PAPER

Irregular Sampling on Shift Invariant Spaces

Kil Hyun KWON[†a)], ***Member and*** **Jaekyu LEE**[†b)], ***Nonmember***

SUMMARY Let $V(\phi)$ be a shift invariant subspace of $L^2(\mathbb{R})$ with a Riesz or frame generator $\phi(t)$. We take $\phi(t)$ suitably so that the regular sampling expansion : $f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n)$ holds on $V(\phi)$. We then find conditions on the generator $\phi(t)$ and various bounds of the perturbation $\{\delta_n\}_{n\in\mathbb{Z}}$ under which an irregular sampling expansion: $f(t) = \sum_{n\in\mathbb{Z}} f(n+\delta_n)S_n(t)$ holds on $V(\phi)$. Some illustrating examples are also provided.

key words: *shift invariant space, irregular sampling, frame, Riesz basis*

1. Introduction

The classical Shannon sampling theorem ([12], [17]) says that any signal $f(t)$ in PW_π, the Paley-Wiener space of finite energy signals, band-limited in $[-\pi, \pi]$ can be recovered uniquely from its uniform sample values as

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}.$$

As a natural extension of the Shannon sampling theorem, many authors have studied the sampling procedure on various signal spaces including shift invariant spaces. See e.g. [1], [3], [14], [18], [20], [23] and references therein. Here, we let $V(\phi)$ be a shift invariant closed subspace of $L^2(\mathbb{R})$, where $\phi(t)$ is a Riesz or frame generator. We put some further restrictions on the generator $\phi(t)$ so that the regular sampling expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n),\ f \in V(\phi)$$

holds. We then consider the irregular sampling expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(n+\delta_n)S_n(t),\ f \in V(\phi)$$

by perturbing the regular sampling instants $\{n\}_{n\in\mathbb{Z}}$ into $\{n+\delta_n\}_{n\in\mathbb{Z}}$ and find conditions on the generator $\phi(t)$ and some bounds on the perturbation $\{\delta_n\}_{n\in\mathbb{Z}}$ so that the above irregular sampling expansion holds on $V(\phi)$. The results obtained here unify and extend some of the previous results on the same topic in [4]–[6], [15], [18], [19], [22]. We also provide some illustrating examples.

Manuscript received October 26, 2009.
Manuscript revised March 2, 2010.

[†]The authors are with the Department of Mathematical Sciences, KAIST, S. Korea.
a) E-mail: khkwon@kaist.edu
b) E-mail: jaekyu.lee@kaist.ac.kr

DOI: 10.1587/transfun.E93.A.1163

2. Preliminaries

A sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ of vectors in a Hilbert space H is

- a Bessel sequence in H(with a bound B) if there is a constant $B > 0$ such that

$$\sum_{n\in\mathbb{Z}} |\langle f, \phi_n\rangle|^2 \le B\,\|f\|^2,\ f \in H;$$

- a frame of H(with bounds (A, B)) if there are constants $B \ge A > 0$ such that

$$A\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, \phi_n\rangle|^2 \le B\,\|f\|^2,\ f \in H;$$

- a Riesz basis of H(with bounds (A, B)) if it is complete in H and there are constants $B \ge A > 0$ such that for any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$,

$$A\,\|\mathbf{c}\|_2^2 \le \left\|\sum_{n\in\mathbb{Z}} c(n)\phi_n\right\|^2 \le B\,\|\mathbf{c}\|_2^2,$$

where $\|\mathbf{c}\|_2^2 := \sum_{n\in\mathbb{Z}} |c(n)|^2$.

Let $\{\phi_n\}_{n\in\mathbb{Z}}$ be a frame of H with bounds (A, B) and

$$S(f) = \sum_{n\in\mathbb{Z}}\langle f, \phi_n\rangle\phi_n,\ f \in H$$

the frame operator of $\{\phi_n\}_{n\in\mathbb{Z}}$. Then $\{S^{-1}(\phi_n)\}_{n\in\mathbb{Z}}$ is also a frame of H with bounds $\left(\frac{1}{B}, \frac{1}{A}\right)$, called the canonical dual frame of $\{\phi_n\}_{n\in\mathbb{Z}}$ and

$$f = \sum_{n\in\mathbb{Z}}\langle f, \phi_n\rangle S^{-1}\phi_n = \sum_{n\in\mathbb{Z}}\langle f, S^{-1}\phi_n\rangle\phi_n,\ f \in H.$$

