifolds

Theorem 9 *Let N be a concave affine 3-manifold in a connected compact real projective 3-manifold M with convex boundary. Suppose that M is neither complete affine nor bihedral. Assume that M has no two-faced submanifold. Then N is irreducible, or M is an affine Hopf 3-manifold and $M = N$. Consequently, N is a prime 3-manifold.*

Proof Let N be a concave affine 3-manifold of type I, and let N_h denote a component of the inverse image of N in M_h. Then we have $N_h = R^o \cup I_R^o$ where R is a hemispherical 3-crescent. Hence, N_h is irreducible.

We assume that $\check{M}_h$ has no hemispherical 3-crescent now. Let N be a concave affine 3-manifold of type II. We follow the proof of Theorems 7 and 8 in Sect. 3.3. We divide into cases (A) and (B).

Let R be a bihedral 3-crescent so that $N_h \subset \Lambda(R)$.

(A) Suppose that there are three mutually overlapping bihedral 3-crescents R_1, R_2, and R_3 with $\{I_{R_i} | i = 1, 2, 3\}$ in general position.

We can have two possibilities:

(i) Suppose that there exists a pair of opposite bihedral 3-crescents $S_1, S_2 \sim R$. (See Sect. 3.3.2.)
(ii) There is no such pair of bihedral 3-crescents. (See Sect. 3.3.3.)

By Lemma 10, (i) implies that M is an affine Hopf manifold. In this case, $\check{M}$ equals a closed hemisphere and equals $\Lambda(R)$ for a crescent R. Thus, $M = N$ for a concave affine manifold N. By Proposition 3, N is a prime 3-manifold.

We now work with (ii). By Theorem 5, $\mathrm{bd}\Lambda(R) \cap M_h$ has no sphere boundary. Any 2-sphere in $\Lambda(R) \cap M_h$ can be isotopied into the dense open submanifold $\Lambda_1(R) \cap M_h$. (See (2.1)). Recall K from (3.1) where K is the complement of the image $\mathbf{dev}(\Lambda_1(R))$ in an 3-hemisphere H. We showed using Lemma 9 in the beginning of (A)(ii) in the proof of Theorem 8 that K cannot be bounded in H^o (see Sect. 3.3.3). We obtain $K \cap \partial H \neq \emptyset$. Since $H - K$ deformation retracts to $\partial H - K$ by projection from a point of K^o, $H - K$ is contractible. Thus, $\Lambda_1(R) \cap M_h$ is contractible, and every immersed sphere is null-homotopic.

Now we go to the case (B) in the proof of Theorem 8 where $\Lambda(R)$ is a union of the segments whose developing image end commonly at the antipodal pair q, q_- (see Sect. 3.3.5). Since the interior of $\Lambda(R) \cap N_h$ fibers over an open surface with fiber homeomorphic to real lines, N is irreducible. □

3.5 Toral π-Submanifolds

Lemma 16 *A toral π-submanifold N of type I is homeomorphic to a solid torus or a solid Klein-bottle and is a concave affine 3-manifold of type I.*

Proof By Theorem 5, there is no boundary component of N homeomorphic to a sphere or a real projective plane.

By Definition 5, N is covered by $R^o \cup I_R^o - \{x\}$ for a hemispherical 3-crescent R and $x \in I_R^o$, and N hence is a concave affine 3-manifold of type I.

Since the deck transformation group acts on the annulus $I_R^o - \{x\}$ properly discontinuously and freely, the deck transformation group of $\tilde{N}$ is isomorphic to a virtually infinite-cyclic group by Lemma 3. By Lemma 2, we are done. □

Lemma 17 *Let M be a connected compact real projective 3-manifold with convex boundary. Suppose that $\check{M}_h$ has no hemispherical 3-crescent. Let N be a concave affine 3-manifold of type II in M covered by $\Lambda(R) \cap N_h$ for a bihedral 3-crescent R. We suppose that*

