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On a problem of Hasse and Ramachandra

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Abstract: Let K be an imaginary quadratic field, and let $\mathfrak{f}$ be a nontrivial integral ideal of K. Hasse and Ramachandra asked whether the ray class field of K modulo $\mathfrak{f}$ can be generated by a single value of the Weber function. We completely resolve this question when $\mathfrak{f} = (N)$ for any positive integer N excluding 2,3,4 and 6.

Keywords: class field theory, complex multiplication, Weber function

MSC: Primary 11R37; Secondary 11G15, 11G16

1 Introduction

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and let E be an elliptic curve with complex multiplication by $\mathcal{O}_K$. When E is given by the affine model

$$y^2 = 4x^3 - g_2x - g_3 \quad \text{with } g_2 = g_2(\mathcal{O}_K) \text{ and } g_3 = g_3(\mathcal{O}_K),$$

the *Weber function* $h : \mathbb{C}/\mathcal{O}_K \to \mathbb{P}^1(\mathbb{C})$ is defined by

$$h(z) = \begin{cases} (g_2^2/\Delta)\wp(z)^2 & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ (g_3/\Delta)\wp(z)^3 & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ (g_2g_3/\Delta)\wp(z) & \text{otherwise}, \end{cases} \tag{1}$$

where $\Delta = g_2^3 - 27g_3^2$ and $\wp(z) = \wp(z;\ \mathcal{O}_K)$. This map gives rise to an isomorphism of $E/\mathrm{Aut}(E)$ onto $\mathbb{P}^1(\mathbb{C})$ ([8, Theorem 7 in Chapter 1]).

Let $\mathfrak{f}$ be a proper nontrivial ideal of $\mathcal{O}_K$. We denote by H the Hilbert class field of K, and by $K_\mathfrak{f}$ the ray class field of K modulo $\mathfrak{f}$. As a consequence of the main theorem of the theory of complex multiplication, Hasse proved in [4] that

$$H = K(j) \text{ with } j = 1728\,\frac{g_2^3}{\Delta} \quad \text{and} \quad K_\mathfrak{f} = H\left(h(z_0)\right) \text{ for some } z_0 \in \mathfrak{f}^{-1}. \tag{2}$$

See also [8, Chapter 10]. In his letter to Hecke, Hasse further asked whether $K_\mathfrak{f}$ can be generated by a single value of h without the j-invariant ([3, p. 91]), and Ramachandra also mentioned this problem later in [10]. It was Sugawara who first gave a partial answer to this question ([12] and [13]), however, it still remains an open question.

In this paper, through careful understanding about the characters on class groups and the second Kronecker limit formula, we shall eventually resolve Hasse-Ramachandra's problem for $\mathfrak{f} = (N)$ with any positive integer N excluding 2,3,4 and 6 (Theorem 5.1).

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2 The second Kronecker limit formula

For $\mathbf{v} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in (\mathbb{Q} \setminus \mathbb{Z})^2$, we define the (first) *Fricke function* $f_{\mathbf{v}}(\tau)$ on the upper half-plane $\mathbb{H}$ by

$$f_{\mathbf{v}}(\tau) = \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp(r_1\tau + r_2), \tag{3}$$

where $g_2(\tau) = g_2([\tau,\ 1])$, $g_3(\tau) = g_3([\tau,\ 1])$, $\Delta(\tau) = \Delta([\tau,\ 1])$ and $\wp(z) = \wp(z;\ [\tau,\ 1])$. This function depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$, and is holomorphic on $\mathbb{H}$ ([8, Chapters 3 and 6]). Furthermore, we define the *Siegel function* $g_{\mathbf{v}}(\tau)$ on $\mathbb{H}$ by the following infinite product

$$g_{\mathbf{v}}(\tau) = -e^{\pi i r_2(r_1-1)} q^{(1/2)(r_1^2-r_1+1/6)}(1 - q^{r_1}e^{2\pi i r_2})\prod_{n=1}^{\infty}(1 - q^{n+r_1}e^{2\pi i r_2})(1 - q^{n-r_1}e^{-2\pi i r_2}),$$

where $q = e^{2\pi i\tau}$. If N is a positive integer so that $N\mathbf{v} \in \mathbb{Z}^2$, then $g_{\mathbf{v}}(\tau)^{12N}$ depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$, and has neither zeros nor poles on $\mathbb{H}$ ([6, §2.1]).