Fourier transform is defined by

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) := \int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt,\ f(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

so that $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\cdot]$ extends to be a unitary operator on $L^2(\mathbb{R})$. For any $\phi(t) \in L^2(\mathbb{R})$, let

$$C_\phi(t) := \sum_{n\in\mathbb{Z}} |\phi(t+n)|^2 \text{ and } G_\phi(\xi) := \sum_{n\in\mathbb{Z}} \left|\widehat{\phi}(\xi + 2n\pi)\right|^2.$$

Then $C_\phi(t) = C_\phi(t+1) \in L^1[0, 1]$ and $G_\phi(\xi) = G_\phi(\xi + 2\pi) \in L^1[0, 2\pi]$. We also let

$$Z_\phi(t,\xi) := \sum_{n\in\mathbb{Z}} \phi(t+n)e^{-in\xi}$$

be the Zak transform of $\phi(t) \in L^2(\mathbb{R})$ ([13]). Then $Z_\phi(t,\xi) \in L^2([0,1]\times[0,2\pi])$ and by the Poisson summation formula (cf. Lemma 6.2 in [2])

$$Z_\phi(t,\xi) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)} \text{ a.e. in } \mathbb{R}^2. \quad (1)$$

Let $V(\phi) := \overline{\text{span}}\{\phi(t-n)\}_{n\in\mathbb{Z}}$ be the shift invariant space generated by $\phi(t) \in L^2(\mathbb{R})$, that is, the closed subspace of $L^2(\mathbb{R})$ spanned by $\{\phi(t-n)\}_{n\in\mathbb{Z}}$. Let

$$A := \left\|G_\phi(\xi)\right\|_0 \text{ and } B := \left\|G_\phi(\xi)\right\|_\infty$$

be the essential infimum and the essential supremum of $G_\phi(\xi)$ on the spectrum $\sigma(V) := \text{supp}G_\phi \cap [0,2\pi]$ of $V(\phi)$ respectively. Here $\text{supp}G_\phi$ is the support of a locally integrable function $G_\phi(\xi)$ as a distribution on $\mathbb{R}$ (cf. [11]), that is, $\mathbb{R}\backslash\text{supp}G_\phi$ is the largest open subset of $\mathbb{R}$ on which $G_\phi(\cdot) = 0$ a.e. Then it is well known ([8]) that $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is

- a Bessel sequence in $V(\phi)$ with the optimal bound B if and only if $B < \infty$;
- a frame of $V(\phi)$ with the optimal bounds (A, B) if and only if $0 < A \le B < \infty$;
- a Riesz basis of $V(\phi)$ with the optimal bounds (A, B) if and only if $0 < A \le B < \infty$ and $\sigma(V) = [0, 2\pi]$.

For a Riesz or frame generator $\phi(t)$, let $\widetilde{\phi}(t) \in L^2(\mathbb{R})$ be such that

$$\widehat{\widetilde{\phi}}(\xi) = \frac{\widehat{\phi}(\xi)}{G_\phi(\xi)}\chi_{\text{supp}G_\phi}(\xi), \quad (2)$$

where $\chi_E(\cdot)$ is the characteristic function of a subset E of $\mathbb{R}$. Then $\{\widetilde{\phi}(t-n)\}_{n\in\mathbb{Z}}$ is the dual Riesz basis or the canonical dual frame of $\{\phi(t-n)\}_{n\in\mathbb{Z}}$, of which the optimal bounds are $\left(\frac{1}{B}, \frac{1}{A}\right)$.

Let

$$T(\mathbf{c}) = (\mathbf{c} * \phi)(t) := \sum_{n\in\mathbb{Z}} c(n)\phi(t-n), \ \mathbf{c} \in l^2$$

be the pre-frame operator of the frame $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ with bounds (A, B). Then T is a bounded linear operator from l^2 onto $V(\phi)$ so that

$$V(\phi) = \{(\mathbf{c} * \phi)(t) : \mathbf{c} \in l^2\} = \{(\mathbf{c} * \phi)(t) : \mathbf{c} \in N(T)^\perp\},$$

where $N(T)$ is the kernel of T and $N(T)^\perp$ is the orthogonal complement of $N(T)$ in l^2. We then have ([8])

$$A\,\|\mathbf{c}\|_2^2 \le \|(\mathbf{c} * \phi)(t)\|^2_{L^2(\mathbb{R})} \le B\,\|\mathbf{c}\|_2^2, \ \mathbf{c} \in N(T)^\perp. \quad (3)$$