- *The Kuiper completion $\check{N}_h$ of some cover N_h of the holonomy cover of N contains a toral bihedral 3-crescent S where a deck transformation g acts on $S^o \cup I_S^o - \{x\} \subset N_h$, fixing a point $x \in I_S^o$ as an attracting fixed point.*
- *The deck transformation group of N is virtually infinite-cyclic.*
- $\mathbf{dev}_h|\Lambda(R)\cap N_h : \Lambda(R)\cap N_h \to H^o - K^o$ *is an embedding to its image containing $H^o - K$ for a compact convex domain K in the 3-hemisphere H with nonempty interior in H^o.*

Then N contains a unique maximal toral π-submanifold of type II, homeomorphic to a solid torus or a solid Klein-bottle, and the interior of every bihedral 3-crescent in $\check{N}_h$ meets the inverse image of the toral π-submanifold in N_h.

Proof By Theorem 5, there is no boundary component of N homeomorphic to a sphere or a real projective plane. By definition, $N_h = \Lambda(S) \cap M_h$ for a bihedral 3-crescent S. We obtain a toral bihedral 3-crescent R in $\check{N}_h$.

(I) The first step is to understand the intersections of two toral bihedral crescents: by assumption, Γ_N is virtually infinite-cyclic. Two bihedral 3-crescents R_1 and R_2 are not opposite since $K^o \neq \emptyset$ holds. Let R_1 and R_2 be two toral bihedral 3-crescents such that $R_1, R_2 \sim R$. Let R_i' denote $R_i^o \cup I_{R_i}^o - \{x_i\}$ for a fixed point x_i of the action of an infinite order generating deck transformation g_i acting on R_i so that $R_i'/\langle g_i \rangle$ is homeomorphic to a solid torus. Here, g_i is the deck transformation associated to R_i. Let F_i, $i = 1, 2$, denote the compact fundamental domain of R_i'. Then the set

$$G_i := \{g \in \Gamma_N | g(F_i) \cap F_i \neq \emptyset\}, i = 1, 2,$$

is finite. We can take a finite index normal subgroup Γ' of the virtually infinite-cyclic group Γ_N so that $\Gamma' \cap G_i := \{e\}$ for both i. Then a cover of the compact submanifold $R_i'/\langle g_i \rangle \cap \Gamma'$ is embedded in N_h/Γ'. Thus, there is some cover N_1 of N so that these lift to embedded submanifolds.

We denote these in N_1 by T_1 and T_2. We may assume that $T_i = R_i'/\langle g_i' \rangle$. x_i is the *fixed point* of R_i and g_i' acts on R_i'.

Suppose that they overlap. Then $R_1 \cap R_2$ is a component of $R_1 - I_{R_2}$ by Theorem 5.4 of Choi (1999). Considering $T_1 \cap T_2$ that must be a solid torus not homotopic to a point in each T_i, we obtain that a nonzero power of g_1', and a nonzero power of g_2' are equal. Therefore, $x_1 = x_2$, and g_i' fixes the point $x_1 = x_2$.

(II) The second step is to build invariant subsets from the union of overlapping toral bihedral 3-crescents and build a toral π-submanifold from it:

We say that two toral bihedral 3-crescents R_1 and R_2 are *equivalent* if they overlap in a cover of a solid torus in N_h. This relation generates an equivalence relation of toral bihedral 3-crescents. We write $R_1 \cong R_2$.

Let S be a toral bihedral 3-crescent in $\check{N}_h$. By this condition, $x = x_i$ for every fixed point x_i of a toral bihedral 3-crescent R_i, $R_i \cong S$ where x is fixed by g'_i associated with R_i. We define

$$\hat{\Lambda}(S) := \bigcup_{R' \cong S} R', \quad \delta_\infty \hat{\Lambda}(S) := \bigcup_{R' \cong S} \alpha_{R'}.$$

We claim that $\hat{\Lambda}(S) \cap N_h$ covers a compact submanifold in N: let T be any bihedral 3-crescent in $\check{N}_h$ where g acts with x as an attracting fixed point. Then $T - \mathrm{Cl}(\alpha_T) - \{x\} \subset N_h$ as in cases (II)(A)(ii) or (II)(B)(i). (See Sects. 3.3.3, 3.3.5).