Lemma 2.1. *Let* $\mathbf{u},\ \mathbf{v} \in (\mathbb{Q} \setminus \mathbb{Z})^2$ *such that* $\mathbf{u} \not\equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$. *Then we have the relation*

$$\left(f_{\mathbf{u}}(\tau) - f_{\mathbf{v}}(\tau)\right)^6 = \frac{j(\tau)^2(j(\tau) - 1728)^3}{2^{30}3^{24}} \frac{g_{\mathbf{u}+\mathbf{v}}(\tau)^6 g_{\mathbf{u}-\mathbf{v}}(\tau)^6}{g_{\mathbf{u}}(\tau)^{12} g_{\mathbf{v}}(\tau)^{12}}.$$

Proof. See [8, Theorem 2 in Chapter 18] and [6, p. 29 and p. 51]. □

Let K be an imaginary quadratic field, let $\mathfrak{f}$ be a proper nontrivial ideal of $\mathcal{O}_K$ and let N (> 1) be the smallest positive integer in $\mathfrak{f}$. We denote by $\mathrm{Cl}(\mathfrak{f})$ the ray class group of K modulo $\mathfrak{f}$. Then $\mathrm{Gal}(K_{\mathfrak{f}}/K)$ is isomorphic to $\mathrm{Cl}(\mathfrak{f})$ via the Artin map $\sigma = \sigma_{\mathfrak{f}} : \mathrm{Cl}(\mathfrak{f}) \to \mathrm{Gal}(K_{\mathfrak{f}}/K)$. Let $C \in \mathrm{Cl}(\mathfrak{f})$. Take any integral ideal $\mathfrak{c}$ in the class C and express

$$\mathfrak{f}\mathfrak{c}^{-1} = [\omega_1,\ \omega_2] \quad \text{for some } \omega_1,\ \omega_2 \in \mathbb{C} \text{ such that } \omega = \frac{\omega_1}{\omega_2} \in \mathbb{H},$$
$$1 = r_1\omega_1 + r_2\omega_2 \quad \text{for some } r_1,\ r_2 \in (1/N)\mathbb{Z}.$$

We define the *Fricke invariant* $f_{\mathfrak{f}}(C)$ and the *Siegel-Ramachandra invariant* $g_{\mathfrak{f}}(C)$ by

$$f_{\mathfrak{f}}(C) = f_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\omega) \quad \text{and} \quad g_{\mathfrak{f}}(C) = g_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\omega)^{12N}, \tag{4}$$

respectively. These values depend only on the class C, not on the choices of $\mathfrak{c}$, ω_1 and ω_2 ([8, §6.2 and §6.3] and [6, §2.1 and 11.1]).

Proposition 2.2. *The invariants* $f_{\mathfrak{f}}(C)$ *and* $g_{\mathfrak{f}}(C)$ *belong to* $K_{\mathfrak{f}}$. *Furthermore, they satisfy*

$$f_{\mathfrak{f}}(C)^{\sigma(C')} = f_{\mathfrak{f}}(CC') \quad \textit{and} \quad g_{\mathfrak{f}}(C)^{\sigma(C')} = g_{\mathfrak{f}}(CC') \quad \textit{for all } C' \in \mathrm{Cl}(\mathfrak{f}).$$

Proof. See [6, Theorem 1.1 in Chapter 11]. □

Let χ be a nonprincipal character of $\mathrm{Cl}(\mathfrak{f})$. We define the *Stickelberger element* $S(\chi) = S_{\mathfrak{f}}(\chi)$ by

$$S(\chi) = \sum_{C \in \mathrm{Cl}(\mathfrak{f})} \chi(C) \ln |g_{\mathfrak{f}}(C)|, \tag{5}$$

and the *L-function* $L_{\mathfrak{f}}(s,\ \chi)$ by

$$L_{\mathfrak{f}}(s,\ \chi) = \sum_{\mathfrak{a}} \frac{\chi([\mathfrak{a}])}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})^s} \quad (s \in \mathbb{C}),$$

where $\mathfrak{a}$ runs over all nontrivial ideals of $\mathcal{O}_K$ prime to $\mathfrak{f}$ and $[\mathfrak{a}]$ stands for the class in $\mathrm{Cl}(\mathfrak{f})$ containing the ideal $\mathfrak{a}$. We shall denote by $\mathfrak{f}_\chi$ the conductor of the character χ.

Proposition 2.3. *Let χ_0 be the primitive character of χ on $\mathrm{Cl}(\mathfrak{f}_\chi)$. If $\mathfrak{f}_\chi \neq \mathcal{O}_K$, then we obtain the relation*

$$\left(\prod_{\substack{\mathfrak{p}\,:\,\text{prime ideals of } \mathcal{O}_K \\ \text{such that } \mathfrak{p} \mid \mathfrak{f},\ \mathfrak{p} \nmid \mathfrak{f}_\chi}} (1 - \overline{\chi_0}([\mathfrak{p}]))\right) L_{\mathfrak{f}_\chi}(1, \chi_0) = -\frac{\pi \chi_0([\gamma \mathfrak{d}_K \mathfrak{f}_\chi])}{3N(\mathfrak{f}_\chi)\sqrt{|d_K|}\,\omega(\mathfrak{f}_\chi) T_\gamma(\overline{\chi_0})} S(\overline{\chi}),$$