For $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ and $\mathbf{d} = \{d(n)\}_{n\in\mathbb{Z}}$ in l^2, let

$$\widehat{\mathbf{c}}(\xi) := \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi} \text{ and } \mathbf{c} * \mathbf{d} := \{(\mathbf{c} * \mathbf{d})(n)\}_{n\in\mathbb{Z}},$$

where $(\mathbf{c} * \mathbf{d})(n) := \sum_{k\in\mathbb{Z}} c(k)d(n-k)$. Then

$$\|\mathbf{c}\|_2^2 = \frac{1}{2\pi}\left\|\widehat{\mathbf{c}}(\xi)\right\|^2_{L^2[0,2\pi]} \text{ and} \quad (4)$$
$$\|\mathbf{c} * \mathbf{d}\|_2^2 = \frac{1}{2\pi}\left\|\widehat{\mathbf{c}}(\xi)\widehat{\mathbf{d}}(\xi)\right\|^2_{L^2[0,2\pi]}.$$

3. The Case of Riesz Generator

In Sect. 3, let $\phi(t) \in L^2(\mathbb{R})$ be a Riesz generator satisfying

- $\phi(t)$ is everywhere well defined on $\mathbb{R}$;
- $C_\phi(t) < \infty, \ t \in \mathbb{R}$.

Then $V(\phi)$ becomes a reproducing kernel Hilbert space (RKHS in short) (see Proposition 2.3 in [14]) with the reproducing kernel

$$q(t,s) := \sum_{k\in\mathbb{Z}} \widetilde{\phi}(t-k)\overline{\phi(s-k)}, \quad (5)$$

where $\widetilde{\phi}(t)$ is given by (2). Since $\|q(\cdot,s)\|^2_{L^2(\mathbb{R})} \le \frac{1}{A}C_\phi(s)$, any sequence in $V(\phi)$, which converges in $L^2(\mathbb{R})$, also converges uniformly on any subset E of $\mathbb{R}$ with $\sup_E C_\phi(t) < \infty$. In particular, for any $\mathbf{c} \in l^2$, $f(t) = (\mathbf{c} * \phi)(t)$ converges absolutely on $\mathbb{R}$ and uniformly on E when $\sup_E C_\phi(t) < \infty$.

We further assume that

$$0 < \alpha := \left\|\widehat{\phi^*}(\xi)\right\|_0 \le \beta := \left\|\widehat{\phi^*}(\xi)\right\|_\infty < \infty, \quad (6)$$

where $\widehat{\phi^*}(\xi) := \sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi}$ and $\left\|\widehat{\phi^*}(\xi)\right\|_0$ and $\left\|\widehat{\phi^*}(\xi)\right\|_\infty$ are the essential infimum and the essential supremum of $\left|\widehat{\phi^*}(\xi)\right|$ on $\sigma(V)(= [0, 2\pi])$ respectively. Then the regular sampling expansion holds on $V(\phi)$ since we have:

Proposition 3.1. *On $V(\phi)$, the following are all equivalent.*

(a) There is a Riesz basis $\{S(t-n)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n). \quad (7)$$

(b) $\{q(t,n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V(\phi)$.
(c) $0 < \left\|\widehat{\phi^*}(\xi)\right\|_0 \le \left\|\widehat{\phi^*}(\xi)\right\|_\infty < \infty$.

Moreover in this case, $\{q(t,n)\}_{n\in\mathbb{Z}}$ is the dual Riesz basis of $\{S(t-n)\}_{n\in\mathbb{Z}}$ and

$$\widehat{S}(\xi) = \frac{\widehat{\phi}(\xi)}{\widehat{\phi^*}(\xi)} \ \textit{a.e. on } \mathbb{R}.$$

Proof. See Theorem 3.3 in [14]. □

Note that in Proposition 3.1, the Riesz bases $\{S(t-n)\}_{n\in\mathbb{Z}}$ and $\{q(t,n)\}_{n\in\mathbb{Z}}$ have bounds $\left(\frac{A}{\beta^2}, \frac{B}{\alpha^2}\right)$ and $\left(\frac{\alpha^2}{B}, \frac{\beta^2}{A}\right)$ respectively.

Lemma 3.2. *Let $\{\phi_n\}_{n\in\mathbb{Z}}$ and $\{\psi_n\}_{n\in\mathbb{Z}}$ be two sequences in a Hilbert space H.*

(a) ([8]) $\{\phi_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence in H with a bound M if and only if

$$\left\| \sum_{n\in\mathbb{Z}} c(n)\phi_n \right\|^2 \le M \|\mathbf{c}\|_2^2, \ \mathbf{c} \in l_F^2,$$

where $l_F^2 := \{\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2 : c(n) = 0$ *for all but finitely many* $n's\}$.

