

RANDOMIZED SCHEDULING ALGORITHM FOR QUEUEING NETWORKS

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There has recently been considerable interests in design of low-complexity, myopic, distributed and stable scheduling policies for constrained queueing network models that arise in the context of emerging communication networks. Here, we consider two representative models. One, a model for the collection of wireless nodes communicating through a shared medium, that represents randomly varying number of packets in the queues at the nodes of networks. Two, a buffered circuit switched network model for an optical core of future Internet, to capture the randomness in calls or flows present in the network. The maximum weight scheduling policy proposed by Tassiulas and Ephremide [32] leads to a myopic and stable policy for the packet-level wireless network model. But computationally it is very expensive (NP-hard) and centralized. It is not applicable to the buffered circuit switched network due to the requirement of non-preemption of the calls in the service. As the main contribution of this paper, we present a stable scheduling algorithm for both of these models. The algorithm is myopic, distributed and performs few logical operations at each node per unit time.

1. Introduction. The primary task of a communication network architect is to provision as well as utilize network resources efficiently to satisfy the demands imposed on it. The main algorithmic problem is that of allocating or scheduling resources among various entities or data units, e.g. packets, flows, that are contending to access them. In recent years, the question of designing a simple, myopic, distributed and high-performance (aka stable) scheduling algorithm has received considerable interest in the context of emerging communication network models. Two such models that we consider this paper are that of a wireless network and a buffered circuit switched network.

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The wireless network consists of wireless transmission capable nodes. Each node receives exogenous demand in form of packets. These nodes communicate these packets through a shared wireless medium. Hence their simultaneous transmission may contend with each other. The purpose of a scheduling algorithm is to resolve these contentions among transmitting nodes so as to utilize the wireless network bandwidth efficiently while keeping the queues at nodes finite. Naturally the desired scheduling algorithm should be distributed, simple/low-complexity and myopic (i.e. utilize only the network state information like queue-sizes).

The buffered circuit switched network can be utilized to model the dynamics of flows or calls in an optical core of future Internet. Here a link capacitated network is given with a collection of end-to-end routes. At the ingress (i.e. input or entry point) of each route, calls arriving as per exogenous process are buffered or queued. Each such call desires resources on each link of its route for a random amount of time duration. Due to link capacity constraints, calls of routes sharing links contend for resources. And, a scheduling algorithm is required to resolve this contention so as to utilize the network links efficiently while keeping buffers or queues at ingress of routes finite. Again, the scheduling algorithm is desired to be distributed, simple and myopic.

An important scheduling algorithm is the maximum weight policy that was proposed by Tassiulas and Ephremides [32]. It was proposed in the context of a packet queueing network model with generic scheduling constraints. It is primarily applicable in a scenario where scheduling decisions are synchronized or made every discrete time. It suggests scheduling queues, subject to constraints, that have the maximum net weight at each time step with the weight of a queue being its queue-size. They established throughput optimality property of this algorithm for this general class of networks. Further, this algorithm, as the description suggests, is myopic. Due to the general applicability and myopic nature, this algorithm and its variants have received a lot of attention in recent years, e.g. [20, 5, 30, 27, 4, 28].

The maximum weight algorithm provides a myopic and stable scheduling algorithm for the wireless network model. However, it requires solving a combinatorial optimization problem, the maximum weight independent set problem, to come up with a schedule every time. And the problem of finding a maximum weight independent set is known to be NP-hard as well as hard to approximate in general [33]. To address this concern, there has been a long line of research conducted to devise implementable approximations of the maximum weight scheduling algorithm, e.g. [19, 31, 12, 7, 23]. A comprehensive survey of such maximum weight inspired and other algo-

rithmic approaches that have been studied over more than four decades in the context of wireless networks can be found in [25, 15].

In the context of buffered circuit switched network, calls have random service requirement. Therefore, scheduling decisions can not be synchronized. Therefore, the maximum weight scheduling algorithm is not applicable. To the best of our knowledge, no other myopic and stable algorithm is known for this network model.

1.1. Contributions. We propose a scheduling algorithm for both wireless and buffered circuit switched network model. The algorithm utilizes only local, queue-size information to make scheduling decisions. That is, the algorithm is myopic and distributed. It requires each queue (or node) in the network to perform few (literally, constant) logical operations per scheduling decision. We establish that it is throughput optimal. That is, the network Markov process is positive Harris recurrent as long as the network is underloaded (or not overloaded).

Philosophically, our algorithm design is motivated by a certain product-form distribution that can be characterized as the stationary distribution of a simple and distributed Markovian dynamics over the space of schedules. For the wireless network, it corresponds to the known Glauber dynamics (cf. [18]) over the space of independent sets of the wireless network interference graph; for the buffered circuit switched network, it corresponds to the known stochastic loss network (cf. [16]).

To establish the stability property of the algorithm, we exhibit an appropriate Lyapunov function. This, along with standard machinery based on identifying an appropriate ‘petit set’, leads to the positive Harris recurrence property of the network Markov process. Technically, this is the most challenging part of our result. It requires proving an effective ‘time scale separation’ between the network queuing dynamics and the scheduling dynamics induced by the algorithm. To make this possible, we use an appropriately slowly increasing function ($\log \log(\cdot + e)$) of queue-size as weight in the scheduling algorithm. Subsequently, the time scale separation follows by studying the mixing property of a specific time varying Markov chain over the space of schedules.

We note that use of Lyapunov function for establishing stability is somewhat classical now (for example, see [32, 30, 27]). Usually difficulty lies in finding an appropriate candidate function followed by establishing that it is indeed a “Lyapunov” function.

1.2. Organization. We start by describing two network models, the wireless network and the buffered circuit switched network in Section 2. We

formally introduce the problem of scheduling and performance metric for scheduling algorithms. The maximum weight scheduling algorithm is described as well. Our randomized algorithm and its throughput optimality for both network models are presented in Section 3. The paper beyond Section 3 is dedicated to establishing the throughput optimality. Necessary technical preliminaries are presented in Section 4. Here we relate our algorithm for both models with appropriate reversible Markov chains on the space of schedules and state useful properties of these Markov chains. We also describe known facts about the positive Harris recurrence as well as state the known Lyapunov drift criteria, to establish positive Harris recurrence. Detailed proofs of our main results are presented in Section 5.

2. Setup.

2.1. Wireless Network. We consider a *single-hop* wireless network of n queues. Queues receive work as per exogenous arrivals and work leaves the system upon receiving service. Specifically, let $Q_i(t) \in \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ denote the amount of work in the i th queue at time $t \in \mathbb{R}_+$ and $\mathbf{Q}(t) = [Q_i(t)]_{1 \leq i \leq n}$; initially $t = 0$ and $\mathbf{Q}(0) = \mathbf{0}^1$. Work arrives to each queue in terms of unit-sized packets as per a discrete-time process. Let $A_i(s, t)$ denote the amount of work arriving to queue i in time interval $[s, t]$ for $0 \leq s < t$. For simplicity, assume that for each i , $A_i(\cdot)$ is an independent Bernoulli process with parameter λ_i , where $A_i(\tau) \triangleq A_i(0, \tau)$. That is, $A_i(\tau+1) - A_i(\tau) \in \{0, 1\}$ and $\Pr(A_i(\tau+1) - A_i(\tau) = 1) = \lambda_i$ for all i and $\tau \in \mathbb{Z}_+ = \{k \in \mathbb{Z} : k \geq 0\}$. Denote the arrival rate vector as $\boldsymbol{\lambda} = [\lambda_i]_{1 \leq i \leq n}$. We assume that arrivals happen at the end of a time slot.

The work from queues is served at the unit rate, but subject to *interference* constraints. Specifically, let $G = (V, E)$ denote the inference graph between the n queues, represented by vertices $V = \{1, \dots, n\}$ and edges E : an $(i, j) \in E$ implies that queues i and j can not transmit simultaneously since their transmission *interfere* with each other. Formally, let $\sigma_i(t) \in \{0, 1\}$ denotes whether the queue i is transmitting at time t , i.e. work in queue i is being served at unit rate at time t and $\boldsymbol{\sigma}(t) = [\sigma_i(t)]$. Then, it must be that for $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) \in \mathcal{I}(G) \triangleq \{\boldsymbol{\rho} = [\rho_i] \in \{0, 1\}^n : \rho_i + \rho_j \leq 1 \text{ for all } (i, j) \in E\}.$$

The total amount of work served at queue i in time interval $[s, t]$ is

$$D_i(s, t) = \int_s^t \sigma_i(y) \mathbf{I}_{\{Q_i(y) > 0\}} dy,$$

¹Bold letters are reserved for vectors; $\mathbf{0}, \mathbf{1}$ represent vectors of all 0s & all 1s respectively.

where $\mathbf{I}_{\{x\}}$ denotes the indicator function.

In summary, the above model induces the following queueing dynamics: for any $0 \leq s < t$ and $1 \leq i \leq n$,

$$Q_i(t) = Q_i(s) - \int_s^t \sigma_i(y) \mathbf{I}_{\{Q_i(y) > 0\}} dy + A_i(s, t).$$

2.2. Buffered Circuit Switched Network. We consider a buffered circuit switched network. Here the network is represented by a capacitated graph $G = (V, E)$ with V being vertices, $E \subset V \times V$ being links (or edges) with each link $e \in E$ having a finite integral capacity $C_e \in \mathbb{N}$. This network is accessed by a fixed set of n routes R_1, \dots, R_n ; each route is a collection of interconnected links. At each route R_i , flows arrive as per an exogenous arrival process. For simplicity, we assume it to be an independent Poisson process of rate λ_i and let $A_i(s, t)$ denote total number of flow arrivals to route R_i in time interval $[s, t]$. Upon arrival of a flow to route R_i , it joins the queue or buffer at the ingress of R_i . Let $Q_i(t)$ denote the number of flows in this queue at time t ; initially $t = 0$ and $Q_i(0) = 0$.

Each flow arriving to R_i , comes with the service requirement of using unit capacity simultaneously on all the links of R_i for a time duration – it is assumed to be distributed independently as per Exponential of unit mean. Now a flow in the queue of route R_i can get simultaneous possession of links along route R_i in the network at time t , if there is a unit capacity available at *all* of these links. To this end, let $z_i(t)$ denote the number of flows that are *active* along route R_i , i.e. posses links along the route R_i . Then, by capacity constraints on the links of the network, it must be that $z(t) = [z_i(t)]$ satisfies

$$z(t) \in \mathcal{X} \triangleq \{z = [z_i] \in \mathbb{Z}_+^n : \sum_{i:e \in R_i} z_i \leq C_e, \quad \forall e \in E\}.$$

This represents the scheduling constraints of the circuit switched network model similar to the interference constraints of the wireless network model.

Finally, a flow active on route R_i , departs the network after the completion of its service requirement and frees unit capacity on the links of R_i . Let $D_i(s, t)$ denote the number of flows which are served (hence leave the system) in time interval $[s, t]$.

2.3. Scheduling Algorithm & Performance Metric. In both models described above, the scheduling is the key operational question. In the wireless network, queues need to decide which of them transmit subject to interference constraints. In the circuit switched network, queues need to agree on

which flows becomes active subject to network capacity constraints. And, a *scheduling algorithm* is required to make these decisions every time.

In wireless network, the scheduling algorithm decides the schedule $\sigma(t) \in \mathcal{I}(G)$ at each time t . We are interested in *distributed* scheduling algorithms, i.e. queue i decides $\sigma_i(t)$ using its local information, such as its queue-size $Q_i(t)$. We assume that queues have instantaneous *carrier sensing* information, i.e. if a queue (or node) j starts transmitting at time t , then all neighboring queues can *listen* to this transmission immediately.

In buffered circuit switched network, the scheduling algorithm decides active flows or schedules $\mathbf{z}(t)$ at time t . Again, our interest is in *distributed* scheduling algorithms, i.e. queue at ingress of route R_i decides $z_i(t)$ using its local information. Each queue (or route) can obtain instantaneous information on whether all links along its route have unit capacity available or not.

In summary, both models need scheduling algorithms to decide when each queue (or its ingress port) will request the network for availability of resources; upon a positive answer (or successful request) from the network, the queue acquires network resources for certain amount of time. And, these algorithm need to be based on local information.

From the perspective of network performance, we would like the scheduling algorithm to be such that the queues in network remain as small as possible for the largest possible range of arrival rate vectors. To formalize this notion of performance, we define the capacity regions for both of these models. Let $\mathbf{\Lambda}_w$ be the capacity region of the wireless network model defined as

$$\begin{aligned} \mathbf{\Lambda}_w &= \text{Conv}(\mathcal{I}(G)) \\ (1) \quad &= \left\{ \mathbf{y} \in \mathbb{R}_+^n : \mathbf{y} \leq \sum_{\sigma \in \mathcal{I}(G)} \alpha_\sigma \sigma, \text{ with } \alpha_\sigma \geq 0, \text{ and } \sum_{\sigma \in \mathcal{I}(G)} \alpha_\sigma \leq 1 \right\}. \end{aligned}$$

And let $\mathbf{\Lambda}_{cs}$ be the capacity region of the buffered circuit switched network defined as

$$\begin{aligned} \mathbf{\Lambda}_{cs} &= \text{Conv}(\mathcal{X}) \\ (2) \quad &= \left\{ \mathbf{y} \in \mathbb{R}_+^n : \mathbf{y} \leq \sum_{\mathbf{z} \in \mathcal{X}} \alpha_{\mathbf{z}} \mathbf{z}, \text{ with } \alpha_{\mathbf{z}} \geq 0, \text{ and } \sum_{\mathbf{z} \in \mathcal{X}} \alpha_{\mathbf{z}} \leq 1 \right\}. \end{aligned}$$

Intuitively, these bounds of capacity regions comes from the fact that any algorithm produces the ‘service rate’ from $\mathcal{I}(G)$ (or \mathcal{X}) each time and hence the time average of the service rate induced by any algorithm must belong

to its convex hull. Therefore, if arrival rates λ can be ‘served well’ by any algorithm then it must belong to $\text{Conv}(\mathcal{I}(G))$ (or $\text{Conv}(\mathcal{X})$).

Motivated by this, we call an arrival rate vector λ admissible if $\lambda \in \Lambda$, and say that an arrival rate vector λ is strictly admissible if $\lambda \in \Lambda^\circ$, where Λ° is the interior of Λ formally defined as

$$\Lambda^\circ = \{ \lambda \in \mathbb{R}_+^n : \lambda < \lambda^* \text{ componentwise, for some } \lambda^* \in \Lambda \}.$$

Equivalently, we may say that the network is *under-loaded*. Now we are ready to define a performance metric for a scheduling algorithm. Specifically, we desire the scheduling algorithm to be throughput optimal as defined below.

Definition 1 (throughput optimal) A scheduling algorithm is called throughput optimal, or stable, or providing 100% throughput, if for any $\lambda \in \Lambda^\circ$ the (appropriately defined) underlying network Markov process is *positive (Harris) recurrent*.

2.4. *The MW Algorithm.* Here we describe a popular algorithm known as the maximum weight or in short MW algorithm that was proposed by Tassiulas and Ephremides [32]. It is throughput optimal for a large class of network models. The algorithm readily applies to the wireless network model. However, it does not apply (exactly or any variant of it) in the case of circuit switched network. Further, this algorithm requires solving a hard combinatorial problem each time slot, e.g. maximum weight independent set for wireless network, which is NP-hard in general. Therefore, it’s far from being practically useful. In a nutshell, the randomized algorithm proposed in this paper will overcome these drawbacks of the MW algorithm while retaining the throughput optimality property. For completeness, next we provide a brief description of the MW algorithm.