Since there are no two-faced submanifolds, we see that either

$$\hat{\Lambda}(S) = g(\hat{\Lambda}(S)) \text{ or } \hat{\Lambda}(S) \cap g(\hat{\Lambda}(S)) \cap N_h = \emptyset \text{ for } g \in \Gamma_N.$$

(This follows as in Lemma 7.2 of Choi 1999). We can also show that the collection

$$\{g(\hat{\Lambda}(S) \cap N_h) | g \in \Gamma_N\}$$

is locally finite in M_h as we did for $\Lambda(S) \cap M_h$ in Chapter 9 of Choi (1999). Hence, the image of $\hat{\Lambda}(S) \cap N_h$ is closed in M_h, and it covers a compact submanifold in M.

Since $\hat{\Lambda}(S)$ is a union of segments from x to

$$\delta_\infty \hat{\Lambda}(S) := \delta_\infty \Lambda(R) \cap \hat{\Lambda}(S),$$

$\mathrm{bd}\hat{\Lambda}(S) \cap N_h$ is on a union L of such segments from x to $\mathrm{Cl}(\delta_\infty \hat{\Lambda}(S))$ passing the set. The open line segments are all in N_h as they are in toral bihedral 3-crescents. Since $\hat{\Lambda}(S)$ is canonically defined, the virtually infinite-cyclic group Γ_N acts on the set. Also, $\hat{\Lambda}(S) \cap N_h$ is connected since we can apply the above paragraph to 3-crescents in $\hat{\Lambda}(S)$ also.

The interior of $\hat{\Lambda}(S) \cap M_h$ is a union of open segments from x to an open surface $\delta_\infty \hat{\Lambda}(S)$. The surface cannot be homeomorphic to a sphere or a real projective plane since a toral π-submanifold has nonempty boundary. Since $\delta_\infty \hat{\Lambda}(S)$ is the complement of ∂H of a compact convex set, it is thus homeomorphic to a 2-cell. Therefore, the interior of $\hat{\Lambda}(S) \cap M_h$ is homeomorphic to a 3-cell. We showed that $\hat{\Lambda}(S) \cap M_h$ covers a compact submanifold in N. We call this T_N.

Suppose that $T_N \subset T'_N$ for any other toral π-submanifold T'_N. We showed that in the above part of the proof that every toral bihedral 3-crescent in a universal cover of T'_N overlapping with ones in $\hat{\Lambda}(S)$ must be in $\hat{\Lambda}(S)$. By considering chains of overlapping bihedral 3-crescents, we obtain $T'_N \subset N$ and $T'_N = T_N$. This shows that T_N is a maximal toral π-submanifold.

Since N has a virtually infinite-cyclic holonomy group by a premise, and $\mathbf{dev}_h|N_h$ is injective, we obtain that the holonomy group image of the deck transformation group acting on $\hat{\Lambda}(S) \cap M_h$ is virtually infinite-cyclic. Since the holonomy homomorphism is injective, the deck transformation group acting on $\hat{\Lambda}(S) \cap M_h$ is virtually infinite-cyclic. By Lemma 2, the toral π-submanifold is homeomorphic to a solid torus or a solid Klein bottle.

(III) Now, we go to the final part: we assumed that $\mathbf{dev}_h|\Lambda(R) \cap N_h$ is an injective map into the complement of a convex domain K^o in H^o. Thus

$$\mathbf{dev}_h(\hat{\Lambda}(S) \cap N_h) = H - K'$$

for a domain K' in H where $K' \supset K^o$. The closure of K' is convex and is a union of segments from $\mathbf{dev}_h(x)$ to a convex domain in ∂H.