(b) ([21]) *Let* $\{\phi_n\}_{n\in\mathbb{Z}}$ *be a Riesz basis of H and* $\{C_n(\cdot)\}_{n\in\mathbb{Z}}$ *the coefficient functionals of* $\{\phi_n\}_{n\in\mathbb{Z}}$. *If*

$$\sum_{n\in\mathbb{Z}} \|C_n(\cdot)\| \, \|\psi_n - \phi_n\| < 1,$$

then $\{\psi_n\}_{n\in\mathbb{Z}}$ *is also a Riesz basis of H.*

(c) (Theorem 5 in [10](resp. Theorem 1 in [7])) *Let* $\{\phi_n\}_{n\in\mathbb{Z}}$ *be a Riesz basis(resp. frame) of H with bounds* (A, B). *If* $\{\psi_n - \phi_n\}_{n\in\mathbb{Z}}$ *is a Bessel sequence in H with a bound* $M < A$, *then* $\{\psi_n\}_{n\in\mathbb{Z}}$ *is also a Riesz basis(resp. frame) of H.*

In the following, we let $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ be a real sequence. We first find conditions on the generator $\phi(t)$ and l^2-bounds on the perturbation $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ for an irregular sampling expansion to hold on $V(\phi)$.

Theorem 3.3. *Assume further that* $\phi(t) \in C(\mathbb{R})$ *and there exists* $\phi'(t)$ *on* $\mathbb{R}\backslash\mathbb{Z}$. *If* $\mathbf{u} = \{u_n\}_{n\in\mathbb{Z}} \in l^2$, *where* $u_n := \sup_{0<|t-n|<\|\delta\|_\infty} |\phi'(t)|$, *then for any* $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ *with* $|\delta_n| < 1$, $n \in \mathbb{Z}$ *and* $\|\delta\|_2 < \sqrt{\frac{A}{B}} \frac{\alpha}{\|\mathbf{u}\|_2}$, *there is a Riesz basis* $\{S_n(t)\}_{n\in\mathbb{Z}}$ *of* $V(\phi)$ *for which the irregular sampling expansion holds:*

$$f(t) = \sum_{n\in\mathbb{Z}} f(n+\delta_n) S_n(t), \ f \in V(\phi). \tag{8}$$

Proof. For any $\mathbf{c} \in l_F^2$, we have by (5)

$$\begin{aligned} \Delta &:= \left\| \sum_{n\in\mathbb{Z}} c(n)(q(t, n+\delta_n) - q(t,n)) \right\|^2 \qquad (9) \\ &= \left\| \sum_{k\in\mathbb{Z}} \left\{ \sum_{n\in\mathbb{Z}} c(n) \left(\overline{\phi(n+\delta_n - k)} - \overline{\phi(n-k)} \right) \right\} \cdot \widetilde{\phi}(t-k) \right\|^2 \\ &\le \frac{1}{A} \sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} c(n) \left(\overline{\phi(n+\delta_n - k)} - \overline{\phi(n-k)} \right) \right|^2 \\ &\le \frac{\|\mathbf{c}\|_2^2}{A} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\phi(n+\delta_n-k) - \phi(n-k)|^2 \\ &= \frac{\|\mathbf{c}\|_2^2}{A} \sum_{n\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |\phi(l+\delta_n) - \phi(l)|^2 \\ &\le \frac{\|\mathbf{c}\|_2^2}{A} \|\delta\|_2^2 \|\mathbf{u}\|_2^2 \end{aligned}$$

since $|\phi(l+\delta_n) - \phi(l)| \le |\delta_n| \, u_l$ by the mean value theorem. Hence by Lemma 3.2(a), $\{q(t, n+\delta_n) - q(t,n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $V(\phi)$ with a bound $\frac{1}{A}\|\delta\|_2^2 \|\mathbf{u}\|_2^2$. Since $\{q(t,n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V(\phi)$ with bounds $\left(\frac{\alpha^2}{B}, \frac{\beta^2}{A}\right)$, $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$ is also a Riesz basis of $V(\phi)$ if $\|\delta\|_2^2 < \frac{A}{B}\frac{\alpha^2}{\|\mathbf{u}\|_2^2}$ by Lemma 3.2(c). Then we have the irregular sampling expansion (8), where $\{S_n(t)\}_{n\in\mathbb{Z}}$ is the dual Riesz basis of $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$ since $f(n+\delta_n) = \langle f(t), q(t, n+\delta_n)\rangle$, $n \in \mathbb{Z}$. □

In the following, $AC_{loc}(\mathbb{R})$ denotes the space of locally absolutely continuous functions on $\mathbb{R}$.