where $\mathfrak{d}_K$ is the different ideal of the extension $K/\mathbb{Q}$, γ is an element of K so that $\gamma\mathfrak{d}_K\mathfrak{f}_\chi$ is a nontrivial ideal of $\mathcal{O}_K$ prime to $\mathfrak{f}_\chi$, $N(\mathfrak{f}_\chi)$ is the least positive integer in $\mathfrak{f}_\chi$, $\omega(\mathfrak{f}_\chi) = |\{\alpha \in \mathcal{O}_K^\star \mid \alpha \equiv 1 \pmod{\mathfrak{f}_\chi}\}|$ and

$$T_\gamma(\overline{\chi_0}) = \sum_{\alpha + \mathfrak{f}_\chi \in (\mathcal{O}_K/\mathfrak{f}_\chi)^*} \overline{\chi_0}([\alpha \mathcal{O}_K]) e^{2\pi \mathrm{i} \mathrm{Tr}_{K/\mathbb{Q}}(\alpha\gamma)}.$$

Proof. See [11, Theorem 9 in Chapter II] or [6, Theorem 2.1 in Chapter 11]. □

Remark 2.4. Since χ_0 is a nonprincipal character of $\mathrm{Cl}(\mathfrak{f}_\chi)$ by the assumption $\mathfrak{f}_\chi \neq \mathcal{O}_K$, we have $L_{\mathfrak{f}_\chi}(1, \chi_0) \neq 0$ ([5, Theorem 10.2 in Chapter V]). Thus, if every prime ideal factor of $\mathfrak{f}$ divides $\mathfrak{f}_\chi$, then we derive by Proposition 2.3 that $S(\overline{\chi}) \neq 0$.

3 Differences of Weber functions

For an imaginary quadratic field K, fix an element τ_K of $\mathbb{H}$ so that $\mathcal{O}_K = [\tau_K, 1]$. From now on, we assume that K is different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, and let $N > 1$. We then have $j(\tau_K) \neq 0,\ 1728$ ([1, p. 261]) and

$$h(r_1\tau_K + r_2) = f_{\left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]}(\tau_K) \quad \text{for all } \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in (\mathbb{Q} \setminus \mathbb{Z})^2$$

by the definitions (1) and (3).

Let H_N be the ring class field of the order of conductor N in K. Then we have a tower of fields

$$K \subseteq H \subseteq H_N \subseteq K_{(N)}$$

([1, §7]). For an integer t prime to N, by $C_t = C_{N,t}$ we mean the class in the ray class group $\mathrm{Cl}(N)$ of K modulo (N) containing the ideal (t). Note that C_1 is the identity element of $\mathrm{Cl}(N)$.

Lemma 3.1. *If t is an integer prime to N, then we get*

$$f_{(N)}(C_t) = f_{\left[\begin{smallmatrix} 0 \\ t/N \end{smallmatrix}\right]}(\tau_K) \quad \text{and} \quad g_{(N)}(C_t) = g_{\left[\begin{smallmatrix} 0 \\ t/N \end{smallmatrix}\right]}(\tau_K)^{12N}.$$

Proof. Since

$$(N\mathcal{O}_K)(t\mathcal{O}_K)^{-1} = (N/t)\mathcal{O}_K = [N\tau_K/t,\ N/t] \quad \text{and} \quad 1 = 0(N\tau_K/t) + (t/N)(N/t),$$

we deduce the lemma by the definition (4). □

For an intermediate field F of the extension $K_{(N)}/K$, we shall denote by $\mathrm{Cl}(K_{(N)}/F)$ the subgroup of $\mathrm{Cl}(N)$ corresponding to $\mathrm{Gal}(K_{(N)}/F)$.

Lemma 3.2. *We have*

$$\mathrm{Cl}(K_{(N)}/H_N) = \{C_t \mid t \in (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}\} \simeq (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}.$$

Proof. See [2, Proposition 3.8]. □

Let t be an integer such that

$$\gcd(N, t) = 1 \quad \text{and} \quad t \not\equiv \pm 1 \pmod{N}.$$

Note that such an integer t always exists except for the four cases $N = 2, 3, 4, 6$. Express $(t+1)/N$ and $(t-1)/N$ as

$$\frac{t+1}{N} = \frac{n_+}{N_+} \quad \text{and} \quad \frac{t-1}{N} = \frac{n_-}{N_-},$$

where n_+, N_+, n_-, N_- are integers such that $N_+, N_- > 0$ and $\gcd(n_+, N_+) = \gcd(n_-, N_-) = 1$. Observe that the condition $t \not\equiv \pm 1 \pmod{N}$ is equivalent to saying that neither N_+ nor N_- is equal to 1.