Given any bihedral 3-crescent R_1 in $\Lambda(S)$, suppose that the open 3-bihedron $\mathbf{dev}_h(R_1^o)$ does not meet $\mathbf{dev}_h(\hat{\Lambda}(S))$. Then $\mathbf{dev}_h(\alpha_{R_1})$ and $\mathbf{dev}_h(\alpha_T)$ for a toral bihedral 3-crescent T, $T \cong S$, have to be 2-hemispheres in ∂H antipodal to each other. Let g_T denote the deck transformation acting on $T^o \cup I_T^o - \{x\}$ for an attracting fixed point x of g_T. Then

$$g_T^i(R_1) \subset g_T^j(R_1) \text{ for } i < j$$

by Proposition 3.9 of Choi (1999) since their images overlap, the image of the latter set contains the former one, and $\mathbf{dev}_h|N_h$ is injective. Hence, the closure of $\bigcup_{i\in\mathbb{N}} g_T^i(R_1)$ is another toral bihedral 3-crescent since g_T acts on it. Then T and R are opposite. This is a contradiction since K then has to have the empty interior. We assumed otherwise in the premise. □

3.6 Proof of Theorem 2

Proof Let M be a connected compact real projective 3-manifold with empty or convex boundary. If M has a non-π_1-injective component of the two-faced totally geodesic submanifold of type I, then M is an affine Hopf manifold by Theorem 6.

Now suppose that M is not an affine Hopf manifold. We split along the two-faced totally geodesic submanifolds of type I now to obtain M^s. Theorem 7 implies the result.

Now assume that there is no hemispheric 3-crescent. To complete, we repeat the above argument for the two-faced totally geodesic submanifold of type II, and Theorem 8 implies the result. □

4 Toral π-Submanifolds and the Decomposition

We now prove a simpler version of Theorem 3.

Theorem 10 *Let M be a connected compact real projective 3-manifold with empty or convex boundary. Suppose that M is neither complete affine nor bihedral, and M is* *not an affine Hopf 3-manifold. Suppose that M has no two-faced submanifold of type I, and M has no concave affine 3-manifold of type I with incompressible boundary. Then the following hold:*

- *each concave affine submanifold of type I in M with compressible boundary contains a unique toral π-submanifold T of type I.*
  - *There are finitely many disjoint toral π-submanifolds*

$$T_1, \dots, T_n$$

    obtained by taking one from each of the concave affine submanifolds of type I in M with compressible boundary.
- *We remove $\bigcup_{i=1}^{n} \mathrm{int} T_i$ from M. Call M' the resulting real projective manifold with convex boundary. Suppose that M' has no two-faced submanifold of type II, and M' has no concave affine 3-manifold of type II with incompressible boundary.*
  - *Each concave affine submanifold of type II in M' with compressible boundary contains a unique toral π-submanifold T of type II where T has compressible boundary.*
  - *There are finitely many disjoint toral π-submanifolds*

$$T_{n+1}, \dots, T_{m+n}$$

    obtained by taking one from each of the concave affine submanifolds in M'.
- *$M - \bigcup_{i=1}^{n+m} \mathrm{int} T_i$ is 2-convex.*

Proof If N is a concave affine 3-manifold of type I with compressible boundary, then its universal cover is in a hemispherical 3-crescent, and N is homeomorphic to a solid torus and is a toral π-submanifold by Lemma 16. These concave affine 3-manifolds are mutually disjoint.

We remove these and denote the result by M'. Then $M - \bigcup_{i=1}^{n} \mathrm{int} T_i$ has totally geodesic boundary. The cover M'_h of M' is given by removing the inverse images of $T_1, \dots, T_n$ from M_h. We take a Kuiper completion $\check{M}'_h$ of M'_h. Now, we consider when N is a concave affine 3-manifold arising from bihedral 3-crescents in $\check{M}'_h$. We obtain toral π-submanifold II in N by Lemma 17.

From M', we remove the union of the interiors of toral π-submanifolds $T_n, \dots, T_{n+m}$. Then $M - \bigcup_{i=1}^{n+m} \mathrm{int} T_i$ has a convex boundary as P_i has concave boundary.