In the wireless network model, the MW algorithm chooses a schedule $\sigma(\tau) \in \mathcal{I}(G)$ every time step $\tau \in \mathbb{Z}_+$ as follows²:

$$\sigma(\tau) \in \arg \max_{\rho \in \mathcal{I}(G)} \mathbf{Q}(\tau) \cdot \rho.$$

In other words, the algorithm changes its decision once in unit time utilizing the information $\mathbf{Q}(\tau)$. The maximum weight property allows one to establish positive recurrence by means of Lyapunov drift criteria (see Lemma 6) when the arrival rate is admissible, i.e. $\lambda \in \Lambda_w^\circ$. However, as indicated above picking such a schedule every time is computationally burdensome.

²Here and everywhere else, we use notation $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^d u_i v_i$ for any d -dimensional vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. That is, $\mathbf{Q}(\tau) \cdot \rho = \sum_i Q_i(\tau) \cdot \rho_i$.

A natural generalization of this, called MW- f algorithm, that uses weight $f(Q_i(\cdot))$ instead of $Q_i(\cdot)$ for an increasing non-negative function f also leads to throughput optimality (cf. see [30, 27, 28]).

For the buffered circuit switched network model, the MW algorithm is not applicable. To understand this, consider the following. The MW algorithm would require the network to schedule active flows as $\mathbf{z}(\tau) \in \mathcal{X}$ where

$$\mathbf{z}(\tau) \in \arg \max_{\mathbf{z} \in \mathcal{X}} \mathbf{Q}(\tau) \cdot \mathbf{z}.$$

This will require the algorithm to possibly preempt some of active flows without the completion of their service requirement. And this is not allowed in this model.

3. Main Result: Simple & Efficient Randomized Algorithms.

As stated above, the MW algorithm is not practical for wireless network and is not applicable to circuit switched network. However, it has the desirable throughput optimality property. As the main result of this paper, we provide a simple, randomized algorithm that is applicable to both wireless and circuit switched network as well as it's throughput optimal. The algorithm requires each node (or queue) to perform only a few logical operations at each time step, it's distributed and effectively it 'simulates' the MW- f algorithm for an appropriate choice of f . In that sense, it's a simple, randomized, distributed implementation of the MW algorithm.

In what follows, we shall describe algorithms for wireless network and buffered circuit switched network respectively. We will state their throughput optimality property. While these algorithms seem different, philosophically they are very similar – also, witnessed in the commonality in their proofs.

3.1. Algorithm for Wireless Network. Let $t \in \mathbb{R}_+$ denote the time index and $\mathbf{W}(t) = [W_i(t)] \in \mathbb{R}_+^n$ be the vector of weights at the n queues. The $\mathbf{W}(t)$ will be a function of $\mathbf{Q}(t)$ to be determined later. In a nutshell, the algorithm described below will choose a schedule $\sigma(t) \in \mathcal{I}(G)$ so that the weight, $\mathbf{W}(t) \cdot \sigma(t)$, is as large as possible.

The algorithm is randomized and asynchronous. Each node (or queue) has an independent Exponential clock of rate 1 (i.e. Poisson process of rate 1). Let the k th tick of the clock of node i happen at time T_k^i ; $T_0^i = 0$ for all i . By definition $T_{k+1}^i - T_k^i, k \geq 0$, are i.i.d. mean 1 Exponential random variables. Each node changes its scheduling decision only at its clock ticks. That is, for node i the $\sigma_i(t)$ remains constant for $t \in (T_k^i, T_{k+1}^i]$. Clearly, with probability 1 no two clock ticks across nodes happen at the same time.

Initially, we assume that $\sigma_i(0) = 0$ for all i . The node i at the k th clock tick, $t = T_k^i$, *listens* to the medium and does the following:

- If any neighbor of i is transmitting, i.e. $\sigma_j(t) = 1$ for some $j \in \mathcal{N}(i) = \{j' : (i, j') \in E\}$, then set $\sigma_i(t^+) = 0$.
- Else, set

$$\sigma_i(t^+) = \begin{cases} 1 & \text{with probability } \frac{\exp(W_i(t))}{1 + \exp(W_i(t))} \\ 0 & \text{otherwise.} \end{cases}$$

Here, we assume that if $\sigma_i(t) = 1$, then node i will always transmit data irrespective of the value of $Q_i(t)$ so that the neighbors of node i can infer $\sigma_i(t)$ by *listening* to the medium.

3.1.1. Throughput Optimality. The above described algorithm for wireless network is throughput optimal for an appropriate choice of weight $\mathbf{W}(t)$. Define weight $W_i(t)$ at node i in the algorithm for wireless network as

(3)

$$W_i(t) = \max \left\{ f(Q_i(\lfloor t \rfloor)), \sqrt{f(Q_{\max}(\lfloor t \rfloor))} \right\},$$

where³ $f(x) = \log \log(x + e)$ and $Q_{\max}(\cdot) = \max_i Q_i(\cdot)$. The non-local information of $Q_{\max}(\lfloor t \rfloor)$ can be replaced by its approximate estimation that can be computed through a very simple distributed algorithm. This does not alter the throughput optimality property of the algorithm. A discussion is provided in Section 6. We state the following property of the algorithm.

Theorem 1 *Suppose the algorithm of Section 3.1 uses the weight as per (3). Then, for any $\lambda \in \Lambda_w^o$, the network Markov process is positive Harris recurrent.*

In this paper, Theorem 1 (as well as Theorem 2) is established for the choice of $f(x) = \log \log(x + e)$. However, the proof technique of this paper extends naturally for any choice of $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that satisfies the following conditions: $f(0) = 0$, f is a monotonically strictly increasing function, $\lim_{x \rightarrow \infty} f(x) = \infty$ and

$$\lim_{x \rightarrow \infty} \exp(f(x)) f'(f^{-1}(\delta f(x))) = 0, \quad \text{for any } \delta \in (0, 1).$$

Examples of such functions includes: $f(x) = \varepsilon(x) \log(x + 1)$, where $\varepsilon(0) = 1$, $\varepsilon(x)$ is monotonically decreasing function to 0 as $x \rightarrow \infty$; $f(x) = \sqrt{\log(x + 1)}$; $f(x) = \log \log(x + e^e)$, etc.

³Unless stated otherwise, here and everywhere else the $\log(\cdot)$ is natural logarithm, i.e. base e .

3.2. *Algorithm for Buffered Circuit Switched Network.* In a buffered circuit switched network, the scheduling algorithm decided when each of the ingress node (or queue) should request the network for availability of resources (links) along its route and upon positive response from the network, it acquires the resources. Our algorithm to make such a decision at each node is described as follows:

- Each ingress node of a route, say R_i , generates request as per a time varying Poisson process whose rate at time t is equal to $\exp(W_i(t))$.
- If the request generated by an ingress node of route, say R_i , is accepted, a flow from the head of its queue leaves the queue and acquire the resources in the network. Else, do nothing.

In above, like the algorithm for wireless network we assume that if the request of ingress node i is accepted, a new flow will acquire resources in the network along its route. This is irrespective of whether queue is empty or not – if queue is empty, a *dummy* flow is generated. This is merely for technical reasons.

3.2.1. *Throughput Optimality.* We describe a specific choice of weight $\mathbf{W}(t)$ for which the algorithm for circuit switched network as described above is throughput optimal. Specifically, for route R_i its weight at time t is defined as

$$(4) \quad W_i(t) = \max \left\{ f(Q_i(\lfloor t \rfloor)), \sqrt{f(Q_{\max}(\lfloor t \rfloor))} \right\},$$

where $f(x) = \log \log(x + e)$. The remark about distributed estimation of $Q_{\max}(\lfloor t \rfloor)$ after (3) applies here as well. We state the following property of the algorithm.

Theorem 2 *Suppose the algorithm of Section 3.2 uses the weight as per (4). Then, for any $\lambda \in \Lambda_{cs}^o$, the network Markov process is positive Harris recurrent.*

4. Technical Preliminaries.

4.1. *Finite State Markov Chain.* Consider a discrete-time, time homogeneous Markov chain over a finite state space Ω . Let its probability transition matrix be $P = [P_{ij}] \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$. If P is irreducible and aperiodic, then the Markov chain is known to have a unique stationary distribution $\pi = [\pi_i] \in \mathbb{R}_+^{|\Omega|}$ and it is ergodic, i.e.

$$\lim_{\tau \rightarrow \infty} P_{ji}^\tau \rightarrow \pi_i, \quad \text{for any } i, j \in \Omega.$$

The adjoint of P , also known as the time-reversal of P , denoted by P^* is defined as follows:

$$(5) \quad \pi_i P_{ij}^* = \pi_j P_{ji}, \quad \text{for any } i, j \in \Omega.$$

By definition, P^* has π as its stationary distribution as well. If $P = P^*$ then P is called *reversible* or *time reversible*.

Similar notions can be defined for a continuous time Markov process over Ω . To this end, let $P(s, t) = [P_{ij}(s, t)] \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$ denote its transition matrix over time interval $[s, t]$. The Markov process is called time-homogeneous if $P(s, t)$ is stationary, i.e. $P(s, t) = P(0, t - s)$ for all $0 \leq s < t$ and is called reversible if $P(s, t)$ is reversible for all $0 \leq s < t$. Further, if $P(0, t)$ is irreducible and aperiodic for all $t > 0$, then this time-homogeneous reversible Markov process has a unique stationary distribution π and it is ergodic, i.e.

$$\lim_{t \rightarrow \infty} P_{ji}(0, t) \rightarrow \pi_i, \quad \text{for any } i, j \in \Omega.$$

4.2. Mixing Time of Markov Chain. Given an ergodic finite state Markov chain, the distribution at time τ converge to the stationary distribution starting from any initial condition as described above. We will need quantitative bounds on the time it takes for them to reach “close” to their stationary distribution. This time to reach stationarity is known as the *mixing time* of the Markov chain. Here we introduce necessary preliminaries related to this notion. We refer an interested reader to survey papers [17, 24]. We start with the definition of distances between probability distributions.

Definition 2 (*Distance of measures*) *Given two probability distributions ν and μ on a finite space Ω , we define the following two distances. The total variation distance, denoted as $\|\nu - \mu\|_{TV}$ is*

$$\|\nu - \mu\|_{TV} = \frac{1}{2} \sum_{i \in \Omega} |\nu(i) - \mu(i)|.$$

The χ^2 distance, denoted as $\left\| \frac{\nu}{\mu} - 1 \right\|_{2, \mu}$ is

$$\left\| \frac{\nu}{\mu} - 1 \right\|_{2, \mu}^2 = \|\nu - \mu\|_{2, \frac{1}{\mu}}^2 = \sum_{i \in \Omega} \mu(i) \left(\frac{\nu(i)}{\mu(i)} - 1 \right)^2.$$

More generally, for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^{|\Omega|}$, we define

$$\|\mathbf{v}\|_{2, \mathbf{u}}^2 = \sum_{i \in \Omega} u_i v_i^2.$$

We make note of the following relation between the two distances defined above: using the Cauchy-Schwarz inequality, we have

(6)

$$\left\| \frac{\nu}{\mu} - 1 \right\|_{2,\mu} \geq 2 \|\nu - \mu\|_{TV}.$$

Next, we define a matrix norm that will be useful in determining the rate of convergence or the mixing time of a finite-state Markov chain.

Definition 3 (Matrix norm) Consider a $|\Omega| \times |\Omega|$ non-negative valued matrix $A \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$ and a given vector $\mathbf{u} \in \mathbb{R}_+^{|\Omega|}$. Then, the matrix norm of A with respect to \mathbf{u} is defined as follows:

$$\|A\|_{\mathbf{u}} = \sup_{\mathbf{v}: \mathbb{E}_{\mathbf{u}}[\mathbf{v}] = 0} \frac{\|A\mathbf{v}\|_{2,\mathbf{u}}}{\|\mathbf{v}\|_{2,\mathbf{u}}},$$

where $\mathbb{E}_{\mathbf{u}}[\mathbf{v}] = \sum_i u_i v_i$.

It can be easily checked that the above definition of matrix norm satisfies the following properties.

P1. For matrices $A, B \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$ and $\pi \in \mathbb{R}_+^{|\Omega|}$

$$\|A + B\|_{\pi} \leq \|A\|_{\pi} + \|B\|_{\pi}.$$

P2. For matrix $A \in \mathbb{R}_+^{|\Omega| \times |\Omega|}$, $\pi \in \mathbb{R}_+^{|\Omega|}$ and $c \in \mathbb{R}$,

$$\|cA\|_{\pi} = |c| \|A\|_{\pi}.$$

P3. Let A and B be transition matrices of reversible Markov chains, i.e. $A = A^*$ and $B = B^*$. Let both of them have π as their unique stationary distribution. Then,

$$\|AB\|_{\pi} \leq \|A\|_{\pi} \|B\|_{\pi}.$$

P4. Let A be the transition matrix of a reversible Markov chain, i.e. $A = A^*$. Then,

$$\|A\| \leq \lambda_{\max},$$

where $\lambda_{\max} = \max\{|\lambda| \neq 1 : \lambda \text{ is an eigenvalue of } A\}$.

For a probability matrix P , we will mostly be interested in the matrix norm of P with respect to its stationary distribution π , i.e. $\|P\|_{\pi}$. Therefore, in this paper if we use a matrix norm for a probability matrix without mentioning

the reference measure, then it is with respect to the stationary distribution. That is, in the above example $\|P\|$ will mean $\|P\|_\pi$.

With these definitions, it follows that for any distribution μ on Ω

$$(7) \quad \left\| \frac{\mu P}{\pi} - 1 \right\|_{2,\pi} \leq \|P^*\| \left\| \frac{\mu}{\pi} - 1 \right\|_{2,\pi},$$

since $\mathbb{E}_\pi \left[\frac{\mu}{\pi} - 1 \right] = 0$, where $\frac{\mu}{\pi} = [\mu(i)/\pi(i)]$. The Markov chain of our interest, Glauber dynamics, is reversible i.e. $P = P^*$. Therefore, for a reversible Markov chain starting with initial distribution $\mu(0)$, the distribution $\mu(\tau)$ at time τ is such that

$$(8) \quad \left\| \frac{\mu(\tau)}{\pi} - 1 \right\|_{2,\pi} \leq \|P\|^\tau \left\| \frac{\mu(0)}{\pi} - 1 \right\|_{2,\pi}.$$

Now starting from any state i , i.e. probability distribution with unit mass on state i , the initial distance $\left\| \frac{\mu(0)}{\pi} - 1 \right\|_{2,\pi}$ in the worst case is bounded above by $\sqrt{1/\pi_{\min}}$ where $\pi_{\min} = \min_i \pi_i$. Therefore, for any $\delta > 0$ we have $\left\| \frac{\mu(\tau)}{\pi} - 1 \right\|_{2,\pi} \leq \delta$ for any τ such that⁴

$$\tau \geq \frac{\log 1/\pi_{\min} + \log 1/\delta}{\log 1/\|P\|} = \Theta \left(\frac{\log 1/\pi_{\min} + \log 1/\delta}{1 - \|P\|} \right).$$

This suggests that the “mixing time”, i.e. time to reach (close to) the stationary distribution of the Markov chain scales inversely with $1 - \|P\|$. Therefore, we will define the “mixing time” of a Markov chain with transition matrix P as $1/(1 - \|P\|)$.

4.3. Glauber Dynamics & Algorithm for Wireless Network. We will describe the relation between the algorithm for wireless network (cf. Section 3.1) and a specific irreducible, aperiodic, reversible Markov chain on the space of independent sets $\mathcal{I}(G)$ or schedules for wireless network with graph $G = (V, E)$. It is also known as the Glauber dynamics, which is used by the standard Metropolis-Hastings [21, 13] sampling mechanism that is described next.