Now, we define

$$\xi_t = \big(h(t/N) - h(1/N)\big)^{12N} = \Big(f_{\left[\begin{smallmatrix}0\\ t/N\end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix}0\\ 1/N\end{smallmatrix}\right]}(\tau_K)\Big)^{12N}. \tag{6}$$

Furthermore, for a character χ of $\mathrm{Cl}(N)$ we denote by

$$S(\chi, \xi_t) = \sum_{C \in \mathrm{Cl}(N)} \chi(C) \ln \left|\xi_t^{\sigma(C)}\right|.$$

Lemma 3.3. *If χ is nontrivial on $\mathrm{Cl}(K_{(N)}/H)$, then we obtain*

$$\begin{aligned} S(\overline{\chi}, \xi_t) = {} & (N/N_+) \sum_{\substack{B_+ \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_+)}))}} \overline{\chi}(B_+) \ln \left|g_{(N_+)}(C_{N_+, n_+})^{\sigma(B_+)}\right| \sum_{A_+ \in \mathrm{Cl}(K_{(N)}/K_{(N_+)})} \overline{\chi}(A_+) \\ & + (N/N_-) \sum_{\substack{B_- \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_-)}))}} \overline{\chi}(B_-) \ln \left|g_{(N_-)}(C_{N_-, n_-})^{\sigma(B_-)}\right| \sum_{A_- \in \mathrm{Cl}(K_{(N)}/K_{(N_-)})} \overline{\chi}(A_-) \\ & - 2(\chi(C_t) + 1) S(\overline{\chi}). \end{aligned}$$

Proof. We derive that

$$\begin{aligned} S(\overline{\chi}, \xi_t) = {} & \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|\left(\frac{j(\tau_K)^{4N}(j(\tau_K) - 1728)^{6N}}{2^{60N} 3^{48N}}\right)^{\sigma(C)}\right| \\ & + \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|\left(g_{\left[\begin{smallmatrix}0\\ n_+/N_+\end{smallmatrix}\right]}(\tau_K)^{12N}\right)^{\sigma(C)}\right| + \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|\left(g_{\left[\begin{smallmatrix}0\\ n_-/N_-\end{smallmatrix}\right]}(\tau_K)^{12N}\right)^{\sigma(C)}\right| \\ & - \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|\left(g_{\left[\begin{smallmatrix}0\\ t/N\end{smallmatrix}\right]}(\tau_K)^{24N}\right)^{\sigma(C)}\right| - \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|\left(g_{\left[\begin{smallmatrix}0\\ 1/N\end{smallmatrix}\right]}(\tau_K)^{24N}\right)^{\sigma(C)}\right| \\ & \text{by the definition (6) and Lemma 2.1} \\ = {} & \sum_{\substack{B \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/H))}} \sum_{A \in \mathrm{Cl}(K_{(N)}/H)} \overline{\chi}(AB) \ln \left|\left(\frac{j(\tau_K)^{4N}(j(\tau_K) - 1728)^{6N}}{2^{60N} 3^{48N}}\right)^{\sigma(AB)}\right| \\ & + (N/N_+) \sum_{\substack{B_+ \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_+)}))}} \sum_{A_+ \in \mathrm{Cl}(K_{(N)}/K_{(N_+)})} \overline{\chi}(A_+B_+) \ln \left|g_{(N_+)}(C_{N_+, n_+})^{\sigma(A_+B_+)}\right| \\ & + (N/N_-) \sum_{\substack{B_- \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_-)}))}} \sum_{A_- \in \mathrm{Cl}(K_{(N)}/K_{(N_-)})} \overline{\chi}(A_-B_-) \ln \left|g_{(N_-)}(C_{N_-, n_-})^{\sigma(A_-B_-)}\right| \\ & - 2 \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|g_{(N)}(C_t)^{\sigma(C)}\right| - 2 \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|g_{(N)}(C_1)^{\sigma(C)}\right| \quad \text{by Lemma 3.1} \\ = {} & \sum_B \overline{\chi}(B) \ln \left|\left(\frac{j(\tau_K)^{4N}(j(\tau_K) - 1728)^{6N}}{2^{60N} 3^{48N}}\right)^{\sigma(B)}\right| \sum_A \overline{\chi}(A) \end{aligned}$$

$$
\begin{aligned}
&+(N/N_+)\sum_{B_+}\overline{\chi}(B_+)\ln\left|g_{(N_+)}(C_{N_+,\,n_+})^{\sigma(B_+)}\right|\sum_{A_+}\overline{\chi}(A_+)\\
&+(N/N_-)\sum_{B_-}\overline{\chi}(B_-)\ln\left|g_{(N_-)}(C_{N_-,\,n_-})^{\sigma(B_-)}\right|\sum_{A_-}\overline{\chi}(A_-)\\
&-2\chi(C_t)\sum_{C}\overline{\chi}(C_tC)\ln\left|g_{(N)}(C_tC)\right|-2\sum_{C}\overline{\chi}(C)\ln\left|g_{(N)}(C)\right| \quad \text{by (2) and Proposition 2.2}\\
=\ &(N/N_+)\sum_{B_+}\overline{\chi}(B_+)\ln\left|g_{(N_+)}(C_{N_+,\,n_+})^{\sigma(B_+)}\right|\sum_{A_+}\overline{\chi}(A_+)\\
&+(N/N_-)\sum_{B_-}\overline{\chi}(B_-)\ln\left|g_{(N_-)}(C_{N_-,\,n_-})^{\sigma(B_-)}\right|\sum_{A_-}\overline{\chi}(A_-)\\
&-2(\chi(C_t)+1)S(\overline{\chi})
\end{aligned}
$$