We claim that this manifold $M - \bigcup_{i=1}^{n+m} \mathrm{int} T_i$ is 2-convex. Suppose not. Then by Theorem 1.1 of Choi (1999), we obtain again a 3-crescent R' in the Kuiper completion of $M_h - p_h^{-1}(\bigcup_{i=1}^{n+m} \mathrm{int} T_i)$. The 3-crescent R' has the interior disjoint from ones we already considered. However, Theorem 8 shows that R'^o must meet the inverse image $p_h^{-1}(\bigcup_{i=1}^{n} \mathrm{int} T_i)$, which is a contradiction.

Lemma 17 shows that each T_i is homeomorphic to a solid torus or a solid Klein bottle. □

Proof of Theorem 3 We may assume that M is not complete or bihedral since then M is convex and the conclusions are true. As stated, $\check{M}_h$ does not contain any hemispherical 3-crescent. By Theorem 10, M either is an affine Hopf 3-manifold, or M^s decomposes into concave affine 3-manifolds with incompressible boundary, toral π-submanifolds of type I, and $M^{(1)}$.

Now $M^{(1)s}$ decomposes into concave affine manifolds of type II with compressible or incompressible boundary. Theorem 0.1 of Choi (2000) shows that a 2-convex affine 3-manifold is irreducible. Toral π-submanifolds and concave affine 3-manifolds of type II with incompressible boundary are irreducible or prime by Lemma 17 and Theorem 9. □

Proposition 6 *Let M be a connected compact real projective manifold with convex boundary. Suppose that M is not an affine Hopf manifold. Then the following hold*:

- *A toral π-submanifolds of type I in M^s is disjoint from the inverse images in M^s of the two-faced submanifolds in M of type I. Hence, it embeds into M.*
- *A toral π-submanifolds of type II is disjoint from the inverse images in $M^{(1)s}$ of the two-faced submanifolds in $M^{(1)}$ of type I.*
- *The image of a toral π-submanifold of type II in M^s is also disjoint from the two-faced submanifolds in M of type I. Hence, it embeds into M.*

Proof Suppose that M^s contains a toral π-submanifold N of type I. Then N^o embeds into M, and N^o is disjoint from the two-faced submanifold F of type I in M by the definition of concave affine manifolds of type I. Suppose that the unique boundary component ∂N of N meets the submanifold F' in M^s mapped to F. Then since F' is totally geodesic and ∂N is concave, it follows that $\partial N \subset F'$. Now, F is non-π_1-injective since ∂N is compressible in N. Theorem 6 shows that M is an affine Hopf 3-manifold. Hence ∂N is disjoint from F', and N embeds into M.

Suppose that $M^{(1)s}$ contains a toral π-submanifold N of type II. Then N^o embeds into $M^{(1)}$. Suppose that ∂N meets the submanifold F_2' in $M^{(1)s}$ mapped to the two-faced submanifold F_2 of type II in $M^{(1)}$. As above, $\partial N \subset F_2'$ for the inverse image F_2' in $M^{(1)s}$ of F_2, and ∂N covers a component F_3 of F_2. Since ∂N is compressible in N, Theorem 6 shows that M is an affine Hopf 3-manifold. Hence, $F_2' \cap \partial N = \varnothing$, and N embeds into $M^{(1)}$. Call the image by the same name.

Again N^o is disjoint from F'. As above N is disjoint from F' or $\partial N \subset F'$. In the second case, Theorem 6 shows that M is an affine Hopf 3-manifold. Thus, N embeds into M. □

Proof of Corollary 1 First, if a 3-hemisphere or a 3-bihedron covers M, then M is irreducible. So, we assume that this is not the case from now on in this proof.

Assume that M is not an affine Hopf 3-manifold. By Proposition 6, if there exists a toral π-submanifold in $M^{(1)s}$ or in M^s, then there is a projectively embedded image in M. Thus, the premise implies that there is no toral π-submanifold in M^s and $M^{(1)s}$.