⁴Throughout this paper, we shall utilize the standard order-notations: for two functions $g, f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g(x) = \omega(f(x))$ means $\liminf_{x \rightarrow \infty} g(x)/f(x) = \infty$; $g(x) = \Omega(f(x))$ means $\liminf_{x \rightarrow \infty} g(x)/f(x) > 0$; $g(x) = \Theta(f(x))$ means $0 < \liminf_{x \rightarrow \infty} g(x)/f(x) \leq \limsup_{x \rightarrow \infty} g(x)/f(x) < \infty$; $g(x) = O(f(x))$ means $\limsup_{x \rightarrow \infty} g(x)/f(x) < \infty$; $g(x) = o(f(x))$ means $\limsup_{x \rightarrow \infty} g(x)/f(x) = 0$.

4.3.1. *Glauber Dynamics & Its Mixing Time.* We shall start off with the definition of the Glauber dynamics followed by a useful bound on its mixing time.

Definition 4 (Glauber dynamics) Consider a graph $G = (V, E)$ of $n = |V|$ nodes with node weights $\mathbf{W} = [W_i] \in \mathbb{R}_+^n$. The Glauber dynamics based on weight \mathbf{W} , denoted by $GD(\mathbf{W})$, is a Markov chain on the space of independent sets of G , $\mathcal{I}(G)$. The transitions of this Markov chain are described next. Suppose the Markov chain is currently in the state $\sigma \in \mathcal{I}(G)$. Then, the next state, say σ' is decided as follows: pick a node $i \in V$ uniformly at random and

- set $\sigma'_j = \sigma_j$ for $j \neq i$,
- if $\sigma_k = 0$ for all $k \in \mathcal{N}(i)$, then set

$$\sigma'_i = \begin{cases} 1 & \text{with probability } \frac{\exp(W_i)}{1 + \exp(W_i)} \\ 0 & \text{otherwise,} \end{cases}$$

- else set $\sigma'_i = 0$.

It can be verified that the Glauber dynamics $GD(\mathbf{W})$ is reversible with stationary distribution π given by

$$(9) \quad \pi_\sigma \propto \exp(\mathbf{W} \cdot \sigma), \quad \text{for any } \sigma \in \mathcal{I}(G).$$

Now we describe bound on the mixing time of Glauber dynamics.

Lemma 3 Let P be the transition matrix of the Glauber dynamics $GD(\mathbf{W})$ with n nodes. Then,

$$(10) \quad \|P\| \leq 1 - \frac{1}{n^2 2^{2n+3} \exp(2(n+1)W_{\max})},$$

$$(11) \quad \left\| e^{n(P-I)} \right\| \leq 1 - \frac{1}{n^2 2^{n+4} \exp(2(n+1)W_{\max})}.$$

Proof. By the property **P4** of the matrix norm and Cheeger's inequality [1, 8, 14, 6, 29], it is well known that $\|P\| \leq \lambda_{\max} \leq 1 - \frac{\Phi^2}{2}$ where Φ is the conductance of P , defined as

$$\Phi = \min_{S \subset \mathcal{I}(G): \pi(S) \leq \frac{1}{2}} \frac{Q(S, S^c)}{\pi(S)\pi(S^c)},$$

where $S^c = \mathcal{I}(G) \setminus S$, $Q(S, S^c) = \sum_{\sigma \in S, \sigma' \in S^c} \pi(\sigma) P(\sigma, \sigma')$. Now we have

$$\begin{aligned}
\Phi &\geq \min_{S \subset \mathcal{I}(G)} Q(S, S^c) \\
&\geq \min_{P(\sigma, \sigma') \neq 0} \pi(\sigma) P(\sigma, \sigma') \\
&\geq \pi_{\min} \cdot \min_i \frac{1}{n} \frac{1}{1 + \exp(W_i)} \\
&\geq \frac{1}{2^n \exp(nW_{\max})} \cdot \frac{1}{n} \frac{1}{1 + \exp(W_{\max})} \\
&\geq \frac{1}{n2^{n+1} \exp((n+1)W_{\max})}.
\end{aligned}$$

Therefore

$$\|P\| \leq 1 - \frac{1}{n^2 2^{2n+3} \exp(2(n+1)W_{\max})}.$$

Now consider $e^{n(P-I)}$. Using properties **P1**, **P2** and **P3** of matrix norm, we have:

$$\begin{aligned}
\|e^{n(P-I)}\| &= \left\| e^{-n} \sum_{k=0}^{\infty} \frac{n^k P^k}{k!} \right\| \\
&\leq e^{-n} \sum_{k=0}^{\infty} \frac{n^k \|P\|^k}{k!} \\
&= e^{n(\|P\|-1)} \\
(12) \quad &\leq 1 - \frac{n(1 - \|P\|)}{2}.
\end{aligned}$$

In the last inequality, we have used the fact that $\|P\| < 1$ and $e^{-x} \leq 1 - x/2$ for all $x \in [0, 1]$. Hence, from the bound of $\|P\|$, we obtain

$$(13) \quad \|e^{n(P-I)}\| \leq 1 - \frac{1}{n2^{2n+4} \exp(2(n+1)W_{\max})}.$$

This completes the proof of Lemma 3. \square

4.3.2. Relation to Algorithm. Now we relate our algorithm for wireless network scheduling described in Section 3.1 with an appropriate continuous time version of the Glauber dynamics with time-varying weights. Recall that $\mathbf{Q}(t)$ and $\sigma(t)$ denote the queue-size vector and schedule at time t . The algorithm changes its scheduling decision, $\sigma(t)$, when a node's exponential clock of rate 1 ticks. Due to memoryless property of exponential distribution and independence of clocks of all nodes, this is equivalent to having a global

exponential clock of rate n and upon clock tick one of the n nodes gets chosen. This node decides its transition as explained in Section 3.1. Thus, the effective dynamics of the algorithm upon a global clock tick is such that the schedule $\sigma(t)$ evolves exactly as per the Glauber dynamics $GD(\mathbf{W}(t))$. Here recall that $\mathbf{W}(t)$ is determined based on $\mathbf{Q}(\lfloor t \rfloor)$. With abuse of notation, let the transition matrix of this Glauber dynamics be denoted by $GD(\mathbf{W}(t))$.

Now consider any $\tau \in \mathbb{Z}_+$. Let $\mathbf{Q}(\tau), \sigma(\tau)$ be the states at time τ . Then,

$$\mathbb{E} \left[\delta_{\sigma(\tau+1)} \mid \mathbf{Q}(\tau), \sigma(\tau) \right] = \sum_{k=0}^{\infty} \delta_{\sigma(\tau)} \Pr(\zeta = k) GD(\mathbf{W}(\tau))^k,$$

where we have used notation δ_{σ} for the distribution with singleton support $\{\sigma\}$ and ζ is a Poisson random variable of mean n . In above, the expectation is taken with respect to the distribution of $\sigma(\tau+1)$ given $\mathbf{Q}(\tau), \sigma(\tau)$. Therefore, it follows that

$$\begin{aligned} \mathbb{E} \left[\delta_{\sigma(\tau+1)} \mid \mathbf{Q}(\tau), \sigma(\tau) \right] &= \delta_{\sigma(\tau)} e^{n(GD(\mathbf{W}(\tau)) - I)} \\ (14) \qquad \qquad \qquad &= \delta_{\sigma(\tau)} P(\tau) \end{aligned}$$

where $P(\tau) \triangleq e^{n(GD(\mathbf{W}(\tau)) - I)}$. In general, for any $\delta \in [0, 1]$

$$(15) \qquad \mathbb{E} \left[\delta_{\sigma(\tau+\delta)} \mid \mathbf{Q}(\tau), \sigma(\tau) \right] = \delta_{\sigma(\tau)} P^\delta(\tau),$$

where $P^\delta(\tau) \triangleq e^{\delta n(GD(\mathbf{W}(\tau)) - I)}$.

4.4. Loss Network \mathcal{E} Algorithm for Circuit Switched Network. For the buffered circuit switched network, the Markov chain of interest is related to the classical stochastic loss network model. This model has been popularly utilized to study the performance of various systems including the telephone networks, human resource allocation, etc. (cf. see [16]). The stochastic loss network model is very similar to the model of the buffered circuit switched network with the only difference that it does not have any buffers at the ingress nodes.

4.4.1. Loss Network \mathcal{E} Its Mixing Time. A loss network is described by a network graph $G = (V, E)$ with capacitated links $[C_e]_{e \in E}$, n routes $\{R_i : R_i \subset E, 1 \leq i \leq n\}$ and without any buffer or queues at the ingress of each route. For each route R_i , there is a dedicated exogenous, independent Poisson arrival process with rate ϕ_i . Let $z_i(t)$ be number of active flows on route i at time t , with notation $\mathbf{z}(t) = [z_i(t)]$. Clearly, $\mathbf{z}(t) \in \mathcal{X}$ due to

network capacity constraints. At time t when a new exogenous flow arrives on route R_i , if it can be accepted by the network, i.e. $\mathbf{z}(t) + e_i \in \mathcal{X}$, then it is accepted with $z_i(t) \rightarrow z_i(t) + 1$. Or else, it is dropped (and hence lost forever). Each flow holds unit amount of capacity on all links along its route for time that is distributed as Exponential distribution with mean 1, independent of everything else. Upon the completion of holding time, the flow departs and frees unit capacity on all links of its own route.

Therefore, effectively this loss network model can be described as a finite state Markov process with state space \mathcal{X} . Given state $\mathbf{z} = [z_i] \in \mathcal{X}$, the possible transitions and corresponding rates are given as

(16)

$$z_i \rightarrow \begin{cases} z_i + 1, & \text{with rate } \phi_i \text{ if } \mathbf{z} + e_i \in \mathcal{X}, \\ z_i - 1, & \text{with rate } x_i. \end{cases}$$

It can be verified that this Markov process is irreducible, aperiodic, and time-reversible. Therefore, it is positive recurrent (due to the finite state space) and has a unique stationary distribution. Its stationary distribution π is known (cf. [16]) to have the following product-form: for any $\mathbf{z} \in \mathcal{X}$,

$$(17) \quad \pi_{\mathbf{z}} \propto \prod_{i=1}^n \frac{\phi_i^{z_i}}{z_i!}.$$

We will be interested in the discrete-time (or *embedded*) version of this Markov processes, which can be defined as follows.

Definition 5 (Loss Network) *A loss network Markov chain with capacitated graph $G = (V, E)$, capacities $C_e, e \in E$ and n routes $R_i, 1 \leq i \leq n$, denoted by $\mathbf{LN}(\phi)$ is a Markov chain on \mathcal{X} . The transition probabilities of this Markov chain are described next. Given a current state $\mathbf{z} \in \mathcal{X}$, the next state $\mathbf{z}^* \in \mathcal{X}$ is decided by first picking a route R_i uniformly at random and performing the following:*

◦ $z_j^* = z_j$ for $j \neq i$ and z_i^* is decided by

$$z_i^* = \begin{cases} z_i + 1 & \text{with probability } \frac{\phi_i}{\mathcal{R}} \cdot \mathbf{1}_{\{\mathbf{z} + e_i \in \mathcal{X}\}} \\ z_i - 1 & \text{with probability } \frac{z_i}{\mathcal{R}} \\ z_i & \text{otherwise.} \end{cases},$$

where $\mathcal{R} = \sum_i \phi_i + C_{\max}$.

$\mathbf{LN}(\phi)$ has the same stationary distribution as in (17), and it is also irreducible, aperiodic, and reversible. Next, we state a bound on the mixing time of the loss network Markov chain $\mathbf{LN}(\phi)$ as follows.

Lemma 4 *Let P be the transition matrix of $\mathbf{LN}(\phi)$ with n routes. If $\phi = \exp(\mathbf{W})$ with ⁵ $W_i \geq 0$ for all i , then,*

$$(18) \quad \|P\| \leq 1 - \frac{1}{8n^4 C_{\max}^{2n C_{\max} + 2n + 2} \exp(2(n C_{\max} + 1)W_{\max})},$$

$$(19) \quad \left\| e^{n\mathcal{R}(P-I)} \right\| \leq 1 - \frac{1}{16n^3 C_{\max}^{2n C_{\max} + 2n + 2} \exp(2(n C_{\max} + 1)W_{\max})}.$$

Proof. Similarly as the proof of Lemma 3, a simple lower bound for the conductance Φ of P is given by

$$(20) \quad \Phi \geq \pi_{\min} \cdot \min_{P_{z,z'} \neq 0} P_{z,z'}.$$

To obtain the lower bound of π_{\min} , recall the equation (17),

$$\pi_{\mathbf{z}} = \frac{1}{Z} \prod_{i=1}^n \frac{\phi_i^{z_i}}{z_i!},$$

where $Z = \sum_{\mathbf{z} \in \mathcal{X}} \prod_{i=1}^n \frac{\phi_i^{z_i}}{z_i!}$, and consider the following:

$$Z \leq |\mathcal{X}| \phi_{\max}^{n C_{\max}} \leq C_{\max}^n \exp(n C_{\max} W_{\max}),$$

and

$$\prod_{i=1}^n \frac{\phi_i^{z_i}}{z_i!} \geq \frac{1}{(C_{\max}!)^n} \geq \frac{1}{C_{\max}^{n C_{\max}}}.$$

By combining the above inequalities, we obtain

$$(21) \quad \pi_{\min} \geq \frac{1}{C_{\max}^{n C_{\max} + n} \exp(n C_{\max} W_{\max})}.$$

On the other hand, one can bound $\min_{P_{z,z'} \neq 0} P_{z,z'}$ as follows:

$$(22) \quad P_{z,z'} \geq \frac{1}{n} \cdot \frac{1}{\mathcal{R}} \geq \frac{1}{n} \cdot \frac{1}{n \phi_{\max} + C_{\max}} \geq \frac{1}{2n^2 C_{\max} \exp(W_{\max})},$$

where we use the fact that $x + y \leq 2xy$ if $x, y \geq 1$. Now, by combining (21) and (22), we have

$$\Phi \geq \frac{1}{2n^2 C_{\max}^{n C_{\max} + n + 1} \exp((n C_{\max} + 1)W_{\max})}.$$

⁵We use the following notation: given a function $g : \mathbb{R} \rightarrow \mathbb{R}$ and a d -dimensional vector $\mathbf{u} \in \mathbb{R}^d$, let $g(\mathbf{u}) = [g(u_i)] \in \mathbb{R}^d$.

Therefore, using the property **P4** of the matrix norm and Cheeger's inequality, we obtain the desired conclusion as

$$\|P\| \leq \lambda_{\max} \leq 1 - \frac{\Phi^2}{2} \leq \frac{1}{8n^4 C_{\max}^{2n C_{\max} + 2n + 2} \exp(2(n C_{\max} + 1)W_{\max})}.$$

Furthermore, using this bound and similar arguments in the proof of Lemma 3, we have

$$\left\| e^{n\mathcal{R}(P-I)} \right\| \leq 1 - \frac{1}{16n^3 C_{\max}^{2n C_{\max} + 2n + 2} \exp(2(n C_{\max} + 1)W_{\max})}.$$

□

4.4.2. Relation to Algorithm. The scheduling algorithm for buffered circuit switched network described in Section 3.2 effectively simulates a stochastic loss network with time-varying arrival rates $\phi(t)$ where $\phi_i(t) = \exp(W_i(t))$. That is, the relation of the algorithm in Section 3.2 with loss network is similar to the relation of the algorithm in Section 3.1 with Glauber dynamics that we explained in the previous section. To this end, for a given $\tau \in \mathbb{Z}_+$, let $\mathbf{Q}(\tau)$ and $\mathbf{z}(\tau)$ be queue-size vector and active flows at time τ . With abuse of notation, let $LN(\exp(\mathbf{W}(\tau)))$ be the transition matrix of the corresponding Loss Network with $\mathbf{W}(\tau)$ dependent on $\mathbf{Q}(\tau)$. Then, for any $\delta \in [0, 1]$

$$(23) \quad \mathbb{E} \left[\delta_{\mathbf{z}(\tau+\delta)} \mid \mathbf{Q}(\tau), \mathbf{z}(\tau) \right] = \delta_{\mathbf{z}(\tau)} e^{n\delta \mathcal{R}(\tau)(LN(\exp(\mathbf{W}(\tau))) - I)},$$

where $\mathcal{R}(\tau) = \sum_i \exp(W_i(\tau)) + C_{\max}$.