by the assumption that χ is nontrivial on $\mathrm{Cl}(K_{(N)}/H)$ and the definition (5).

□

4 Lemmas on characters of class groups

If we set

$$F = K\left(h(1/N)\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/N\end{smallmatrix}\right]}(\tau_K)\right),$$

then we obtain by (2) that

$$\mathrm{Cl}(K_{(N)}/H)\cap\mathrm{Cl}(K_{(N)}/F) = \mathrm{Cl}(K_{(N)}/HF) = \mathrm{Cl}(K_{(N)}/K_{(N)}) = \{C_1\}. \tag{7}$$

In this section, we shall prove the existence of certain characters of class groups under the assumption that F is properly contained in $K_{(N)}$.

Lemma 4.1. *Assume that*

$$\gcd(72,\,N)\in\{1,\,8,\,9,\,72\}.$$

Then, there is a character χ of $\mathrm{Cl}(N)$ *satisfying the following properties*:

(A1) *It is trivial on* $\mathrm{Cl}(K_{(N)}/H_N)$.
(A2) $\chi(C)\neq 1$ *for any chosen* $C\in\mathrm{Cl}(K_{(N)}/H)\setminus\mathrm{Cl}(K_{(N)}/H_N)$.
(A3) *Every prime ideal factor of* (N) *divides the conductor* $(N)_\chi$.

Proof. See [7, Lemma 3.4 and Remark 4.5]. □

Lemma 4.2. *Suppose that F is properly contained in $K_{(N)}$. Then, there is a character ρ of* $\mathrm{Cl}(N)$ *satisfying the following properties*:

- *It is trivial on* $\mathrm{Cl}(K_{(N)}/H)$, *and so* $(N)_\rho = \mathcal{O}_K$.
- *It is nontrivial on* $\mathrm{Cl}(K_{(N)}/F)$.

Here, $(N)_\rho$ stands for the conductor of the character ρ.

Proof. Since $|\mathrm{Cl}(K_{(N)}/F)|\geq 2$ and $\mathrm{Cl}(K_{(N)}/H)\cap\mathrm{Cl}(K_{(N)}/F)=\{C_1\}$ by (7), one can take a class $C\in\mathrm{Cl}(K_{(N)}/F)\setminus\mathrm{Cl}(K_{(N)}/H)$. Thus, if we let $\mu:\mathrm{Cl}(N)\to\mathrm{Cl}(N)/\mathrm{Cl}(K_{(N)}/H)$ be the canonical homomorphism, then there is a character ψ of $\mathrm{Cl}(N)/\mathrm{Cl}(K_{(N)}/H)$ such that $\psi(\mu(C))\neq 1$.

Now, defining a character ρ of $\mathrm{Cl}(N)$ by $\rho=\psi\circ\mu$, we see that it is trivial on $\mathrm{Cl}(K_{(N)}/H)$. Since

$$\mathrm{Cl}(N)/\mathrm{Cl}(K_{(N)}/H)\simeq\mathrm{Cl}(H/K)=\mathrm{Cl}(\mathcal{O}_K),$$

we get $(N)_\rho=\mathcal{O}_K$. Moreover, $\rho(C)=\psi(\mu(C))\neq 1$ implies that ρ is nontrivial on $\mathrm{Cl}(K_{(N)}/F)$. □

Proposition 4.3. *Assume that*

$$\gcd(72,\ N) \in \{1,\ 8,\ 9,\ 72\} \text{ and } F \text{ is properly contained in } K_{(N)}. \tag{8}$$

Then, there is a character χ of Cl(N) *and an integer t which satisfy the following properties*:

- (B1) χ *is nontrivial on* $\mathrm{Cl}(K_{(N)}/F)$.
- (B2) $\gcd(N,\ t) = 1$ *and* $t \not\equiv \pm 1 \pmod{N}$.
- (B3) $S(\overline{\chi},\ \xi_t) \neq 0$.

Proof. We divide the proof into three cases in accordance with $\gcd(72, N)$.