Hence, $M^{(1)s}$ decomposes into concave affine 3-manifolds of type II with incompressible boundary and 2-convex affine 3-manifolds. Since these are irreducible and each boundary component is not homeomorphic to a sphere by Theorem 5, it follows that $M^{(1)s}$ is irreducible. Any embedded sphere S in $M^{(1)}$ meets F_2 in a disjoint union of circles after perturbations. Since two-faced submanifold F_2 is π_1-injective by Theorem 6, any disk component of $S - F_2$ can be isotopied away since such a disk lifts to one in $M^{(1)s}$ with boundary in the incompressible surface F_2' which are the inverse images of F_2 under the splitting process. (See Hempel 2004). By induction, we may assume that S is in $M^{(1)} - F_2$. Hence, it bounds a 3-ball. Thus, we obtained that $M^{(1)}$ is irreducible as well.

Now, M^s is a union of $M^{(1)}$ and a concave affine manifold of type I with incompressible boundary. Similar argument shows that M^s and M are irreducible and M. □

Proof of Corollary 2 Suppose that M has an embedded sphere S. The connected open set Ω contains a lift S' of S. If S is nonseparating, then Corollary 4 shows that M is an affine Hopf manifold.

Suppose that S is separating. Then S bounds a 3-ball B in M by Theorem 1.1 of Wu (2012). □

Appendix A: Contraction Maps

Here, we will discuss contraction maps in $\mathbb{R}^n$. A *contracting map* $f : X \to X$ for a metric space X with metric d is a map so that $d(f(x), f(y)) < d(x, y)$ for $x, y \in X$.

Lemma 18 *A linear map L has the property that all the norms of the eigenvalues are < 1. if and only if L is a contracting map for the distance induced by a norm.*

Proof See Corollary 1.2.3 of Katok and Hasselblatt (1995). □

Proposition 7 *$\langle g\rangle$ acts on $\mathbb{R}^n - \{O\}$ (resp. $U - \{O\}$ for the upper half space $U \subset \mathbb{R}^n$) properly if and only if the all the norms of the eigenvalues of g are > 1 or < 1.*

Proof Suppose that $\langle g\rangle$ acts on $\mathbb{R}^n - \{O\}$ properly. Fix a standard norm on $\mathbb{R}^n$. For a unit sphere S with center 0, the image $g^i(S)$ is inside a unit ball B for some integer i by the properness of the action. This implies that $g^i(B) \subset B$, and the norms of the eigenvalues of g^i are < 1 by Lemma 18. Hence the conclusion follows for g. The case of the half space U is similar.

For the converse, by replacing g with g^{-1} if necessary, we assume that all norms of eigenvalues < 1. Lemma 18 shows that $g(B) \subset B$ for a unit ball corresponding to a norm. This implies the result. □

Given two subsets A, B in an affine subspace $\mathbb{R}^n$, we denote by $A * B$ the union of all segments with end points in A and B respectively.

Proposition 8 *Let D be a connected open set in $\mathbb{S}^n$ in an affine patch $\mathbb{R}^n$. Let g be a projective automorphism acting on D and an affine patch $\mathbb{R}^n$. We assume the following:*

- *S is a compact connected smoothly embedded submanifold of codimension-one of D so that $D - S$ has two components D_1 and D_2 where D_1 is bounded in an affine patch $\mathbb{R}^n$ in $\mathbb{S}^n$.*

- *g acts with a fixed point $x \in \mathbb{R}^n$ in the closure of D_1.*
- $g(\mathrm{Cl}(D_1)) \subset D_1$.
- *Every complete affine line containing x meets S at at least one point.*
- *$D_1 \subset \{x\} * S$ where $\{x\} * S$ is the union of all segments from x ending at S.*

Then x is the global attracting fixed point of g in $\mathbb{R}^n$.