4.4.3. Positive Harris Recurrence & Its Implication. For completeness, we define the well known notion of positive Harris recurrence (e.g. see [2, 3]). We also state its useful implications to explain its desirability. In this paper, we will be concerned with discrete-time, time-homogeneous Markov process or chain evolving over a complete, separable metric space \mathbf{X} . Let $\mathcal{B}_{\mathbf{X}}$ denote the Borel σ -algebra on \mathbf{X} . We assume that the space \mathbf{X} be endowed with a norm⁶, denoted by $|\cdot|$. Let $X(\tau)$ denote the state of Markov chain at time $\tau \in \mathbb{Z}_+$.

Consider any $A \in \mathcal{B}_{\mathbf{X}}$. Define stopping time $T_A = \inf\{\tau \geq 1 : X(\tau) \in A\}$. Then the set A is called Harris recurrent if

$$\Pr_{\mathbf{x}}(T_A < \infty) = 1 \quad \text{for any } \mathbf{x} \in \mathbf{X},$$

⁶One may assume it to be induced by the metric of \mathbf{X} , denoted by d . For example, for any $\mathbf{x} \in \mathbf{X}$, $|\mathbf{x}| = d(\mathbf{0}, \mathbf{x})$ with respect to a fixed $\mathbf{0} \in \mathbf{X}$.

where $\Pr_{\mathbf{x}}(\cdot) \equiv \Pr(\cdot | X(0) = \mathbf{x})$. A Markov chain is called Harris recurrent if there exists a σ -finite measure μ on $(\mathsf{X}, \mathcal{B}_{\mathsf{X}})$ such that whenever $\mu(A) > 0$ for $A \in \mathcal{B}_{\mathsf{X}}$, A is Harris recurrent. It is well known that if X is Harris recurrent then an essentially unique invariant measure exists (e.g. see Gettoor [11]). If the invariant measure is finite, then it may be normalized to obtain a unique invariant probability measure (or stationary probability distribution); in this case X is called positive Harris recurrent.

Now we describe a useful implication of positive Harris recurrence. Let π be the unique invariant (or stationary) probability distribution of the positive Harris recurrent Markov chain X . Then the following ergodic property is satisfied: for any $x \in \mathsf{X}$ and non-negative measurable function $f : \mathsf{X} \rightarrow \mathbb{R}_+$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{\tau=0}^{T-1} f(X(\tau)) \rightarrow \mathbb{E}_{\pi}[f], \quad \Pr_{\mathbf{x}}\text{-almost surely.}$$

Here $\mathbb{E}_{\pi}[f] = \int f(z)\pi(z)$. Note that $\mathbb{E}_{\pi}[f]$ may not be finite.

4.4.4. Criteria for Positive Harris Recurrence. Here we introduce a well known criteria for establishing the positive Harris recurrence based on existence of a Lyapunov function and an appropriate petit set.

We will need some definitions to begin with. Given a probability distribution (also called sampling distribution) a on \mathbb{N} , the a -sampled transition matrix of the Markov chain, denoted by K_a is defined as

$$K_a(\mathbf{x}, B) = \sum_{\tau \geq 0} a(\tau) P^{\tau}(\mathbf{x}, B), \quad \text{for any } \mathbf{x} \in \mathsf{X}, B \in \mathcal{B}_{\mathsf{X}}.$$

Now, we define a notion of a *petite* set. A non-empty set $A \in \mathcal{B}_{\mathsf{X}}$ is called μ_a -*petite* if μ_a is a non-trivial measure on $(\mathsf{X}, \mathcal{B}_{\mathsf{X}})$ and a is a probability distribution on \mathbb{N} such that for any $\mathbf{x} \in A$,

$$K_a(\mathbf{x}, \cdot) \geq \mu_a(\cdot).$$

A set is called a *petite* set if it is μ_a -petite for some such non-trivial measure μ_a . A known sufficient condition to establish positive Harris recurrence of a Markov chain is to establish positive Harris recurrence of closed petite sets as stated in the following lemma. We refer an interested reader to the book by Meyn and Tweedie [22] or the recent survey by Foss and Konstantopoulos [9] for details.

Lemma 5 *Let B be a closed petite set. Suppose B is Harris recurrent, i.e. $\Pr_{\mathbf{x}}(T_B < \infty) = 1$ for any $\mathbf{x} \in \mathsf{X}$. Further, let*

$$\sup_{\mathbf{x} \in B} \mathbb{E}_{\mathbf{x}}[T_B] < \infty.$$

Then the Markov chain is positive Harris recurrent. Here $\mathbb{E}_{\mathbf{x}}$ is defined with respect to $\text{Pr}_{\mathbf{x}}$.

Lemma 5 suggests that to establish the positive Harris recurrence of the network Markov chain, it is sufficient to find a closed petite set that satisfies the conditions of Lemma 5. To establish positive recurrence of a closed petite set, we shall utilize the *drift criteria* based on an appropriate Lyapunov function stated in the following Lemma (cf. [9, Theorem 1]).

Lemma 6 *Let $L : \mathsf{X} \rightarrow \mathbb{R}_+$ be a function such that $L(\mathbf{x}) \rightarrow \infty$ as $|\mathbf{x}| \rightarrow \infty$. For any $\kappa > 0$, let $B_\kappa = \{\mathbf{x} : L(\mathbf{x}) \leq \kappa\}$. And let there exist functions $h, g : \mathsf{X} \rightarrow \mathbb{Z}_+$ such that for any $\mathbf{x} \in \mathsf{X}$,*

$$\mathbb{E}[L(X(g(\mathbf{x}))) - L(X(0)) | X(0) = \mathbf{x}] \leq -h(\mathbf{x}),$$

that satisfy the following conditions:

- (a) $\inf_{\mathbf{x} \in \mathsf{X}} h(\mathbf{x}) > -\infty$.
- (b) $\liminf_{L(\mathbf{x}) \rightarrow \infty} h(\mathbf{x}) > 0$.
- (c) $\sup_{L(\mathbf{x}) \leq \gamma} g(\mathbf{x}) < \infty$ for all $\gamma > 0$.
- (d) $\limsup_{L(\mathbf{x}) \rightarrow \infty} g(\mathbf{x})/h(\mathbf{x}) < \infty$.

Then, there exists constant $\kappa_0 > 0$ so that for all $\kappa_0 < \kappa$, the following holds:

$$(24) \quad \mathbb{E}_{\mathbf{x}} [T_{B_\kappa}] < \infty, \quad \text{for any } \mathbf{x} \in \mathsf{X}$$

$$(25) \quad \sup_{\mathbf{x} \in B_\kappa} \mathbb{E}_{\mathbf{x}} [T_{B_\kappa}] < \infty.$$

That is, B_κ is positive recurrent.

5. Proofs of Theorems 1 & 2. This section provides proofs of Theorems 1 and 2. We shall start with necessary formalism followed by a summary of the entire proof. This summary will utilize a series of Lemmas whose proofs will follow.

5.1. Network Markov Process. We describe discrete time network Markov processes under both algorithms that we shall utilize throughout. Let $\tau \in \mathbb{Z}_+$ be the time index. Let $\mathbf{Q}(\tau) = [Q_i(\tau)]$ be the queue-size vector at time τ , $\mathbf{x}(\tau)$ be the schedule at time τ with $\mathbf{x}(\tau) = \boldsymbol{\sigma}(\tau) \in \mathcal{I}(G)$ for the wireless network and $\mathbf{x}(\tau) = \mathbf{z}(\tau) \in \mathcal{X}$ for the circuit switched network. It can be checked that the tuple $X(\tau) = (\mathbf{Q}(\tau), \mathbf{x}(\tau))$ is the Markov state of the network for both setups. Here $X(\tau) \in \mathsf{X}$ where $\mathsf{X} = \mathbb{R}_+^n \times \mathcal{I}(G)$ or $\mathsf{X} = \mathbb{Z}_+^n \times \mathcal{X}$. Clearly, X is a Polish space endowed with the natural product topology. Let

\mathcal{B}_X be the Borel σ -algebra of X with respect to this product topology. For any $\mathbf{x} = (\mathbf{Q}, \mathbf{x}) \in X$, we define norm of \mathbf{x} denoted by $|\mathbf{x}|$ as

$$|\mathbf{x}| = |\mathbf{Q}| + |\mathbf{x}|,$$

where $|\mathbf{Q}|$ denotes the standard ℓ_1 norm while $|\mathbf{x}|$ is defined as its index in $\{0, \dots, |\Omega| - 1\}$, which is assigned arbitrarily. Since $|\mathbf{x}|$ is always bounded, $|\mathbf{x}| \rightarrow \infty$ if and only if $|\mathbf{Q}| \rightarrow \infty$. Theorems 1 and 2 wish to establish that the Markov process $X(\tau)$ is positive Harris recurrent.

5.2. Proof Plan. To establish the positive Harris recurrence of $X(\tau)$, we will utilize the Lyapunov drift criteria to establish the positive recurrence property of an appropriate petit set (cf. Lemma 5). To establish the existence of such a Lyapunov function, we shall study properties of our randomized scheduling algorithms. Specifically, we shall show that in a nutshell our schedule algorithms are *simulating* the maximum weight scheduling algorithm with respect to an appropriate weight, function of the queue-size. This will lead to the desired Lyapunov function and a drift criteria. The detailed proof of positive Harris recurrence that follows this intuition is stated in four steps. We briefly give an overview of these four steps.

To this end, recall that the randomized algorithms for wireless or circuit switched network are effectively asynchronous, continuous versions of the time-varying $GD(\mathbf{W}(t))$ or $\mathbf{LN}(\exp(\mathbf{W}(t)))$ respectively. Let $\pi(t)$ be the stationary distribution of the Markov chain $GD(\mathbf{W}(t))$ or $\mathbf{LN}(\exp(\mathbf{W}(t)))$; $\mu(t)$ be the distribution of the schedule, either $\sigma(t)$ or $\mathbf{z}(t)$, under our algorithm at time t . In the first step, roughly speaking we argue that the weight of schedule sampled as per the stationary distribution $\pi(t)$ is close to the weight of maximum weight schedule for both networks (with an appropriately defined weight). In the second step, roughly speaking we argue that indeed the distribution $\mu(t)$ is close enough to that of $\pi(t)$ for all time t . In the third step, using these two properties we establish the Lyapunov drift criteria for appropriately defined Lyapunov function (cf. Lemma 6). In the fourth and final step, we show that this implies positive recurrence of an appropriate closed petit set. Therefore, due to Lemma 5 this will imply the positive Harris recurrence property of the network Markov process.

5.3. Formal Proof. To this end, we are interested in establishing Lyapunov drift criteria (cf. Lemma 6). For this, consider Markov process starting at time 0 in state $X(0) = (\mathbf{Q}(0), \mathbf{x}(0))$ and as per hypothesis of both Theorems, let $\lambda \in (1 - \varepsilon) \text{Conv}(\Omega)$ with some $\varepsilon > 0$ and $\Omega = \mathcal{I}(G)$ (or \mathcal{X}). Given this, we will go through four steps to prove positive Harris recurrence.

5.3.1. *Step One.* Let $\pi(0)$ be the stationary distribution of $GD(\mathbf{W}(0))$ or $LN(\exp(\mathbf{W}(0)))$. The following Lemma states that the average weight of schedule as per $\pi(0)$ is essentially as good as that of the maximum weight schedule with respect to weight $f(\mathbf{Q}(0))$.

Lemma 7 *Let \mathbf{x} be distributed over Ω as per $\pi(0)$ given $\mathbf{Q}(0)$. Then,*

$$\mathbb{E}_{\pi(0)}[f(\mathbf{Q}(0)) \cdot \mathbf{x}] \geq \left(1 - \frac{\varepsilon}{4}\right) \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}(0)) \cdot \mathbf{y}\right) - O(1). \quad (26)$$

The proof of Lemma 7 is based on the variational characterization of distribution in the exponential form. Specifically, we state the following proposition which is a direct adaptation of the known results in literature (cf. [10]).

Proposition 8 *Let $T : \Omega \rightarrow \mathbb{R}$ and let $\mathcal{M}(\Omega)$ be space of all distributions on Ω . Define $F : \mathcal{M}(\Omega) \rightarrow \mathbb{R}$ as*

$$F(\mu) = \mathbb{E}_{\mu}(T(\mathbf{x})) + H_{ER}(\mu),$$

where $H_{ER}(\mu)$ is the standard discrete entropy of μ . Then, F is uniquely maximized by the distribution ν , where

$$\nu_{\mathbf{x}} = \frac{1}{Z} \exp(T(\mathbf{x})), \quad \text{for any } \mathbf{x} \in \Omega,$$

where Z is the normalization constant (or partition function). Further, with respect to ν , we have

$$\mathbb{E}_{\nu}[T(\mathbf{x})] \geq \left[\max_{\mathbf{x} \in \mathcal{X}} T(\mathbf{x}) \right] - \log |\Omega|.$$

Proof. Observe that the definition of distribution ν implies that for any $\mathbf{x} \in \Omega$,

$$T(\mathbf{x}) = \log Z + \log \nu_{\mathbf{x}}.$$

Using this, for any distribution μ on Ω , we obtain

$$\begin{aligned}
F(\mu) &= \sum_{\mathbf{x}} \mu_{\mathbf{x}} T(\mathbf{x}) - \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \mu_{\mathbf{x}} \\
&= \sum_{\mathbf{x}} \mu_{\mathbf{x}} (\log Z + \log \nu_{\mathbf{x}}) - \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \mu_{\mathbf{x}} \\
&= \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log Z + \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \frac{\nu_{\mathbf{x}}}{\mu_{\mathbf{x}}} \\
&= \log Z + \sum_{\mathbf{x}} \mu_{\mathbf{x}} \log \frac{\nu_{\mathbf{x}}}{\mu_{\mathbf{x}}} \\
&\leq \log Z + \log \left(\sum_{\mathbf{x}} \mu_{\mathbf{x}} \frac{\nu_{\mathbf{x}}}{\mu_{\mathbf{x}}} \right) \\
&= \log Z
\end{aligned}$$

with equality if and only if $\mu = \nu$. To complete other claim of proposition, consider $\mathbf{x}^* \in \arg \max T(\mathbf{x})$. Let μ be Dirac distribution $\mu_{\mathbf{x}} = \mathbf{1}_{[\mathbf{x}=\mathbf{x}^]}$. Then, for this distribution

$$F(\mu) = T(\mathbf{x}^*).$$

But, $F(\nu) \geq F(\mu)$. Also, the maximal entropy of any distribution on Ω is $\log |\Omega|$. Therefore,

$$\begin{aligned}
T(\mathbf{x}^*) &\leq F(\nu) \\
&= \mathbb{E}_{\nu}[T(\mathbf{x})] + H_{ER}(\nu) \\
(27) \quad &\leq \mathbb{E}_{\nu}[T(\mathbf{x})] + \log |\Omega|.
\end{aligned}$$