Case 1. First, consider the case where $\gcd(72,\ N) \in \{8,\ 72\}$. Let C be the class in Cl(N) containing the ideal $((N/2)\tau_K + 1)$. We observe by Lemma 3.2 that

$$C \in \mathrm{Gal}(K_{(N)}/K_{(N/2)}) \setminus \mathrm{Gal}(K_{(N)}/H_N). \tag{9}$$

Then, by Lemma 4.1 there is a character χ of Cl(N) satisfying (A1)–(A3). If χ is trivial on $\mathrm{Cl}(K_{(N)}/F)$, then we replace χ by $\chi\rho$, where ρ is a character of Cl(N) given in Lemma 4.2. The new character χ is nontrivial on $\mathrm{Cl}(K_{(N)}/F)$ and preserves the properties (A1)–(A3). Take any integer t such that $\gcd(N,\ t) = 1$ and $t \not\equiv \pm 1 \pmod{N}$. Since N, $t+1$ and $t-1$ are all even, we see that N_+ and N_- divide $N/2$, from which it follows that

$$\mathrm{Cl}(K_{(N)}/K_{(N/2)}) \subseteq \mathrm{Cl}(K_{(N)}/K_{(N_+)}) \cap \mathrm{Cl}(K_{(N)}/K_{(N_-)}). \tag{10}$$

We then achieve that

$$\begin{aligned}
S(\overline{\chi},\ \xi_t) = (N/N_+) & \sum_{\substack{B_+ \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_+)}))}} \overline{\chi}(B_+) \ln \left| g_{(N_+)}(C_{N_+,\, n_+})^{\sigma(B_+)} \right| \sum_{A_+ \in \mathrm{Cl}(K_{(N)}/K_{(N_+)})} \overline{\chi}(A_+) \\
& + (N/N_-) \sum_{\substack{B_- \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_-)}))}} \overline{\chi}(B_-) \ln \left| g_{(N_-)}(C_{N_-,\, n_-})^{\sigma(B_-)} \right| \sum_{A_- \in \mathrm{Cl}(K_{(N)}/K_{(N_-)})} \overline{\chi}(A_-) \\
& - 2(\chi(C_t) + 1) S(\overline{\chi}) \quad \text{by Lemma 3.3} \\
& = -2(\chi(C_t) + 1) S(\overline{\chi}) \quad \text{since } \chi \text{ is nontrivial on } \mathrm{Cl}(K_{(N)}/K_{(N_+)}) \text{ and } \mathrm{Cl}(K_{(N)}/K_{(N_-)}) \\
& \qquad \text{by (9), (10) and (A2)} \\
& = -4S(\overline{\chi}) \quad \text{by (A1) and Lemma 3.2} \\
& \neq 0 \quad \text{by Proposition 2.3 and Remark 2.4.}
\end{aligned}$$

Case 2. Second, consider the case where $\gcd(72,\ N) = 9$. If we let C be the class in Cl(N) containing the ideal $((N/3)\tau_K + 1)$, then we see that

$$C \in \mathrm{Gal}(K_{(N)}/K_{(N/3)}) \setminus \mathrm{Gal}(K_{(N)}/H_N) \tag{11}$$

by Lemma 3.2. By Lemma 4.1, there exists a character χ of Cl(N) satisfying (A1)–(A3). In a similar way to the above Case 1, we may assume that χ is nontrivial on $\mathrm{Cl}(K_{(N)}/F)$. Take $t = 2$, and then we get

$$n_+ = 1,\ N_+ = \frac{N}{3} \quad \text{and} \quad n_- = 1,\ N_- = N.$$

So, we derive that

$$\begin{aligned}
S(\overline{\chi},\ \xi_t) = 3 & \sum_{\substack{B_+ \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N/3)}))}} \overline{\chi}(B_+) \ln \left| g_{(N/3)}(C_{(N/3),\, 1})^{\sigma(B_+)} \right| \sum_{A_+ \in \mathrm{Cl}(K_{(N)}/K_{(N/3)})} \overline{\chi}(A_+) \\
& + S(\overline{\chi}) - 2(\chi(C_t) + 1) S(\overline{\chi}) \quad \text{by Lemma 3.3} \\
& = -(2\chi(C_t) + 1) S(\overline{\chi}) \quad \text{since } \chi \text{ is nontrivial on } \mathrm{Cl}(K_{(N)}/K_{(N/3)}) \text{ by (11) and (A2)} \\
& = -3S(\overline{\chi}) \quad \text{by (A1) and Lemma 3.2} \\
& \neq 0 \quad \text{by Proposition 2.3 and Remark 2.4.}
\end{aligned}$$