Proof Choose the coordinate system on $\mathbb{R}^n$ so that x is the origin. Let $L(g)$ denote the linear part of the g in this coordinate system. Suppose that there is a norm of the eigenvalue of $L(g)$ greater than or equal to 1. Then there is a subspace V of dimension 1 or 2 so that $V \otimes \mathbb{C}$ is an eigenspace in $\mathbb{C}^n$ associated with an eigenvalue of norm ≥ 1. We obtain $S_V := V \cap S \neq \emptyset$ by the above paragraph. Let $\Theta(S_V)$ denote the set of directions of S_V from x. $L(g)$ acts on the space of directions from x. Since $\{x\} * g(S) \subset \{x\} * S$, we obtain $L(g)(\Theta(S_V)) \subset \Theta(S_V)$. Hence, $\Theta(S_V)$ is either the set of a point, the set of a pair of antipodal points, or a subset of a circle where every point or its antipode are in it. Now, V has a Riemannian metric where g acts as a rotation times a scalar map. There is a point t of S_V where a maximal radius of S_V takes place under this metric. Then $g(t) \in g(S_V)$ must meet $D_2 \cup S_V$, a contradiction.

Thus, the norms of eigenvalues of $L(g)$ are < 1. By Lemma 18, $L(g)$ has a fixed point x as an attracting fixed point. □

Now, we prove without a g-invariant affine subspace.

Proposition 9 *Let D be a connected open set in $\mathbb{S}^n$ in an affine subspace $\mathbb{R}^n$. Let g be a projective automorphism of $\mathbb{S}^n$ acting on D. We assume the following:*

- *S is a compact connected smoothly embedded $(n-1)$-sphere of D so that $D - S$ has two components D_1 and D_2 where D_1 is bounded in an affine path $\mathbb{R}^n$ in $\mathbb{S}^n$.*
- $g(\mathrm{Cl}(D_1)) \subset D_1$.

Then g acts on an open affine subspace $\mathbb{R}^n$ containing D_1, and g has the global attracting fixed point x in $\mathbb{R}^n$, and

Proof By the Schoenflies theorem, a component of $\mathbb{S}^3 - S$ is a 3-cell D_1' bounded in $\mathbb{R}^n$. So, D_1' is in a cell in $\mathbb{R}^n$. Then $g(D_1') \subset D_1'$ since $g(S) \subset D_1'$ and the external component of $\mathbb{S}^2 - g(S)$ is not contained in a properly convex domain. By the Brouwer fixed-point theorem, g fixes a point in the interior of D_1'.

The convex hull C' of D_1 is still in an affine patch and is a properly convex domain. An easy argument shows that $g(C')$ is a compact subset of the interior of C' since every pair of points of $g(D_1)$ has a pair of convex open neighborhoods in the interior of D_1 useful for taking segments.

Then $C'' := \bigcup_{i=1}^{\infty} g^{-n}(C')$ is a convex open subset of $\mathbb{S}^n$, and hence is in an open hemisphere by Proposition 2.3 of Choi (1999) since C'' cannot be a great sphere. We claim that C'' is an open n-hemisphere. Suppose that C'' is not an open n-hemisphere. C'' has a family of open i-hemispheres foliating C'' for $1 \leq i < n$ or is properly convex by Proposition 2.4 of Choi (1999). The space of i-hemispheres forms a properly convex open domain K of dimension $n-i < n$ as shown in Section 1.4 of Chae et al. (1993). Let Π_K denote the projection $C'' \to K$. When C'' is properly convex, we let Π_K be the identity map. Again g acts on K with a fixed point x' in the interior of $\Pi_K(C')$

so that $g(\Pi_K(C'))$ into the interior of $\Pi_K(C')$. By the existence of Hilbert metric Kobayashi (1984) for a properly convex domain, if g fixes an interior point, then $g(\Pi_K(C'))$ cannot go into the interior of $\Pi_K(C')$ by the existence of the points of maximal distance from x at $\Pi_K(C')$. Hence, C'' is an affine patch where g acts on.