Theorem 3.4. *Assume further that* $\phi(t) \in AC_{loc}(\mathbb{R})$ *and* $C_{\phi'}(t) \in L^\infty(\mathbb{R})$. *Then for any* $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ *with* $\|\delta\|_2 < \sqrt{\frac{A}{B}} \frac{\alpha}{\sqrt{\|C_{\phi'}(t)\|_{L^\infty(\mathbb{R})}}}$, *there is a Riesz basis* $\{S_n(t)\}_{n\in\mathbb{Z}}$ *of* $V(\phi)$ *for which the irregular sampling expansion (8) holds on* $V(\phi)$.

Proof. Since $\phi(t) \in AC_{loc}(\mathbb{R})$, there exists $\phi'(t)$ a.e. so that

$$\phi(l+\delta_n) - \phi(l) = \int_0^{\delta_n} \phi'(t+l)dt, \ l \in \mathbb{Z}$$

by the fundamental theorem of calculus. Then

$$\sum_{l\in\mathbb{Z}} |\phi(l+\delta_n) - \phi(l)|^2 \le |\delta_n|^2 \left\| C_{\phi'}(t) \right\|_{L^\infty(\mathbb{R})}$$

so that as in (9)

$$\begin{aligned} \Delta &= \left\| \sum_{n\in\mathbb{Z}} c(n) \left(q(t, n+\delta_n) - q(t,n) \right) \right\|^2 \\ &\le \frac{\|\mathbf{c}\|_2^2}{A} \|\delta\|_2^2 \left\| C_{\phi'}(t) \right\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Hence, by Lemma 3.2(a) $\{q(t, n+\delta_n) - q(t,n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $V(\phi)$ with a bound $\frac{1}{A}\|\delta\|_2^2 \left\| C_{\phi'}(t) \right\|_{L^\infty(\mathbb{R})}$. Therefore, the conclusion again follows from Lemma 3.2(c). □

If $\|\delta\|_\infty \ge \frac{1}{2}$ in Theorem 3.3, then $\phi(t) \in AC_{loc}(\mathbb{R})$ so that Theorem 3.3 is a consequence of Theorem 3.4 since $\left\| C_{\phi'}(t) \right\|_{L^\infty(\mathbb{R})}^{1/2} \le \|\mathbf{u}\|_2$.

Note that when $\phi'(t) \in L^2(\mathbb{R})$, $C_{\phi'}(t) = \frac{1}{2\pi} \left\| Z_{\phi'}(t, \cdot) \right\|^2_{L^2[0,2\pi]}$ and by (1)

$$\left| Z_{\phi'}(t, \xi) \right| \le H_{\phi'}(\xi) := \sum_{n\in\mathbb{Z}} \left| \widehat{\phi'}(\xi + 2n\pi) \right| \text{ a.e. in } \mathbb{R}^2. \tag{10}$$

Hence $C_{\phi'}(t) \in L^\infty(\mathbb{R})$ and $\left| C_{\phi'}(t) \right| \le \frac{1}{2\pi} \left\| H_{\phi'}(\xi) \right\|^2_{L^2[0,2\pi]}$ a.e. if $H_{\phi'}(\xi) \in L^2[0, 2\pi]$.

As it was done in [6] (where $\phi(t)$ is a continuous Riesz generator with $\phi(t) = O\left((1+|t|)^{-r}\right)$, $r > 1$ so that $\sup_{\mathbb{R}} C_\phi(t) < \infty$), we may apply Lemma 3.2(b) to the pairs $\{\widehat{q}(\xi, n)\}_{n\in\mathbb{Z}}$ and $\{\widehat{q}(\xi, n+\delta_n)\}_{n\in\mathbb{Z}}$. Then for $\phi(t)$ and $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ as in Theorem 3.3, we obtain : if $\|\delta\|_1 := \sum_{n\in\mathbb{Z}} |\delta_n| < \sqrt{\frac{A}{B}} \frac{\alpha}{\|\mathbf{u}\|_1}$, then there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which (8) holds. But, this is a consequence of Theorem 3.3 since $\|\delta\|_1 \ge \|\delta\|_2$ for any sequence $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$.

We now find conditions on the generator $\phi(t)$ and l^∞-bounds on the perturbation $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ for the irregular sampling expansion (8) to hold on $V(\phi)$.

Theorem 3.5. *For any $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$, let*

$$\Phi(t) := \sum_{j\in\mathbb{Z}} |\phi(j+t) - \phi(j)| \text{ and}$$
$$M := \sup_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\phi(k+\delta_{k+n}) - \phi(k)|.$$

If $M < \infty$ and $\sup_{|t|\le\|\delta\|_\infty} \Phi(t) < \frac{A}{B}\frac{\alpha^2}{M}$, where $\|\delta\|_\infty := \sup_{n\in\mathbb{Z}} |\delta_n|$, then there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which the irregular sampling expansion (8) holds on $V(\phi)$.