Case 3. Lastly, consider the case where $\gcd(72, N) = 1$. By Lemma 4.1, there is a character χ of $\mathrm{Cl}(N)$ satisfying (A1)–(A3) for any chosen $C \in \mathrm{Cl}(K_{(N)}/H) \setminus \mathrm{Cl}(K_{(N)}/H_N)$. In like manner as above, we may assume that χ is nontrivial on $\mathrm{Cl}(K_{(N)}/F)$. Take $t = 2$, then it follows that

$$n_+ = 3,\ N_+ = N \quad \text{and} \quad n_- = 1,\ N_- = N.$$

Therefore, we obtain

$$\begin{aligned} S(\overline{\chi}, \xi_t) &= \chi(C_{n_+})S(\overline{\chi}) + S(\overline{\chi}) - 2(\chi(C_t) + 1)S(\overline{\chi}) \quad \text{by Lemma 3.3} \\ &= -2S(\overline{\chi}) \quad \text{by (A1) and Lemma 3.2} \\ &\neq 0 \quad \text{by Proposition 2.3 and Remark 2.4.} \end{aligned}$$

This proves the lemma.

□

Lemma 4.4. *Assume that*

$$\gcd(72, N) \in \{2, 3, 4, 6, 12, 18, 24, 36\} \quad \textit{and} \quad N \neq 2, 3, 4, 6. \tag{12}$$

Then, there exists an integer t satisfying the following properties:

(C1) $\gcd(N, t) = 1$ *and* $t \not\equiv \pm 1 \pmod{N}$.

(C2) *There are prime factors* p_+, p_- *of* N (*not necessarily distinct*) *such that* $\gcd(p_\pm, N_\pm) = 1$ (*Note that* $N_\pm$ *depends on the choice of* t).

Proof. Let ℓ be an integer such that $\ell > 1$ and $\gcd(6, \ell) = 1$. One can take t as listed in Table 1.

Table 1: An integer t satisfying (C1) and (C2)

N	t	N_+	N_-	p_+	p_-
12	5	2	3	3	2
18	5	3	9	2	2
24	7	3	4	2	3
36	17	2	9	3	2
2ℓ	$\ell + 2$	ℓ	ℓ	2	2
4ℓ	$2\ell + 1$	ℓ	2	2	a prime factor of ℓ
$2^a 3^b \ell$ with $a \geq 0,\ b \geq 1$	a solution of $\begin{cases} x \equiv 1 \pmod{2^a \ell}, \\ x \equiv -1 \pmod{3^b} \end{cases}$	a divisor of $2^a \ell$	a divisor of 3^b	3	a prime factor of ℓ

□

Let $(N) = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$ be the prime ideal factorization of (N). Then we get

$$[K_{(N)} : H] = \frac{\omega(N)}{2} \prod_{\mathfrak{p} \mid (N)} \left(\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{p}) - 1\right) \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{p})^{n_{\mathfrak{p}} - 1},$$

where $\omega(N)$ is the number of roots of unity in K which are congruent to 1 modulo (N) ([9, Theorem 1 in Chapter VI]). One can then readily deduce that

$$K_{(N)} = K_{(M)} \text{ for a proper divisor } M \text{ of } N \quad \Longleftrightarrow \quad 2 \parallel N \text{ and } 2 \text{ splits in } K.$$

In this case, we have

$$K_{(N)} = K_{(N/2)}. \tag{13}$$

Furthermore, it is well known that

$$[H_N : H] = N \prod_{p \mid N} \left(1 - \left(\frac{d_K}{p}\right)\frac{1}{p}\right), \tag{14}$$

where (d_K/p) is the Legendre symbol for an odd prime p, and $(d_K/2)$ is the Kronecker symbol ([1, Theorem 7.24]).

Lemma 4.5. *Assume that if* $2 \parallel N$, *then* 2 *does not split in* K. *Let* p *be a prime factor of* N *with* $p^e \parallel N$. *Then, there is a nontrivial character* χ_p *of* $\mathrm{Cl}(N)$ *satisfying the following properties*:

- *It is trivial on* $\mathrm{Cl}(K_{(N)}/H_{p^e})$, *and so* $(N)_{\chi_p}$ *divides* (p^e).
- $(N)_{\chi_p}$ *is divisible by every prime ideal factor of* (p).

Proof. Note that the assumption implies $[H_{p^e} : H] \geq 2$ by (14). Therefore, the lemma is an immediate consequence of [7, Lemma 3.3]. □

Proposition 4.6. *Assume that*

N satisfies (12) *and F is properly contained in* $K_{(N)}$.

Under this assumption instead of (8), Proposition 4.3 *also holds.*

Proof. Let

$$\chi = \prod_{p \mid N} \chi_p,$$

where χ_p is a character of $\mathrm{Cl}(N)$ given in Lemma 4.5 for each prime factor p of N. If χ is trivial on $\mathrm{Cl}(K_{(N)}/F)$, then we replace χ by $\chi\rho$ where ρ is a character of $\mathrm{Cl}(N)$ given in Lemma 4.2. Then, χ satisfies the following properties:

(i) It is trivial on $\mathrm{Cl}(K_{(N)}/H_N)$.
(ii) It is nontrivial on $\mathrm{Cl}(K_{(N)}/F)$.
(iii) $(N)_\chi$ is divisible by every prime ideal factor of (N).