Re-arrangement of terms in (27) will imply the second claim of Proposition 8. This completes the proof of Proposition 8. \square

Proof of Lemma 7. The proof is based on known observations in the context of classical Loss networks literature (cf. see [16]). In what follows, for simplicity we use $\pi = \pi(0)$ for a given $\mathbf{Q} = \mathbf{Q}(0)$. From (9) and (17), it follows that for both network models, the stationary distribution π has the following form: for any $\mathbf{x} \in \Omega$,

$$\pi_{\mathbf{x}} \propto \prod_i \frac{\exp(W_i x_i)}{x_i!} = \exp \left(\sum_i W_i x_i - \log(x_i!) \right).$$

To apply Proposition 8, this suggest the choice of function $T : \mathcal{X} \rightarrow \mathbb{R}$ as

$$T(\mathbf{x}) = \sum_i W_i x_i - \log(x_i!), \quad \text{for any } \mathbf{x} \in \Omega.$$

Observe that for any $\mathbf{x} \in \Omega$, x_i takes one of the finitely many values in wireless or circuit switched network for all i . Therefore, it easily follows that

$$0 \leq \sum_i \log(x_i!) \leq O(1),$$

where the constant may depend on n and the problem parameter (e.g. C_{\max} in circuit switched network). Therefore, for any $\mathbf{x} \in \Omega$,

$$\begin{aligned} T(\mathbf{x}) &\leq \sum_i W_i x_i \\ (28) \quad &\leq T(\hat{\mathbf{x}}) + O(1). \end{aligned}$$

Define $\hat{\mathbf{x}} = \arg \max_{\mathbf{x} \in \Omega} \sum_i W_i x_i$. From (28) and Proposition 8, it follows that

$$\begin{aligned} \mathbb{E}_\pi \left[\sum_i W_i x_i \right] &\geq \mathbb{E}_\pi [T(\mathbf{x})] \\ &\geq \max_{\mathbf{x} \in \Omega} T(\mathbf{x}) - \log |\Omega| \\ &\geq T(\hat{\mathbf{x}}) - \log |\Omega| \\ &= \left(\sum_i W_i \hat{x}_i \right) - O(1) - \log |\Omega| \\ (29) \quad &= \left(\max_{\mathbf{x} \in \Omega} \mathbf{W} \cdot \mathbf{x} \right) - O(1). \end{aligned}$$

From the definition of weight in both algorithms ((3) and (4)) for a given \mathbf{Q} , weight $\mathbf{W} = [W_i]$ is defined as

$$W_i = \max \left\{ f(Q_i), \sqrt{f(Q_{\max})} \right\}.$$

Define $\eta \triangleq \frac{\varepsilon}{4 \max_{\mathbf{x} \in \Omega} \|\mathbf{x}\|_1}$. To establish the proof of Lemma 7, we will consider Q_{\max} such that it is large enough satisfying

$$\eta f(Q_{\max}) \geq \sqrt{f(Q_{\max})}.$$

For smaller Q_{\max} we do not need to argue as in that case (26) (due to $O(1)$ term) is straightforward. Therefore, in the remainder we assume Q_{\max} large enough. For this large enough Q_{\max} , it follows that for all i ,

$$(30) \quad 0 \leq W_i - f(Q_i) \leq \sqrt{f(Q_{\max})} \leq \eta f(Q_{\max})$$

Using (30), for any $\mathbf{x} \in \Omega$,

$$\begin{aligned}
0 &\leq \mathbf{W} \cdot \mathbf{x} - f(\mathbf{Q}) \cdot \mathbf{x} = (\mathbf{W} - f(\mathbf{Q})) \cdot \mathbf{x} \\
&\leq \|\mathbf{x}\|_1 \|\mathbf{W} - f(\mathbf{Q})\|_\infty \\
&\leq \|\mathbf{x}\|_1 \times \eta f(Q_{\max}) \\
&\stackrel{(a)}{\leq} \frac{\varepsilon}{4} f(Q_{\max}) \\
(31) \quad &\stackrel{(b)}{\leq} \frac{\varepsilon}{4} \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}) \cdot \mathbf{y} \right),
\end{aligned}$$

where (a) is from our choice of $\eta = \frac{\varepsilon}{4 \max_{\mathbf{x} \in \Omega} \|\mathbf{x}\|_1}$. For (b), we use the fact that the singleton set $\{i\}$, i.e. independent set $\{i\}$ for wireless network and a single active on route i for circuit switched network, is a valid schedule. And, for $i = \arg \max_j Q_j$, it has weight $f(Q_{\max})$. Therefore, the weight of the maximum weighted schedule among all possible schedules in Ω is at least $f(Q_{\max})$. Finally, using (29) and (31) we obtain

$$\begin{aligned}
\mathbb{E}_\pi [f(\mathbf{Q}) \cdot \mathbf{x}] &\geq \mathbb{E}_\pi [\mathbf{W} \cdot \mathbf{x}] - \frac{\varepsilon}{4} \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}) \cdot \mathbf{y} \right) \\
&\geq \left(\max_{\mathbf{y} \in \Omega} \mathbf{W} \cdot \mathbf{y} \right) - O(1) - \frac{\varepsilon}{4} \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}) \cdot \mathbf{y} \right) \\
&\geq \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}) \cdot \mathbf{y} \right) - O(1) - \frac{\varepsilon}{4} \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}) \cdot \mathbf{y} \right) \\
&= \left(1 - \frac{\varepsilon}{4} \right) \left(\max_{\mathbf{y} \in \Omega} f(\mathbf{Q}) \cdot \mathbf{y} \right) - O(1).
\end{aligned}$$

This completes the proof of Lemma 7.

5.3.2. Step Two. Let $\mu(t)$ be the distribution of schedule $\mathbf{x}(t)$ over Ω at time t , given initial state $X(0) = (\mathbf{Q}(0), \mathbf{x}(0))$. We wish to show that for any initial condition $\mathbf{x}(0) \in \Omega$, for t large (but not too large) enough, $\mu(t)$ is close to $\pi(0)$ if $Q_{\max}(0)$ is large enough. Formal statement is as follows.

Lemma 9 *For a large enough $Q_{\max}(0)$,*

$$(32) \quad \|\mu(t) - \pi(0)\|_{TV} < \varepsilon/4,$$

for $t \in I = [b_1(Q_{\max}(0)), b_2(Q_{\max}(0))]$, where b_1, b_2 are integer-valued functions on \mathbb{R}_+ such that

$$b_1, b_2 = \text{polylog}(Q_{\max}(0)) \quad \text{and} \quad b_2/b_1 = \Theta(\log(Q_{\max}(0))).$$

In above the constants may depend on ε, C_{\max} and n .

The notation $\text{polylog}(z)$ represents a positive real-valued function of z that scales no faster than a finite degree polynomial of $\log z$.

Proof of Lemma 9. We shall prove this Lemma for the wireless network. The proof of buffered circuit switch network follows in an identical manner. Hence we shall skip it. Therefore, we shall assume $\Omega = \mathcal{I}(G)$ and $\mathbf{x}(t) = \boldsymbol{\sigma}(t)$.

First, we establish the desired claim for integral times. The argument for non integral times will follow easily as argued near the end of this proof. For $t = \tau \in \mathbb{Z}_+$, we have

$$\begin{aligned} \mu(\tau + 1) &= \mathbb{E} [\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau+1)}] \\ &= \mathbb{E} [\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \cdot P(\tau)], \end{aligned}$$

where recall that $P(\tau) = e^{n(GW(\mathbf{W}(\tau)) - I)}$ and the last equality follows from (14). Again recall that the expectation is with respect to the joint distribution of $\{\mathbf{Q}(\tau), \boldsymbol{\sigma}(\tau)\}$. Hence, it follows that

$$\begin{aligned} \mu(\tau + 1) &= \mathbb{E} [\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \cdot P(\tau)] \\ &= \mathbb{E} \left[\mathbb{E} [\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \cdot P(\tau) \mid \mathbf{Q}(\tau)] \right] \\ &\stackrel{(a)}{=} \mathbb{E} \left[\mathbb{E} [\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \mid \mathbf{Q}(\tau)] \cdot P(\tau) \right] \\ &= \mathbb{E} [\tilde{\mu}(\tau) \cdot P(\tau)], \end{aligned}$$

where

$$\tilde{\mu}(\tau) = \tilde{\mu}(\mathbf{Q}(\tau)) \triangleq \mathbb{E} [\boldsymbol{\delta}_{\boldsymbol{\sigma}(\tau)} \mid \mathbf{Q}(\tau)].$$

In above the expectation is taken with respect to the conditional marginal distribution of $\boldsymbol{\sigma}(\tau)$ given $\mathbf{Q}(\tau)$; (a) follows since $P(\tau)$ is a function of $\mathbf{Q}(\tau)$. Next, we establish the relation between $\mu(\tau)$ and $\mu(\tau + 1)$.

$$\begin{aligned} \mu(\tau + 1) &= \mathbb{E} [\tilde{\mu}(\tau) \cdot P(\tau)] \\ &= \mathbb{E} [\tilde{\mu}(\tau) \cdot P(0)] + \mathbb{E} [\tilde{\mu}(\tau) \cdot (P(\tau) - P(0))] \\ &= \mathbb{E} [\tilde{\mu}(\tau)] \cdot P(0) + e(\tau) \\ &= \mu(\tau) \cdot P(0) + e(\tau), \end{aligned}$$

where $e(\tau) \triangleq \mathbb{E} [\tilde{\mu}(\tau) \cdot (P(\tau) - P(0))]$. Here the expectation is with respect to $\mathbf{Q}(\tau)$. Similarly,

$$\begin{aligned} \mu(\tau + 1) &= \mu(\tau) \cdot P(0) + e(\tau) \\ &= (\mu(\tau - 1) \cdot P(0) + e(\tau - 1)) \cdot P(0) + e(\tau) \\ &= \mu(\tau - 1) \cdot P(0)^2 + e(\tau - 1) \cdot P(0) + e(\tau). \end{aligned}$$

Therefore, recursively we obtain

$$(33) \quad \mu(\tau + 1) = \mu(0) \cdot P(0)^{\tau+1} + \sum_{s=0}^{\tau} e(\tau - s) \cdot P(0)^s.$$

We will choose b_1 (which will depend on $Q_{\max}(0)$) such that for $\tau \geq b_1$,

$$(34) \quad \|\mu(0) \cdot P(0)^\tau - \pi(0)\|_{TV} \leq \varepsilon/8.$$

That is, b_1 is the *mixing time* of $P(0)$. Using inequalities (8), (6) and Lemma 3, it follows that

$$b_1 \equiv b_1(Q_{\max}(0)) = \text{polylog}(Q_{\max}(0)).$$

In above, constants may depend on n and ε . Therefore, from (33) and (34), it suffices to show that

$$(35) \quad \left\| \sum_{s=0}^{\tau-1} e(\tau - 1 - s) \cdot P(0)^s \right\|_1 \leq \varepsilon/8,$$

for $\tau \in I = [b_1, b_2]$ with an appropriate choice of $b_2 = b_2(Q_{\max}(0))$. To this end, we choose

$$b_2 \equiv b_2(Q_{\max}(0)) = \lceil b_1 \log(Q_{\max}(0)) \rceil.$$

Thus, $b_2(Q_{\max}(0)) = \text{polylog}(Q_{\max}(0))$ as well. With this choice of b_2 , we obtain the following bound on $e(\tau)$ to conclude (35).

$$(36) \quad \begin{aligned} \|e(\tau)\|_1 &= \|\mathbb{E}[\tilde{\mu}(\tau) \cdot (P(\tau) - P(0))]\|_1 \\ &\leq \mathbb{E}[\|\tilde{\mu}(\tau) \cdot (P(\tau) - P(0))\|_1] \\ &\stackrel{(a)}{\leq} O(\mathbb{E}[\|P(\tau) - P(0)\|_\infty]) \\ &\stackrel{(b)}{\leq} O(\mathbb{E}[\|GW(\mathbf{W}(\tau)) - GW(\mathbf{W}(0))\|_\infty]) \\ &\stackrel{(c)}{\leq} O\left(\mathbb{E}\left[\max_i \left| \frac{1}{1 + \exp(W_i(\tau))} - \frac{1}{1 + \exp(W_i(0))} \right| \right]\right) \\ &\stackrel{(d)}{\leq} O\left(\mathbb{E}\left[\max_i |W_i(\tau) - W_i(0)|\right]\right) \\ &\stackrel{(e)}{\leq} O\left(\max_i \mathbb{E}[|W_i(\tau) - W_i(0)|]\right). \end{aligned}$$

In above, (a) follows from the standard norm inequality and the fact that $\|\tilde{\mu}(\tau)\|_1 = 1$, (b) follows from Lemma 11 in Appendix, (c) follows

directly from the definition of transition matrix $GD(\mathbf{W})$, (d) follows from 1-Lipschitz⁷ property of function $1/(1 + e^x)$ and (e) follows from the fact that vector $\mathbf{W}(\tau)$ being $O(1)$ dimensional⁸.

Next, we will show that for all i and $\tau \leq b_2$,

$$(37) \quad \mathbb{E} [|W_i(\tau) - W_i(0)|] = O\left(\frac{1}{\text{superpolylog}(Q_{\max}(0))}\right),$$

the notation $\text{superpolylog}(z)$ represents a positive real-valued function of z that scales faster than any finite degree polynomial of $\log z$. This is enough to conclude (35) (hence complete the proof of Lemma 9) since

$$\begin{aligned} \left\| \sum_{s=0}^{\tau-1} e(\tau-1-s) \cdot P(0)^s \right\|_1 &\leq \sum_{s=0}^{\tau-1} \|e(\tau-1-s) \cdot P(0)^s\|_1 \\ &= \sum_{s=0}^{\tau-1} O(\|e(\tau-1-s)\|_1) \\ &\stackrel{(a)}{=} O\left(\frac{\tau}{\text{superpolylog}(Q_{\max}(0))}\right) \\ &\stackrel{(b)}{\leq} \frac{\varepsilon}{4}, \end{aligned}$$

where we use (36) and (37) to obtain (a), (b) holds for large enough $Q_{\max}(0)$ and $\tau \leq b_2 = \text{polylog}(Q_{\max}(0))$.

Now to complete the proof, we only need to establish (37). This is the step that utilizes ‘slowly varying’ property of function $f(x) = \log \log(x + e)$. First, we provide an intuitive sketch of the argument. Somewhat involved details will be follow. To explain the intuition behind (37), let us consider a simpler situation where i is such that $Q_i(0) = Q_{\max}(0)$ and $f(Q_i(\tau)) > \sqrt{f(Q_{\max}(\tau))}$ for a given $\tau \in [0, b_2]$. That is, let $W_i(\tau) = f(Q_i(\tau))$. Now,

⁷A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is k -Lipschitz if $|f(s) - f(t)| \leq k|s - t|$ for all $s, t \in \mathbb{R}$.