Proof. For any $\mathbf{c} \in l^2_F$,

$$\begin{aligned}\Delta &= \left\| \sum_{n\in\mathbb{Z}} c(n)\left(q(t, n+\delta_n) - q(t,n)\right)\right\|^2 \\ &\le \frac{1}{A}\sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} c(n)\left(\overline{\phi(n+\delta_n-k)} - \overline{\phi(n-k)}\right)\right|^2 \\ &\quad \text{(cf. (9))} \\ &= \frac{1}{A}\sum_{n,l\in\mathbb{Z}} c(n)\overline{c(l)} b_{n,l} \\ &\le \frac{1}{A}\sum_{n,l\in\mathbb{Z}} |c(n)|^2 \left|b_{n,l}\right| \le \frac{1}{A}\|\mathbf{c}\|_2^2 \sup_{n\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \left|b_{n,l}\right|,\end{aligned}$$

where $b_{n,l} = \overline{b_{l,n}} := \sum_{k\in\mathbb{Z}} \left(\overline{\phi(n+\delta_n-k)} - \overline{\phi(n-k)}\right)\cdot(\phi(l+\delta_l-k) - \phi(l-k))$.

Now

$$\begin{aligned}&\sum_{l\in\mathbb{Z}} \left|b_{n,l}\right| \\ &\le \sum_{j\in\mathbb{Z}} |\phi(j+\delta_n) - \phi(j)| \sum_{k\in\mathbb{Z}} \left|\phi(k+\delta_{k+n-j}) - \phi(k)\right| \\ &\le M \sup_{|t|\le\|\delta\|_\infty} \Phi(t)\end{aligned}$$

so that

$$\begin{aligned}&\left\| \sum_{n\in\mathbb{Z}} c(n)\left(q(t,n+\delta_n) - q(t,n)\right)\right\|^2 \\ &\le \frac{M}{A}\|\mathbf{c}\|_2^2 \sup_{|t|\le\|\delta\|_\infty} \Phi(t), \ \mathbf{c}\in l^2_F.\end{aligned} \tag{11}$$

Hence $\{q(t, n+\delta_n) - q(t,n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $V(\phi)$ with a bound $\frac{M}{A}\sup_{|t|\le\|\delta\|_\infty}\Phi(t)$ so that the conclusion follows from Lemma 3.2(c). □

It's interesting to note that $\phi(t)$ needs not be continuous on $\mathbb{R}$ in Theorem 3.5.

Corollary 3.6. *Assume further that $\phi(t) \in C(\mathbb{R})$ and $\phi(t) = O((1+|t|)^{-r})$, $r > 1$. If $\|\delta\|_\infty$ is small enough, then there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which the irregular sampling expansion (8) holds on $V(\phi)$.*

Proof. Assume $\phi(t) = O((1+|t|)^{-r})$, $r > 1$ and $\|\delta\|_\infty \le 1$. Then with M and $\Phi(t)$ as in Theorem 3.5, $M \le \widetilde{M} := \sup_{\|\sigma\|_\infty\le 1}\sum_{k\in\mathbb{Z}} |\phi(k+\sigma_k) - \phi(k)| < \infty$ and $\Phi(t) \in C(\mathbb{R})$ since $\Phi(t) = \sum_{j\in\mathbb{Z}} |\phi(j+t) - \phi(j)|$ converges locally uniformly on $\mathbb{R}$ and $\phi(t) \in C(\mathbb{R})$. Hence (cf. (11)) $\{q(t, n+\delta_n) - q(t,n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $V(\phi)$ with a bound $\frac{\widetilde{M}}{A}\sup_{|t|\le\|\delta\|_\infty}\Phi(t)$, which converges to 0 as $\|\delta\|_\infty$ tends to 0 since $\Phi(t) \in C(\mathbb{R})$. Therefore if $\|\delta\|_\infty$ is so small that $\sup_{|t|\le\|\delta\|_\infty}\Phi(t) < \frac{A}{B}\frac{\alpha^2}{\widetilde{M}}$, then $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$ is also a Riesz basis of $V(\phi)$ by Lemma 3.2(c) and the conclusion follows. □

Remark 3.7. If $\phi(t) = O((1+|t|)^{-r})$, $r > 1$, is continuous on $\mathbb{R}$, then $C_\phi(t) = C_\phi(t+1) \in C[0,1]$ so $\sup_{\mathbb{R}} C_\phi(t) < \infty$ and $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$. Hence $\widehat{\phi}^*(\xi) = \widehat{\phi}^*(\xi+2\pi) \in C[0,2\pi]$ so that the condition (6) holds if and only if $\widehat{\phi}^*(\xi) \ne 0$ on $[0,2\pi]$.