Now, take an integer t satisfying (C1) and (C2) in Lemma 4.4. We then derive that

$$\begin{aligned} S(\overline{\chi}, \xi_t) = {} & (N/N_+) \sum_{\substack{B_+ \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_+)}))}} \overline{\chi}(B_+) \ln \left| g_{(N_+)}(C_{N_+,\, n_+})^{\sigma(B_+)} \right| \sum_{A_+ \in \mathrm{Cl}(K_{(N)}/K_{(N_+)})} \overline{\chi}(A_+) \\ & + (N/N_-) \sum_{\substack{B_- \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/K_{(N_-)}))}} \overline{\chi}(B_-) \ln \left| g_{(N_-)}(C_{N_-,\, n_-})^{\sigma(B_-)} \right| \sum_{A_- \in \mathrm{Cl}(K_{(N)}/K_{(N_-)})} \overline{\chi}(A_-) \\ & - 2(\chi(C_t) + 1) S(\overline{\chi}) \quad \text{by Lemma 3.3} \\ = {} & -4S(\overline{\chi}) \quad \text{because } \chi \text{ is nontrivial on } \mathrm{Cl}(K_{(N)}/K_{(N_+)}) \text{ and } \mathrm{Cl}(K_{(N)}/K_{(N_-)}) \\ & \qquad \text{by (iii) and (C2) and } \chi(C_t) = 1 \text{ by (i) and Lemma 3.2} \\ \neq {} & 0 \quad \text{by Proposition 2.3, Remark 2.4 and (iii).} \end{aligned}$$

□

5 Main theorem

Now, we are ready to prove our main theorem. Note by (2) that the problem of Hasse and Ramachandra is trivial if the class number of K is one.

Theorem 5.1. *Let K be an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, and let $N > 1$ be an integer such that $N \neq 2, 3, 4, 6$. Then we have*

$$K_{(N)} = \begin{cases} K(h(1/N)) \text{ if } 2 \nmid\nmid N \text{ or } 2 \text{ does not split in } K, \\ K(h(2/N)) \text{ otherwise.} \end{cases}$$

Proof. First, consider the case where $2 \nmid\nmid N$ or 2 does not split in K. Suppose on the contrary that $F = K\left(h(1/N)\right)$ is properly contained in $K_{(N)}$. Then, by Propositions 4.3 and 4.6, there exist a character χ of $\mathrm{Cl}(N)$ and an integer t such that

(B1) χ is nontrivial on $\mathrm{Cl}(K_{(N)}/F)$,
(B2) $\gcd(N, t) = 1$ and $t \not\equiv \pm 1 \pmod N$,
(B3) $S(\overline{\chi}, \xi_t) \neq 0$.

On the other hand, since F is a Galois extension of K, it contains the Galois conjugate $h(1/N)^{\sigma(C_t)}$ of $h(1/N)$. We then see by Proposition 2.2 and Lemma 3.1 that

$$h(1/N)^{\sigma(C_t)} = f_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{\sigma(C_t)} = f_{(N)}(C_1)^{\sigma(C_t)} = f_{(N)}(C_t) = f_{\left[\begin{smallmatrix} 0 \\ t/N \end{smallmatrix}\right]}(\tau_K) = h(t/N).$$

Thus F contains the element $\xi_t = \left(h(t/N) - h(1/N)\right)^{12N}$. Now, we derive that

$$\begin{aligned} S(\overline{\chi}, \xi_t) &= \sum_{C \in \mathrm{Cl}(N)} \overline{\chi}(C) \ln \left|\xi_t^{\sigma(C)}\right| \\ &= \sum_{\substack{B \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/F))}} \sum_{A \in \mathrm{Cl}(K_{(N)}/F)} \overline{\chi}(AB) \ln \left|\xi_t^{\sigma(AB)}\right| \\ &= \sum_{\substack{B \in \mathrm{Cl}(N) \\ (\mathrm{mod}\ \mathrm{Cl}(K_{(N)}/F))}} \overline{\chi}(B) \ln \left|\xi_t^{\sigma(B)}\right| \sum_{A \in \mathrm{Cl}(K_{(N)}/F)} \overline{\chi}(A) \quad \text{because } \xi_t \in F \\ &= 0 \quad \text{by (B1)}, \end{aligned}$$

which contradicts (B3). Hence, we have $K_{(N)} = K\left(h(1/N)\right)$ as desired.

Second, consider the case where $2 \,\|\, N$ and 2 splits in K. Then we have

$$\begin{aligned} K_{(N)} &= K_{(N/2)} \quad \text{as mentioned in (13)} \\ &= K(h(2/N)) \quad \text{by the first case of the theorem.} \end{aligned}$$

This completes the proof. □

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