Since S is a separating sphere, every complete affine ray starting from x meets S at at least one point. Since every point of D_1 is on a complete affine ray starting from x, $D_1 \subset \{x\} * S$ where $\{x\} * S$ is the union of all segments from x ending at S. Since g acts on an affine patch C'', and $D_1 \subset C''$, Proposition 8 in Appendix 1 implies that x is an attracting fixed point of g on C''. □

Appendix B: The Boundary of a Concave Affine Manifolds is Not Strictly Concave

The following is the easy generalization of the maximum property in Section 6.2 of Choi (1994). Let N be an affine manifold with boundary. Hence, each boundary point has a chart going to an affine space where the boundary subset of the open set where the chart is defined maps to a submanifold of codimension-one. A *strictly concave boundary point* of an affine manifold N is a boundary point y where a totally geodesic open disk D contains y, $y \in D^o$, and $D - \{y\} \subset N^o$.

Theorem 11 *Let N be a concave affine 3-manifold of type II in a compact real projective manifold M with convex boundary. Then ∂N has no strictly concave point.*

Proof Let M_h be a cover as in the main part of the paper. Let N_h be a component of the inverse image of N in M_h.

Suppose that the conclusion does not hold. Then there is a boundary point y of N_h with a disk D as above. Then if y is a boundary point of M_h, then D must be in ∂M_h since ∂M_h is convex: we can use a chart of a point of ∂M_h to a convex subset of $\mathbb{S}^n$ mapping into an affine patch and deduce by looking these as graphs of convex functions. This contradicts the premise since $D - \{y\} \subset N^o$.

Now, N is covered by $\Lambda(R) \cap M_h$ for bihedral 3-crescent R in $\check{M}_h$. Since y is not a boundary point of M_h, we take a convex compact neighborhood $B(y)$ of the convex point y so that $\mathbf{dev}_h(B(y))$ is an ϵ-$\mathbf{d}$-ball for some $\epsilon > 0$. Then $B(y) - \Lambda(R)$ is a properly convex domain with the image $\mathbf{dev}_h(B(y) - \Lambda(R))$ is properly convex. For each point $z \in \mathrm{bd}\Lambda(R) \cap B(y)$, let S_z, $S_z \sim R$, be a bihedral 3-crescent containing z. Since $\Lambda(R)$ is maximal, $\mathbf{dev}_h(I_{S_z})$ is a supporting plane at $\mathbf{dev}_h(z)$ of $\mathbf{dev}_h(B(y) - \Lambda(R))$.

We perturb a small convex disk $D \subset I_{S_y}$ containing y away from y, so that the perturbed convex disk D' is such that the closure of $D' \cap B(y) - \Lambda(R)$ is a small compact disk D'' with

$$\partial D'' \subset \mathrm{bd}\Lambda(R) \cap M_h \text{ and } D''^o \cap \Lambda(R) = \varnothing.$$

Moreover, $\partial D''$ bounds a compact disk B' in $\mathrm{bd}\Lambda(R) \cap B(y)$. Choose a point z_0 in the interior of D''. For each point $z \in B'$, $I^o_{S_z}$ is transversal to $\overline{z_0 z}$ since $z_0 \notin S_z$. Since S^o_z is further away from z_0 than z, we can choose a maximal segment $s_z \subset S_z$ starting from

z_0 passing z ending at a point $\delta_+ s_z$ of α_{S_z}. We obtain a compact 3-ball $B_{z_0} = \bigcup_{z \in B'} s_z$ with its boundary in $\delta_\infty \Lambda(R)$. The boundary is the union of $D_{z_0} := \bigcup_{z \in \partial D''} s_z$, a compact disk, and an open disk

$$\alpha_{z_0} := \bigcup_{z \in B^o_{z_0}} \delta_+ s_z \subset \delta_\infty \Lambda(R).$$

The injectivity of $\mathbf{dev}_h | B_{z_0}$ is clear since we are using maps $\mathbf{dev}_h | s_z$, $z \in B'$ which are radiant from $\mathbf{dev}_h(z_0)$. Hence, B_{z_0} is a bihedral 3-crescent.

Since B_{z_0} is a union of segments from z_0 passing points of B' containing a segment passing y and transversal to $I^o_{S_y}$, it overlaps with S_y. Since B_{z_0} is a bihedral 3-crescent $\sim S_y$, $S_y \sim R$, we obtain $B_{z_0} \subset \Lambda(R)$. This contradicts our choice of y and D''. □

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