⁸We note here that the $O(\cdot)$ notation means existences of constants that do not depend scaling quantities such as time τ and $\mathbf{Q}(0)$; however it may depend on the fixed system parameters such as number of queues. The use of this terminology is to retain the clarity of exposition.

consider following sequence of inequalities:

$$\begin{aligned}
|W_i(\tau) - W_i(0)| &= |f(Q_i(\tau)) - f(Q_i(0))| \\
&\stackrel{(a)}{\leq} f'(\zeta)|Q_i(\tau) - Q_i(0)|, \quad \text{for some } \zeta \text{ around } Q_i(0) \\
&\stackrel{(b)}{\leq} f'(\min\{Q_i(\tau), Q_i(0)\})O(\tau) \\
&\stackrel{(c)}{\leq} f'(Q_i(0) - O(\tau))O(\tau) \\
(38) \quad &\stackrel{(d)}{=} O\left(\frac{\tau}{Q_i(0)}\right).
\end{aligned}$$

In above, (a) follows from the mean value theorem; (b) follows from monotonicity of f' and Lipschitz property of $Q_i(\cdot)$ (as a function of τ) – which holds deterministically for wireless network and probabilistically for circuit switched network; (c) uses the same Lipschitz property; and (d) uses the fact that $\tau \leq b_2$ and $b_2 = \text{polylog}(Q_{\max}(0))$, $Q_{\max}(0) = Q_i(0)$. Therefore, effectively the bound of (38) is $O(1/\text{superpolylog}(Q_{\max}(0)))$.

The above explains the gist of the argument that is to follow. However, to make it precise, we will need to provide lots more details. Toward this, we consider the following two cases: (i) $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$, and (ii) $f(Q_i(0)) < \sqrt{f(Q_{\max}(0))}$. In what follows, we provide detailed arguments for (i). The arguments for case (ii) are similar in spirit and will be provided later in the proof.

Case (i): Consider an i such that $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$. Then,

$$\begin{aligned}
&\mathbb{E}[|W_i(\tau) - W_i(0)|] \\
&= \mathbb{E}[|W_i(\tau) - f(Q_i(0))|] \\
&= \mathbb{E}\left[|f(Q_i(\tau)) - f(Q_i(0))| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))}\}}\right] \\
(39) \quad &+ \mathbb{E}\left[|\sqrt{f(Q_{\max}(\tau))} - f(Q_i(0))| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))}\}}\right],
\end{aligned}$$

where each equality follows from (3). The first term in (39) can be bounded as follows

$$\begin{aligned}
& \mathbb{E} \left[|f(Q_i(\tau)) - f(Q_i(0))| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))}\}} \right] \\
& \leq \mathbb{E} [|f(Q_i(\tau)) - f(Q_i(0))|] \\
& \stackrel{(o)}{\leq} \mathbb{E} [f'(\min\{Q_i(\tau), Q_i(0)\}) |Q_i(\tau) - Q_i(0)|] \\
& \stackrel{(a)}{\leq} \sqrt{\mathbb{E} [f'(\min\{Q_i(\tau), Q_i(0)\})^2]} \cdot \sqrt{\mathbb{E} [(Q_i(\tau) - Q_i(0))^2]} \\
& \stackrel{(b)}{\leq} \sqrt{f' \left(\frac{Q_i(0)}{2} \right)^2 + \Theta \left(\frac{\tau}{Q_i(0)} \right)} \cdot O(\tau) \\
& \stackrel{(c)}{\leq} \sqrt{f' \left(\frac{1}{2} f^{-1} \left(\sqrt{f(Q_{\max}(0))} \right) \right)^2 + \Theta \left(\frac{\tau}{f^{-1} \left(\sqrt{f(Q_{\max}(0))} \right)} \right)} \cdot O(\tau) \\
(40) \stackrel{(d)}{=} O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right).
\end{aligned}$$

In above, (o) follows from concavity of f . For (a) we use the standard Cauchy-Schwarz inequality $\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]} \sqrt{\mathbb{E}[Y^2]}$. For (b), note that given $Q_i(0)$, $\mathbb{E}[(Q_i(0) - Q_i(\tau))^2] = O(\tau^2)$ for both network models – for wireless network, it is deterministically true due to Lipschitz property of $\mathbf{Q}(\cdot)$; for circuit switched network, it is due to the fact that the arrival as well as (the overall) departure processes are bounded rate Poisson processes. Given this, using Markov's inequality it follows that

$$\Pr \left(\min\{Q_i(\tau), Q_i(0)\} \leq \frac{Q_i(0)}{2} \right) = O \left(\frac{\tau}{Q_i(0)} \right).$$

Finally, using the fact that $\sup_{y \in \mathbb{R}_+} f'(y) = O(1)$, we obtain (b). Now (c) follows from the condition of $Q_i(0)$ that $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$. And, (d) is implied by $\tau \leq b_2 = \text{polylog}(Q_{\max}(0))$, $f(x) = \log \log(x + e)$.

Next, we bound the second term in (39). We will use notation

$$\begin{aligned}
A(\tau) &= \left\{ f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))} \ \& \ \sqrt{f(Q_{\max}(\tau))} \geq f(Q_i(0)) \right\}, \\
B(\tau) &= \left\{ f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))} \ \& \ \sqrt{f(Q_{\max}(\tau))} < f(Q_i(0)) \right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
& \mathbb{E} \left[\left| \sqrt{f(Q_{\max}(\tau))} - f(Q_i(0)) \right| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(\tau))}\}} \right] \\
&= \mathbb{E} \left[\left(\sqrt{f(Q_{\max}(\tau))} - f(Q_i(0)) \right) \cdot \mathbf{I}_{A(\tau)} \right] \\
&\quad + \mathbb{E} \left[\left(f(Q_i(0)) - \sqrt{f(Q_{\max}(\tau))} \right) \cdot \mathbf{I}_{B(\tau)} \right] \\
&\stackrel{(a)}{\leq} \mathbb{E} \left[\left(\sqrt{f(Q_{\max}(\tau))} - \sqrt{f(Q_{\max}(0))} \right) \cdot \mathbf{I}_{A(\tau)} \right] \\
&\quad + \mathbb{E} \left[\left(f(Q_i(0)) - f(Q_i(\tau)) \right) \cdot \mathbf{I}_{B(\tau)} \right] \\
&\stackrel{(b)}{\leq} \mathbb{E} [|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|] + \mathbb{E} [|f(Q_i(0)) - f(Q_i(\tau))|] \\
(41) \quad &= O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right).
\end{aligned}$$

In above, (a) follows because we are considering case (i) with $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$ and definition of event $B(\tau)$; (b) follows from 1-Lipschitz property of $\sqrt{\cdot}$ function and appropriate removal of indicator random variables. For the final conclusion, we observe that the arguments used to establish (40) imply the $O(1/\text{superpolylog}(Q_{\max}(0)))$ bound on both the terms in very similar manner: for the term corresponding to $|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|$, one has to adapt arguments of (40) by essentially replacing the index i by \max . This concludes the proof of (37) for case (i) of $f(Q_i(0)) \geq \sqrt{f(Q_{\max}(0))}$.

Case (ii): Now consider i such that $f(Q_i(0)) < \sqrt{f(Q_{\max}(0))}$. Then,

$$\begin{aligned}
& \mathbb{E} [|W_i(\tau) - W_i(0)|] = \mathbb{E} \left[\left| W_i(\tau) - \sqrt{f(Q_{\max}(0))} \right| \right] \\
&= \mathbb{E} \left[\left| f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(0))}\}} \right] \\
(42) \quad &+ \mathbb{E} \left[\left| \sqrt{f(Q_{\max}(\tau))} - \sqrt{f(Q_{\max}(0))} \right| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(0))}\}} \right].
\end{aligned}$$

First observe that by 1-Lipschitz property of $\sqrt{\cdot}$ function, the second term can be bounded as (similar to (41))

$$\begin{aligned}
& \mathbb{E} \left[\left| \sqrt{f(Q_{\max}(\tau))} - \sqrt{f(Q_{\max}(0))} \right| \cdot \mathbf{I}_{\{f(Q_i(\tau)) < \sqrt{f(Q_{\max}(0))}\}} \right] \\
&\leq \mathbb{E} [|f(Q_{\max}(\tau)) - f(Q_{\max}(0))|] \\
(43) \quad &= O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right).
\end{aligned}$$

Therefore, we are left with proving the first term of (42). We will follow similar line of arguments as those used for (41). Define

$$A'(\tau) = \left\{ f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))} \ \& \ \sqrt{f(Q_{\max}(0))} \geq f(Q_i(\tau)) \right\},$$

$$B'(\tau) = \left\{ f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))} \ \& \ \sqrt{f(Q_{\max}(0))} < f(Q_i(\tau)) \right\}.$$

Then,

$$\begin{aligned} & \mathbb{E} \left[\left| f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right| \cdot \mathbf{I}_{\{f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))}\}} \right] \\ &= \mathbb{E} \left[\left(\sqrt{f(Q_{\max}(0))} - f(Q_i(\tau)) \right) \cdot \mathbf{I}_{A(\tau)} \right] \\ &\quad + \mathbb{E} \left[\left(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right) \cdot \mathbf{I}_{B(\tau)} \right] \\ &\stackrel{(a)}{\leq} \mathbb{E} \left[\left(\sqrt{f(Q_{\max}(0))} - \sqrt{f(Q_{\max}(\tau))} \right) \cdot \mathbf{I}_{A(\tau)} \right] \\ &\quad + \mathbb{E} \left[\left(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right) \cdot \mathbf{I}_{B(\tau)} \right] \\ &\stackrel{(b)}{\leq} O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right) \\ &\quad + \mathbb{E} \left[\left(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right) \cdot \mathbf{I}_{B(\tau)} \right]. \end{aligned} \tag{44}$$

In above, (a) follows because we are considering case (i) with $f(Q_i(\tau)) \geq \sqrt{f(Q_{\max}(\tau))}$ and definition of event $B(\tau)$; (b) follows from 1-Lipschitz property of $\sqrt{\cdot}$ function and appropriate removal of indicator random variables as follows:

$$\begin{aligned} & \mathbb{E} \left[\left(\sqrt{f(Q_{\max}(0))} - \sqrt{f(Q_{\max}(\tau))} \right) \cdot \mathbf{I}_{A(\tau)} \right] \\ &\leq \mathbb{E} \left[|f(Q_{\max}(\tau)) - f(Q_{\max}(0))| \right] \\ &= O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right). \end{aligned} \tag{45}$$

Finally, to complete the proof of case (ii) using (42), we wish to establish

$$\mathbb{E} \left[\left(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right) \cdot \mathbf{I}_{B(\tau)} \right] = O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right). \tag{46}$$

Now suppose $x \in \mathbb{R}_+$ be such that $f(x) = \sqrt{f(Q_{\max}(0))}$. Then,

$$\begin{aligned}
& \mathbb{E} \left[\left(f(Q_i(\tau)) - \sqrt{f(Q_{\max}(0))} \right) \cdot \mathbf{I}_{B(\tau)} \right] \\
&= \mathbb{E} \left[(f(Q_i(\tau)) - f(x)) \cdot \mathbf{I}_{B(\tau)} \right] \\
&\stackrel{(a)}{\leq} \mathbb{E} \left[f'(x)(Q_i(\tau) - x) \cdot \mathbf{I}_{B(\tau)} \right] \\
&= f'(x) \mathbb{E} \left[(Q_i(\tau) - x) \cdot \mathbf{I}_{B(\tau)} \right] \\
&\stackrel{(b)}{\leq} f'(x) \mathbb{E} \left[(Q_i(\tau) - Q_i(0)) \cdot \mathbf{I}_{B(\tau)} \right] \\
&\leq f'(x) \mathbb{E} [|Q_i(\tau) - Q_i(0)|] \\
&\stackrel{(c)}{=} f'(x) O(\tau) \\
(47) \quad &\stackrel{(d)}{=} O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right).
\end{aligned}$$

In above, (a) follows from concavity of f ; (b) from $Q_i(0) \leq x$ and $Q_i(\tau) \geq x$ implied by case (ii) and $B(\tau)$ respectively; (c) follows from arguments used earlier that for any i , $\mathbb{E}[(Q_i(\tau) - Q_i(0))^2] = O(\tau^2)$; (d) follows from $\tau \leq b_2 = \text{polylog}(Q_{\max}(0))$ and

$$f'(x) = O \left(\frac{1}{\text{superpolylog}(Q_{\max}(0))} \right).$$

This complete the proof of (37) for both cases and the proof of Lemma 9 for integral time steps. A final remark validity of this result about non-integral times is in order.

Consider $t \in I$ and $t \notin \mathbb{Z}_+$. Let $\tau = \lfloor t \rfloor$ and $t = \tau + \delta$ for $\delta \in (0, 1)$. Then, it follows that (using formal definition P^δ as in (15))

$$\begin{aligned}
(48) \quad \mu(t) &= \mu(\tau + \delta) = \mu(\tau)P^\delta(0) + \mathbb{E} \left[\tilde{\mu}(\tau)(P^\delta(\tau) - P^\delta(0)) \right] \\
&= \mu(0)P(0)^\tau P^\delta(0) + e(\tau + \delta).
\end{aligned}$$

Now it can be checked that $P^\delta(0)$ is a probability matrix and has $\pi(0)$ as its stationary distribution for any $\delta > 0$; and we have argued that for τ large enough $\mu(0)P(0)^\tau$ is close to $\pi(0)$. Therefore, $\mu(0)P(0)^\tau P^\delta(0)$ is also equally close to $\pi(0)$. For $e(\tau + \delta)$, it can be easily argued that the bound obtained in (36) for $e(\tau + 1)$ will dominate the bound for $e(\tau + \delta)$. Therefore, the statement of Lemma holds for any non-integral t as well. This complete the proof of Lemma 9.

5.3.3. *Step Three: Wireless Network.* In this section, we prove Lemma 6 for the wireless network model. For Markov process $X(t) = (\mathbf{Q}(t), \boldsymbol{\sigma}(t))$, we consider Lyapunov function

$$L(X(t)) = \sum_i F(Q_i(t)),$$

where $F(x) = \int_0^x f(y) dy$ and recall that $f(x) = \log \log(x + e)$. For this Lyapunov function, it suffices to find appropriate functions h and g as per Lemma 6 for a large enough $Q_{\max}(0)$. Therefore, we assume that $Q_{\max}(0)$ is large enough so that it satisfies the conditions of Lemma 9. To this end, from Lemma 9, we have that for $t \in I$,

$$|\mathbb{E}_{\pi(0)}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}] - \mathbb{E}_{\mu(t)}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}]| \leq \frac{\varepsilon}{4} \left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho} \right).$$

Thus from Lemma 7, it follows that
(49)

$$\mathbb{E}_{\mu(t)}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}] \geq \left(1 - \frac{\varepsilon}{2}\right) \left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho} \right) - O(1).$$

Now we can bound the difference between $L(X(\tau + 1))$ and $L(X(\tau))$ as follows.

$$\begin{aligned} L(X(\tau + 1)) - L(X(\tau)) &= (F(\mathbf{Q}(\tau + 1)) - F(\mathbf{Q}(\tau))) \cdot \mathbf{1} \\ &\leq f(\mathbf{Q}(\tau + 1)) \cdot (\mathbf{Q}(\tau + 1) - \mathbf{Q}(\tau)), \\ &\leq f(\mathbf{Q}(\tau)) \cdot (\mathbf{Q}(\tau + 1) - \mathbf{Q}(\tau)) + n, \end{aligned}$$

where the first inequality is from the convexity of F and the last inequality follows from the fact that $f(Q)$ is 1-Lipschitz. Therefore,

$$\begin{aligned} L(X(\tau + 1)) - L(X(\tau)) &= (F(\mathbf{Q}(\tau + 1)) - F(\mathbf{Q}(\tau))) \cdot \mathbf{1} \\ &\leq f(\mathbf{Q}(\tau)) \cdot \left(A(\tau, \tau + 1) - \int_{\tau}^{\tau+1} \boldsymbol{\sigma}(y) \mathbf{1}_{\{Q_i(y) > 0\}} dy \right) + n \\ &\stackrel{(a)}{\leq} f(\mathbf{Q}(\tau)) \cdot A(\tau, \tau + 1) - \int_{\tau}^{\tau+1} f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y) \mathbf{1}_{\{Q_i(y) > 0\}} dy + 2n \\ (50) \quad &= f(\mathbf{Q}(\tau)) \cdot A(\tau, \tau + 1) - \int_{\tau}^{\tau+1} f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y) dy + 2n, \end{aligned}$$

where again (a) follows from the fact that $f(Q)$ is 1-Lipschitz. Given initial state $X(0) = \mathbf{x}$, taking the expectation of (50) for $\tau, \tau + 1 \in I$,