Corollary 3.6 was first proved in [5], where $V(\phi) = V_0$ and $\phi(t)$ is a continuous scaling function of an MRA $\{V_m\}_{m\in\mathbb{Z}}$ (see Theorem 5 in [5]). Authors in [5], [6] implicitly assumed that $\phi(t)$ is real-valued and claimed (see Lemma 4 in [5]) that $\{\sum_{n\in\mathbb{Z}}\phi(t-n)\phi(k-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V(\phi)$. But it is not true unless $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is a real orthonormal basis of $V(\phi)$. Although we mainly follow ideas in [5] to estimate Δ, the proof of Corollary 3.6 here is much simpler than the one for Theorem 5 in [5], where they used the biorthogonal scaling functions.

Lemma 3.8. *Let $\psi(t) \in L^2(\mathbb{R}) \cap AC_{loc}(\mathbb{R})$ be a frame generator and $\psi'(t) \in L^2(\mathbb{R})$. Then*

(i) $C_\psi(t) \in AC_{loc}(\mathbb{R})$ so $\sup_{\mathbb{R}} C_\psi(t) < \infty$;
(ii) $V(\psi) \subset L^2(\mathbb{R}) \cap AC_{loc}(\mathbb{R})$;
(iii) for any $f(t) = (\mathbf{c} * \psi)(t) \in V(\psi)$ with $\mathbf{c} \in l^2$, $f'(t) = (\mathbf{c} * \psi')(t)$ a.e.

Proof. See Lemma 3 in [18]. □

Theorem 3.9. *Assume further that $\phi(t) \in AC_{loc}(\mathbb{R})$, $\phi'(t) \in L^2(\mathbb{R})$ and $Z_{\phi'}(t,\xi) \in L^\infty(\mathbb{R}^2)$. If $\|\delta\|_\infty < \alpha\sqrt{\frac{A}{2B}}\frac{1}{\|Z_{\phi'}(t,\xi)\|_{L^\infty(\mathbb{R}^2)}}$, then there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which the irregular sampling expansion (8) holds on $V(\phi)$.*

Proof. Since $V(\phi) \subset AC_{loc}(\mathbb{R})$ by Lemma 3.8, we have by the fundamental theorem of calculus, for any $f \in V(\phi)$ and any $n \in \mathbb{Z}$

$$\begin{aligned}|f(n+\delta_n) - f(n)| &= \left|\int_0^{\delta_n} f'(t+n)dt\right| \\ &\le \sqrt{\|\delta\|_\infty}\left(\int_{-\|\delta\|_\infty}^{\|\delta\|_\infty} \left|f'(t+n)\right|^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

so that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} |\langle f(t), q(t,n+\delta_n) - q(t,n)\rangle|^2 \\ &\le \|\delta\|_\infty \int_{-\|\delta\|_\infty}^{\|\delta\|_\infty} \sum_{n\in\mathbb{Z}} \left|f'(t+n)\right|^2 dt.\end{aligned}$$

Now $f(t) = (\mathbf{c} * \phi)(t)$, $\mathbf{c} \in N(T)^\perp$ so that by Lemma 3.8

$$f'(t+n) = (\mathbf{c} * \phi')(t+n) = \sum_{k\in\mathbb{Z}} c(k)\phi'(t+n-k)$$

$$= (\mathbf{c} * \mathbf{d_t})(n) \text{ a.e.},$$

where $\mathbf{d_t} := \{\phi'(t+n)\}_{n\in\mathbf{Z}} \in l^2$ for a.e. t since $\phi'(t) \in L^2(\mathbb{R})$. Then by (3) and (4)

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left|f'(t+n)\right|^2 &= \|\mathbf{c} * \mathbf{d_t}\|_2^2 \\ &= \frac{1}{2\pi}\left\|\widehat{\mathbf{c}}(\xi)\widehat{\mathbf{d_t}}(\xi)\right\|^2_{L^2[0,2\pi]} \\ &\le \left\|\widehat{\mathbf{d_t}}(\xi)\right\|^2_{L^\infty[0,2\pi]}\|\mathbf{c}\|_2^2 \\ &\le \frac{1}{A}\left\|\widehat{\mathbf{d_t}}(\xi)\right\|^2_{L^\infty[0,2\pi]}\|f\|^2_{L^2(\mathbb{R})}\end{aligned}$$