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}}[L(X(\tau + 1)) - L(X(\tau))] \\ & \leq \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(\tau)) \cdot A(\tau, \tau + 1)] - \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y)] dy + 2n \\ & = \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(\tau)) \cdot \boldsymbol{\lambda}] - \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(y)) \cdot \boldsymbol{\sigma}(y)] dy + 2n, \end{aligned}$$

where the last equality follows from the independence between $\mathbf{Q}(\tau)$ and $A(\tau, \tau + 1)$ (recall, Bernoulli arrival process). Therefore,

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}}[L(X(\tau + 1)) - L(X(\tau))] \\ & \leq \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(\tau)) \cdot \boldsymbol{\lambda}] - \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}(y)] dy \\ & \quad - \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}}[(f(\mathbf{Q}(y)) - f(\mathbf{Q}(0))) \cdot \boldsymbol{\sigma}(y)] dy + 2n \\ & \stackrel{(a)}{\leq} f(\mathbf{Q}(0) + \tau \cdot \mathbf{1}) \cdot \boldsymbol{\lambda} - \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}}[f(\mathbf{Q}(0)) \cdot \boldsymbol{\sigma}(y)] dy \\ & \quad - \int_{\tau}^{\tau+1} (f(\mathbf{Q}(0) - y \cdot \mathbf{1}) - f(\mathbf{Q}(0))) \cdot \mathbf{1} dy + 2n \\ & \stackrel{(b)}{\leq} f(\mathbf{Q}(0)) \cdot \boldsymbol{\lambda} + f(\tau \cdot \mathbf{1}) \cdot \boldsymbol{\lambda} - \left(1 - \frac{\varepsilon}{2}\right) \left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\right) \\ & \quad + \int_{\tau}^{\tau+1} f(y \cdot \mathbf{1}) \cdot \mathbf{1} dy + O(1) \\ & \leq f(\mathbf{Q}(0)) \cdot \boldsymbol{\lambda} - \left(1 - \frac{\varepsilon}{2}\right) \left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\right) + 2nf(\tau + 1) + O(1). \end{aligned}$$

In above, (a) uses Lipschitz property of $\mathbf{Q}(\cdot)$ (as a function of τ); (b) follows from (49) and the inequality that for $f(x) = \log \log(x + e)$, $f(x) + f(y) + \log 2 \geq f(x + y)$ for all $x, y \in \mathbb{R}_+$. The $O(1)$ term is constant, dependent on n , and captures the constant from (49).

Now since $\boldsymbol{\lambda} \in (1 - \varepsilon) \text{Conv}(\mathcal{I}(G))$, we obtain

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}}[L(X(\tau + 1)) - L(X(\tau))] \\ & \leq -\frac{\varepsilon}{2} \left(\max_{\boldsymbol{\rho} \in \mathcal{I}(G)} f(\mathbf{Q}(0)) \cdot \boldsymbol{\rho}\right) + 2nf(\tau + 1) + O(1) \\ & \leq -\frac{\varepsilon}{2} f(Q_{\max}(0)) + 2nf(\tau + 1) + O(1). \end{aligned}$$

Therefore, summing τ from $b_1 = b_1(Q_{\max}(0))$ to $b_2 = b_2(Q_{\max}(0))$, we have

$$\begin{aligned}
& \mathbb{E}_{\mathbf{x}} [L(X(b_2)) - L(X(b_1))] \\
& \leq -\frac{\varepsilon}{2}(b_2 - b_1)f(Q_{\max}(0)) + 2n \sum_{\tau=b_1}^{b_2-1} f(\tau + 1) + O(b_2 - b_1) \\
(51) \quad & \leq -\frac{\varepsilon}{2}(b_2 - b_1)f(Q_{\max}(0)) + 2n(b_2 - b_1)f(b_2) + O(b_2 - b_1).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& \mathbb{E}_{\mathbf{x}} [L(X(b_2)) - L(X(0))] \\
& = \mathbb{E}_{\mathbf{x}} [L(X(b_1)) - L(X(0))] + \mathbb{E}_{\mathbf{x}} [L(X(b_2)) - L(X(b_1))] \\
& \stackrel{(a)}{\leq} \mathbb{E}_{\mathbf{x}} [f(\mathbf{Q}(b_1)) \cdot (\mathbf{Q}(b_1) - \mathbf{Q}(0))] - \frac{\varepsilon}{2}(b_2 - b_1)f(Q_{\max}(0)) \\
& \quad + 2n \sum_{\tau=b_1}^{b_2-1} f(\tau + 1) + O(b_2 - b_1) \\
& \stackrel{(b)}{\leq} nb_1 f(Q_{\max}(0) + b_1) - \frac{\varepsilon}{2}(b_2 - b_1)f(Q_{\max}(0)) \\
(52) \quad & \quad + 2n(b_2 - b_1)f(b_2) + O(b_2 - b_1),
\end{aligned}$$

where (a) follows from the convexity of L and (b) is due to the 1-Lipschitz property of \mathbf{Q} . Now if we choose $g(\mathbf{x}) = b_2$ and

$$h(\mathbf{x}) = -nb_1 f(Q_{\max}(0) + b_1) + \frac{\varepsilon}{2}(b_2 - b_1)f(Q_{\max}(0)) - 2n(b_2 - b_1)f(b_2) - O(b_2 - b_1),$$

the desired inequality follows:

$$\mathbb{E}_{\mathbf{x}} [L(X(g(\mathbf{x}))) - L(X(0))] \leq -h(\mathbf{x}).$$

The desired conditions of Lemma 6 can be checked as follows. First observe that with respect to $Q_{\max}(0)$, the function h scales as $b_2(Q_{\max}(0))f(Q_{\max}(0))$ due to $b_2/b_1 = \Theta(\log Q_{\max}(0))$ as per Lemma 9. Further, h is a function that is lower bounded and its value goes to ∞ as $Q_{\max}(0)$ goes to ∞ . Therefore, h/g scales as $f(Q_{\max}(0))$. These properties will imply the verification conditions of Lemma 6.

5.3.4. Step Three: Buffered Circuit Switched Network. In this section, we prove Lemma 6 for the circuit switched network model. Similar to wireless network, we are interested in large enough $Q_{\max}(0)$ that satisfies condition

of Lemma 9. Given the state $X(t) = (\mathbf{Q}(t), \mathbf{z}(t))$ of the Markov process, we shall consider the following Lyapunov function :

$$L(X(t)) = \sum_i F(R_i(t)).$$

Here $\mathbf{R}(t) = [R_i(t)]$ with $R_i(t) = Q_i(t) + z_i(t)$ and as before $F(x) = \int_0^x f(y) dy$. Now we proceed towards finding appropriate functions h and g as desired in Lemma 6. For any $\tau \in \mathbb{Z}_+$,

$$\begin{aligned} & L(X(\tau+1)) - L(X(\tau)) \\ &= (F(\mathbf{R}(\tau+1)) - F(\mathbf{R}(\tau))) \cdot \mathbf{1} \\ &\leq f(\mathbf{R}(\tau+1)) \cdot (\mathbf{R}(\tau+1) - \mathbf{R}(\tau)), \\ &= f(\mathbf{R}(\tau) + A(\tau, \tau+1) - D(\tau, \tau+1)) \cdot (A(\tau, \tau+1) - D(\tau, \tau+1)) \\ &\leq f(\mathbf{R}(\tau)) \cdot (A(\tau, \tau+1) - D(\tau, \tau+1)) + \|A(\tau, \tau+1) - D(\tau, \tau+1)\|_2^2. \end{aligned}$$

Given initial state $X(0) = \mathbf{x}$, taking expectation for $\tau, \tau+1 \in I$, we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}}[L(X(\tau+1)) - L(X(\tau))] \\ &\leq \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot A(\tau, \tau+1)] - \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot D(\tau, \tau+1)] \\ &\quad + \mathbb{E}_{\mathbf{x}}[\|A(\tau, \tau+1) - D(\tau, \tau+1)\|_2^2] \\ (53) \quad &= \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot \boldsymbol{\lambda}] - \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot D(\tau, \tau+1)] + O(1). \end{aligned}$$

The last equality follows from the fact that arrival process is Poisson with rate vector $\boldsymbol{\lambda}$ and $\mathbf{R}(\tau)$ is independent of $A(\tau, \tau+1)$. In addition, the overall departure process for any i , $D_i(\cdot)$, is governed by a Poisson process of rate at most C_{\max} . Therefore, the second moment of the difference of arrival and departure processes in unit time is $O(1)$. Now,

$$(54) \quad \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot \boldsymbol{\lambda}] = f(\mathbf{R}(0)) \cdot \boldsymbol{\lambda} + \mathbb{E}_{\mathbf{x}}[(f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))) \cdot \boldsymbol{\lambda}].$$

And,

$$(55) \quad \begin{aligned} & \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(\tau)) \cdot D(\tau, \tau+1)] \\ &= \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(0)) \cdot D(\tau, \tau+1)] + \mathbb{E}_{\mathbf{x}}[(f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))) \cdot D(\tau, \tau+1)]. \end{aligned}$$

The first term on the right hand side in (54) can be bounded as

$$(56) \quad \begin{aligned} f(\mathbf{R}(0)) \cdot \boldsymbol{\lambda} &\leq (1 - \varepsilon) \left(\max_{\mathbf{y} \in \mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y} \right) \\ &\leq -\frac{3\varepsilon}{4} \left(\max_{\mathbf{y} \in \mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y} \right) + \mathbb{E}_{\pi(0)}[f(\mathbf{R}(0)) \cdot \mathbf{z}] + O(1), \end{aligned}$$

where the first inequality is due to $\boldsymbol{\lambda} \in (1 - \varepsilon) \text{Conv}(\mathcal{X})$ and the second inequality follows from Lemma 7 with the fact that $|f_i(\mathbf{R}(\tau)) - f_i(\mathbf{Q}(\tau))| < f(C_{\max}) = O(1)$ for all i . On the other hand, the first term in the right hand side of (55) can be bounded below as

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}} [f(\mathbf{R}(0)) \cdot D(\tau, \tau + 1)] &= f(\mathbf{R}(0)) \cdot \mathbb{E}_{\mathbf{x}} [D(\tau, \tau + 1)] \\
&\geq f(\mathbf{R}(0)) \cdot \int_{\tau}^{\tau+1} \mathbb{E}_{\mathbf{x}} [\mathbf{z}(s)] ds \\
(57) \qquad \qquad \qquad &= \int_{\tau}^{\tau+1} \mathbb{E}_{\mu(s)} [f(\mathbf{R}(0)) \cdot \mathbf{z}] ds.
\end{aligned}$$

In above, we have used the fact that $D_i(\cdot)$ is a Poisson process with rate given by $z_i(\cdot)$. Further, the second term in the right hand side of (53) can be bounded as follows.

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}} [\|f(\mathbf{R}(\tau)) - f(\mathbf{R}(0))\|_1] &\leq \mathbb{E}_{\mathbf{x}} [f(|\mathbf{R}(\tau) - \mathbf{R}(0)|)] + O(1) \\
&\leq f(\mathbb{E}_{\mathbf{x}} [|\mathbf{R}(\tau) - \mathbf{R}(0)|]) + O(1) \\
&\leq nf(C_{\max}\tau) + O(1) \\
(58) \qquad \qquad \qquad &= O(f(\tau)),
\end{aligned}$$

The first inequality follows from $f(x+y) \leq f(x) + f(y) + 2$ for any $x, y \in \mathbb{R}_+$. This is because $\log(x+y+e) \leq \log(x+e) + \log(y+e)$ for any $x, y \geq \mathbb{R}_+$, $\log a + b \leq 2 + \log a + \log b$ for any $a, b \geq 1$ and $f(x) = \log \log(x+e)$. The second inequality follows by applying Jensen's inequality for concave function f . Combining (53)-(58), we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}} [L(X(\tau+1)) - L(X(\tau))] \\
&\leq -\frac{3\varepsilon}{4} \left(\max_{\mathbf{y} \in \mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y} \right) + \mathbb{E}_{\pi(0)} [f(\mathbf{R}(0)) \cdot \mathbf{z}] \\
&\quad - \int_{\tau}^{\tau+1} \mathbb{E}_{\mu(s)} [f(\mathbf{R}(0)) \cdot \mathbf{z}] ds + O(f(\tau)) \\
&\leq -\frac{3\varepsilon}{4} \left(\max_{\mathbf{y} \in \mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y} \right) \\
&\quad + \int_{\tau}^{\tau+1} \left(\max_{\mathbf{y} \in \mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y} \right) \|\mu(s) - \pi(0)\|_{TV} ds + O(f(\tau)) \\
&\stackrel{(a)}{\leq} -\frac{\varepsilon}{2} \left(\max_{\mathbf{y} \in \mathcal{X}} f(\mathbf{R}(0)) \cdot \mathbf{y} \right) + O(f(\tau)) \\
&\leq -\frac{\varepsilon}{2} f(Q_{\max}(0)) + O(f(\tau)),
\end{aligned}$$

where (a) follows from Lemma 9. Summing this for $\tau \in I = [b_1, b_2 - 1]$,
(59)

$$\mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \leq -\frac{\varepsilon}{2}f(Q_{\max}(0))(b_2 - b_1) + O((b_2 - b_1)f(b_2)).$$

Therefore, we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(0))] \\ &= \mathbb{E}_{\mathbf{x}}[L(X(b_1)) - L(X(0))] + \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\ &\stackrel{(a)}{\leq} \mathbb{E}_{\mathbf{x}}[f(\mathbf{R}(b_1)) \cdot (\mathbf{R}(b_1) - \mathbf{R}(0))] + \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\ &= \sum_i \mathbb{E}_{\mathbf{x}}[f(R_i(b_1)) \cdot (R_i(b_1) - R_i(0))] + \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\ &\stackrel{(b)}{\leq} \sum_i \sqrt{\mathbb{E}_{\mathbf{x}}[f(R_i(b_1))^2]} \sqrt{\mathbb{E}_{\mathbf{x}}[(R_i(b_1) - R_i(0))^2]} + \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\ &\stackrel{(c)}{\leq} \sum_i \sqrt{f(\mathbb{E}_{\mathbf{x}}[R_i(b_1)])^2 + O(1)} \cdot O(b_1) + \mathbb{E}_{\mathbf{x}}[L(X(b_2)) - L(X(b_1))] \\ &\stackrel{(d)}{=} n f(Q_{\max}(0) + O(b_1)) \cdot O(b_1) - \frac{\varepsilon}{2}f(Q_{\max}(0))(b_2 - b_1) \\ &\quad + O((b_2 - b_1)f(b_2)) \\ &\stackrel{\triangle}{=} -h(\mathbf{x}). \end{aligned}$$

Here (a) follows from convexity of L ; (b) from Cauchy-Schwarz, (c) is due to the bounded second moment $\mathbb{E}_{\mathbf{x}}[|R_i(b_1) - R_i(0)|] = O(b_1)$ as argued earlier in the proof and observing that there exists a concave function g such that $f^2 = g + O(1)$ over \mathbb{R}_+ , subsequently Jensen's inequality can be applied; (d) follows from (59). Finally, choose $g(\mathbf{x}) = b_2$.

With these choices of h and g , the desired conditions of Lemma 6 can be checked as follows. First observe that with respect to $Q_{\max}(0)$, the function h scales as $b_2(Q_{\max}(0))f(Q_{\max}(0))$ due to $b_2/b_1 = \Theta(\log Q_{\max}(0))$ as per Lemma 9. Further, h is a function that is lower bounded and its value goes to ∞ as $Q_{\max}(0)$ goes to ∞ . Therefore, h/g scales as $f(Q_{\max}(0))$. These properties will imply the verification conditions of Lemma 6.

5.3.5. Step Four. For completing the proof of the positive Harris recurrence of both algorithms, it only remains to show that for $\kappa > 0$, the set $B_\kappa = \{\mathbf{x} \in \mathcal{X} : L(\mathbf{x}) \leq \kappa\}$ is a closed petit. This is because other conditions of Lemma 5 follow from Lemma 6. And the Step Three exhibited choice of Lyapunov function L and desired 'drift' functions h, g .

To this end, first note that B_κ is closed by definition. To establish that it is a petit set, we need to find a non-trivial measure μ on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ and

sampling distribution a on \mathbb{Z}_+ so that for any $\mathbf{x} \in B_\kappa$,

$$K_a(\mathbf{x}, \cdot) \geq \mu(\cdot).$$

To construct such a measure μ , we shall use the following Lemma.

Lemma 10 *Let the network Markov chain $X(\cdot)$ start with the state $\mathbf{x} \in B_\kappa$ at time 0 i.e. $X(0) = \mathbf{x}$. Then, there exists $T_\kappa \geq 1$ and $\gamma_\kappa > 0$ such that*

$$\sum_{\tau=1}^{T_\kappa} \Pr_{\mathbf{x}}(X(\tau) = \mathbf{0}) \geq \gamma_\kappa, \quad \forall \mathbf{x} \in B_\kappa.$$

Here $\mathbf{0} = (\mathbf{0}, \mathbf{0}) \in \mathbf{X}$ denote the state where all components of \mathbf{Q} are 0 and the schedule is the empty independent set.

Proof. We establish this for wireless network. The proof for circuit switched network is identical and we skip the details. Consider any $\mathbf{x} \in B_\kappa$. Then by definition $L(\mathbf{x}) \leq \kappa + 1$ for given $\kappa > 0$. Hence by definition of $L(\cdot)$ it can be easily checked that each queue is bounded above by κ . Consider some large enough (soon to be determined) T_κ . By the property of Bernoulli (or Poisson for circuit switched network) arrival process, there is a positive probability $\theta_\kappa^0 > 0$ of no arrivals happening to the system during time interval of length T_κ . Assuming that no arrival happens, we will show that in large enough time t_κ^1 , with probability $\theta_\kappa^1 > 0$ each queue receives at least κ amount of service; and after that in additional time t^2 with positive probability $\theta^2 > 0$ the empty set schedule is reached. This will imply that by defining $T_\kappa \triangleq t_\kappa^1 + t^2$ the state $\mathbf{0} \in \mathbf{X}$ is reached with probability at least

$$\gamma_\kappa \triangleq \theta_\kappa^0 \theta_\kappa^1 \theta^2 > 0.$$

And this will immediately imply the desired result of Lemma 10. To this end, we need to show existence of $t_\kappa^1, \theta_\kappa^1$ and t^2, θ^2 with properties stated above to complete the proof of Lemma 10.

First, existence of $t_\kappa^1, \theta_\kappa^1$. For this, note that the Markov chain corresponding to the scheduling algorithm has time varying transition probabilities and is irreducible over the space of all independent sets, $\mathcal{I}(G)$. If there are no new arrivals and initial $\mathbf{x} \in B_\kappa$, then clearly queue-sizes are uniformly bounded by κ . Therefore, the transition probabilities of all feasible transitions for this time varying Markov chain is uniformly lower bounded by a strictly positive constant (dependent on κ, n). It can be easily checked that the transition probability induced graph on $\mathcal{I}(G)$ has diameter at most $2n$ and Markov chain transits as per Exponential clock of overall rate n . Therefore,

it follows that starting from any initial scheduling configuration, there exists finite time \widehat{t}_κ such that a schedule is reached so that any given queue i is scheduled for at least unit amount of time with probability at least $\widehat{\theta}_\kappa > 0$. Here, both $\widehat{t}_\kappa, \widehat{\theta}_\kappa$ depend on n, κ . Therefore, it follows that in time $t_\kappa^1 \triangleq \kappa n \widehat{t}_\kappa$ all queues become empty with probability at least $\theta_\kappa^1 \triangleq (\widehat{\theta}_\kappa)^{n\kappa}$. Next, to establish existence of t^2, θ^2 as desired, observe that once the system reaches empty queues, it follows that in the absence of new arrivals the empty schedule $\mathbf{0}$ is reached after some finite time t^2 with probability $\theta^2 > 0$ by similar properties of the Markov chain on $\mathcal{I}(G)$ when all queues are 0. Here t^2 and θ^2 are dependent on n only. This completes the proof of Lemma 10. \square

In what follows, Lemma 10 will be used to complete the proof that B_κ is a closed petit. To this end, consider Geometric(1/2) as the sampling distribution a , i.e.

$$a(\ell) = 2^{-\ell}, \quad \ell \geq 1.$$

Let $\delta_{\mathbf{0}}$ be the Dirac distribution on element $\mathbf{0} \in \mathcal{X}$. Then, define μ as

$$\mu = 2^{-T_\kappa} \gamma_k \delta_{\mathbf{0}}, \quad \text{that is } \mu(\cdot) = 2^{-T_\kappa} \gamma_k \delta_{\mathbf{0}}(\cdot).$$

Clearly, μ is non-trivial measure on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$. With these definitions of a and μ , Lemma 10 immediately implies that for any $\mathbf{x} \in B_\kappa$,

$$K_a(\mathbf{x}, \cdot) \geq \mu(\cdot).$$

This establishes that set B_κ is a closed petit set.

6. Discussion. This paper introduced a new randomized scheduling algorithm for two constrained queueing network models: wireless network and buffered circuit switched network. The algorithm is simple, distributed, myopic and throughput optimal. The main reason behind the throughput optimality property of the algorithm is two folds: (1) The relation of algorithm dynamics to the Markovian dynamics over the space of schedules that have a certain product-form stationary distribution, and (2) choice of slowly increasing weight function $\log \log(\cdot + e)$ that allows for an effective time scale separation between algorithm dynamics and the queueing dynamics. We chose wireless network and buffered circuit switched network model to explain the effectiveness of our algorithm because (a) they are becoming of great interest [26, 34] and (b) they represent two different, general class of network models: synchronized packet network model and asynchronous flow network model.

Now we turn to discuss the distributed implementation of our algorithm. As described in Section 3.1, given the weight information at each wireless node (or ingress of a route), the algorithm completely distributed. The weight, as defined in (3) (or (4)), depends on the local queue-size as well as the Q_{\max} information. As is, Q_{\max} is global information. To keep the exposition simpler, we have used the precise Q_{\max} information to establish the throughput property. However, as remarked earlier in the Section 3.1 (soon after (3)), the Q_{\max} can be replaced by its appropriate distributed estimation without altering the throughput optimality property. Such a distributed estimation can be obtained through an extremely simple Markovian like algorithm that require each node to perform broadcast of exactly one number in unit time. A detailed description of such an algorithm can be found in Section 3.3 of [25].

On the other hand, consider the algorithm that does not use Q_{\max} information. That is, instead of (3) or (4), let weight be

$$W_i(t) = f(Q_i(\lfloor t \rfloor)).$$

We conjecture that this algorithm is throughput optimal.

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REFERENCES

- [1] J. Cheeger. A lower bound for the smallest eigenvalue of the laplacian. *Problems in analysis (Papers dedicated to S. Bochner, 1969)*, pages 195–199, 1970.
- [2] J. G. Dai. On positive harris recurrence of multiclass queueing networks. *Annals of Applied Probability*, 5:49–77, 1995.
- [3] J. G. Dai. Stability of fluid and stochastic processing networks. *Lecture notes, Center for Mathematical Physics and Stochastics, Denmark*, 1999.
- [4] J. G. Dai and W. Lin. Asymptotic optimality of maximum pressure policies in stochastic processing networks. *Annals of Applied Probability*, 18(6):2239–2299, 2008.
- [5] J. G. Dai and B. Prabhakar. The throughput of switches with and without speed-up. In *Proceedings of IEEE Infocom*, pages 556–564, 2000.
- [6] P. Diaconis and D. Stroock. Geometric bounds on eigenvalues of markov chains. *Annals of applied probability*, pages 36–61, 1991.
- [7] A. Dimakis and J. Walrand. Sufficient conditions for stability of longest-queue-first scheduling: second-order properties using fluid limits. *Advances in Applied Probability*, 38(2):505, 2006.
- [8] M. Dyer, A. Frieze, and R. Kannan. A random polynomial-time algorithm for approximating the volume of convex bodies. *J. ACM*, 38(1):1–17, 1991.
- [9] S. Foss and T. Konstantopoulos. An overview of some stochastic stability methods. *Journal of Operations Research, Society of Japan*, 47(4), 2004.

- [10] H. O. Georgii. *Gibbs measures and phase transitions*. Walter de Gruyter, 1988.
- [11] R. K. Gettoor. Transience and recurrence of markov processes. In *Azma, J. and Yor, M., editors, Sminaire de Probabilits XIV*, page 397409, 1979.
- [12] P. Giaccone, B. Prabhakar, and D. Shah. Randomized scheduling algorithms for high-aggregate bandwidth switches. *IEEE Journal on Selected Areas in Communications High-performance electronic switches/routers for high-speed internet*, 21(4):546–559, 2003.
- [13] W.K. Hastings. Monte carlo sampling methods using markov chains and their applications. *Biometrika*, 57:97–109, 1970.
- [14] M. Jerrum and A. Sinclair. Polynomial-time approximation algorithms for the ising model. *SIAM Journal on Computing*, 22:1087–1116, 1993.
- [15] L. Jiang, D. Shah, J. Shin, and J. Walrand. Distributed random access algorithm: Scheduling and congestion control. Available at: arxiv.org/abs/0907.122.
- [16] F.P. Kelly. Loss networks. *The annals of applied probability*, 1(3):319–378, 1991.
- [17] L. Lovasz and P. Winkler. Mixing times. *Microsurveys in Discrete Probability (ed. D. Aldous and J. Propp), DIMACS Series in Discrete Math. and theor. Comp. Sci.*, pages 85–133, 1998.
- [18] F. Martinelli. Lectures on glauber dynamics for discrete spin models. *Lectures on probability theory and statistics (Saint-Flour 1997), Lecture notes in Mathematics*, 1717.
- [19] N. McKeown. iSLIP: a scheduling algorithm for input-queued switches. *IEEE Transaction on Networking*, 7(2):188–201, 1999.
- [20] N. McKeown, V. Anantharam, and J. Walrand. Achieving 100% throughput in an input-queued switch. In *Proceedings of IEEE Infocom*, pages 296–302, 1996.
- [21] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller, and E. Teller. Equations of state calculations by fast computing machines. *Journal of Chemical Physics*, 21:1087–1092, 1953.
- [22] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, London, 1993.
- [23] E. Modiano, D. Shah, and G. Zussman. Maximizing throughput in wireless network via gossiping. In *ACM SIGMETRICS/Performance*, 2006.
- [24] R. Montenegro and P. Tetali. Mathematical aspects of mixing times in markov chains. *Found. Trends Theor. Comput. Sci.*, 1(3):237–354, 2006.
- [25] S. Rajagopalan, D. Shah, and J. Shin. Network adiabatic theorem: an efficient randomized protocol for contention resolution. In *Proceedings of the eleventh international joint conference on Measurement and modeling of computer systems*, pages 133–144. ACM New York, NY, USA, 2009.
- [26] Microsoft research lab. Self organizing neighborhood wireless mesh networks. <http://research.microsoft.com/mesh/>.
- [27] D. Shah and D. J. Wischik. Optimal scheduling algorithm for input queued switch. In *Proceeding of IEEE INFOCOM*, 2006.
- [28] D. Shah and D. J. Wischik. The teleology of scheduling algorithms for switched networks under light load, critical load, and overload. <http://web.mit.edu/devavrat/www/shahwischik.pdf>, 2007-09.
- [29] A. Sinclair. *Algorithms for Random Generation and Counting: A Markov Chain Approach*. Birkhäuser, Boston, 1993.

- [30] A. L. Stolyar. MaxWeight scheduling in a generalized switch: State space collapse and workload minimization in heavy traffic. *Annals of Applied Probability*, 14(1):1–53, 2004.
- [31] L. Tassiulas. Linear complexity algorithms for maximum throughput in radio networks and input queued switches. In *IEEE INFOCOM*, volume 2, pages 533–539, 1998.
- [32] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Transactions on Automatic Control*, 37:1936–1948, 1992.
- [33] L. Trevisan. Non-approximability results for optimization problems on bounded degree instances. In *ACM STOC*, 2001.
- [34] S. Verma, H. Chaskar, and R. Ravikanth. Optical burst switching: a viable solution for terabit ip backbone, 2000.

APPENDIX A: A USEFUL LEMMA

Lemma 11 *Let $P_1, P_2 \in \mathbb{R}^{N \times N}$. Then,*

$$\|e^{P_1} - e^{P_2}\|_\infty \leq e^{NM} \|P_1 - P_2\|_\infty,$$

where $M = \max\{\|P_1\|_\infty, \|P_2\|_\infty\}$.

Proof. Using mathematical induction, we first establish that for any $k \in \mathbb{N}$,

$$(60) \quad \|P_1^k - P_2^k\|_\infty \leq k(NM)^{k-1} \|P_1 - P_2\|_\infty.$$

□

To this end, the base case $k = 1$ follows trivially. Suppose it is true for some $k \geq 1$. Then, the inductive step can be justified as follows.

$$\begin{aligned} \|P_1^{k+1} - P_2^{k+1}\|_\infty &= \left\| P_1 \left(P_1^k - P_2^k \right) + (P_1 - P_2) P_2^k \right\|_\infty \\ &\leq \left\| P_1 \left(P_1^k - P_2^k \right) \right\|_\infty + \left\| (P_1 - P_2) P_2^k \right\|_\infty \\ &\stackrel{(a)}{\leq} N \|P_1\|_\infty \left\| P_1^k - P_2^k \right\|_\infty + N \|P_1 - P_2\|_\infty \left\| P_2^k \right\|_\infty \\ &\stackrel{(b)}{\leq} NM \times k(NM)^{k-1} \|P_1 - P_2\|_\infty + N \|P_1 - P_2\|_\infty \times N^{k-1} M^k \\ &= (k+1)(NM)^k \|P_1 - P_2\|_\infty. \end{aligned}$$

In above, (a) follows from an easily verifiable fact that for any $Q_1, Q_2 \in \mathbb{R}^{N \times N}$,

$$\|Q_1 Q_2\|_\infty \leq N \|Q_1\|_\infty \|Q_2\|_\infty.$$

We use induction hypothesis to justify (b). Using (60), we have

$$\begin{aligned}
\|e^{P_1} - e^{P_2}\|_\infty &= \left\| \sum_k \frac{1}{k!} (P_1^k - P_2^k) \right\|_\infty \\
&\leq \sum_k \frac{1}{k!} \|P_1^k - P_2^k\|_\infty \\
&\leq \sum_k \frac{1}{k!} k(NM)^{k-1} \|P_1 - P_2\|_\infty \\
&= e^{NM} \|P_1 - P_2\|_\infty.
\end{aligned}$$