for a.e. t, where $\widehat{\mathbf{d_t}}(\xi) = Z_{\phi'}(t,\xi)$. Hence

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} |\langle f(t), q(t, n+\delta_n) - q(t,n)\rangle|^2 \\ &\quad\le \frac{2\,\|\delta\|_\infty^2}{A}\left\|Z_{\phi'}(t,\xi)\right\|^2_{L^\infty(\mathbb{R}^2)}\|f\|^2_{L^2(\mathbb{R})}\end{aligned}$$

so that $\{q(t, n+\delta_n) - q(t,n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $V(\phi)$ with a bound $\frac{2\|\delta\|_\infty^2}{A}\left\|Z_{\phi'}(t,\xi)\right\|^2_{L^\infty(\mathbb{R}^2)}$. Hence if $\frac{2\|\delta\|_\infty^2}{A}\left\|Z_{\phi'}(t,\xi)\right\|^2_{L^\infty(\mathbb{R})} < \frac{\alpha^2}{B}$, then $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V(\phi)$ by Lemma 3.2(c) and the conclusion follows. □

The proof of Theorem 3.9 can be easily modified to yield : if either all $\delta_n \ge 0$ or all $\delta_n \le 0$, then (8) holds on $V(\phi)$ when $\|\delta\|_\infty < \alpha\sqrt{\frac{A}{B}}\frac{1}{\|Z_{\phi'}(t,\xi)\|_{L^\infty(\mathbb{R}^2)}}$.

Note that when $\phi'(t) \in L^2(\mathbb{R})$, we have by (1)

$$\begin{aligned}Z_{\phi'}(t,\xi) &= \sum_{n\in\mathbb{Z}} \widehat{\phi'}(\xi + 2n\pi)e^{it(\xi+2n\pi)} \\ &= i\sum_{n\in\mathbb{Z}}(\xi + 2n\pi)\widehat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)} \text{ a.e. in } \mathbb{R}^2\end{aligned}$$

so that $Z_{\phi'}(t,\xi) \in L^\infty(\mathbb{R}^2)$ and $\left|Z_{\phi'}(t,\xi)\right| \le \left\|H_{\phi'}(\xi)\right\|_{L^\infty(\mathbb{R})}$ a.e. in $\mathbb{R}^2$ if $\widehat{\phi}(\xi) = O((1+|\xi|)^{-r})$, $r > 2$. For example, the latter condition holds for the B-spline $\phi_n(t) = (\phi_0 * \phi_{n-1})(t)$ of degree $n \ge 3$, where $\phi_0(t) = \chi_{[0,1)}(t)$ is the Haar scaling function.

4. The Case of Frame Generator

In Sect. 4, let $\phi(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ be a frame generator satisfying $\sup_{\mathbb{R}} C_\phi(t) < \infty$. Then $V(\phi)$ becomes an RKHS with the reproducing kernel $q(t,s)$ given by (5) and any $f(t) = (\mathbf{c} * \phi)(t) \in V(\phi)$ with $\mathbf{c} \in l^2$ converges uniformly on $\mathbb{R}$ to $f(t) \in C(\mathbb{R})$. We also assume that the condition (6) holds. Then by the frame version of Proposition 3.1(see Theorem 3.4 in [14] or Theorem 1 in [23]), $\{q(t,n)\}_{n\in\mathbb{Z}}$ is a frame of $V(\phi)$ so that the regular sampling expansion (7) holds on $V(\phi)$, where $\{S(t-n)\}_{n\in\mathbb{Z}}$ is the canonical dual frame of $\{q(t,n)\}_{n\in\mathbb{Z}}$.

Therefore all results (Theorems 3.3–3.9) in Sect. 3 remain true on $V(\phi)$, where $\phi(t)$ is a frame generator as above. However, in the case of frame generator, we may improve bounds on δ by using Lemma 4.1 below instead of Lemma 3.2(c).

Lemma 4.1. *Let $\{\phi_n\}_{n\in\mathbb{Z}}$ be a frame of a Hilbert space H with bounds (A,B) and $\{\psi_n\}_{n\in\mathbb{Z}} \in H$. If there is a constant λ with $0 \le \lambda < 1$ and*

$$\sum_{n\in\mathbb{Z}} |\langle f, \psi_n - \phi_n\rangle|^2 \le \lambda\sum_{n\in\mathbb{Z}}|\langle f,\phi_n\rangle|^2, \ f \in H, \tag{12}$$

then $\{\psi_n\}_{n\in\mathbb{Z}}$ is also a frame of H.

Proof. The inequality (12) implies by the triangular inequality

$$\begin{aligned}(1-\sqrt{\lambda})^2\sum_{n\in\mathbb{Z}}|\langle f,\phi_n\rangle|^2 &\le \sum_{n\in\mathbb{Z}}|\langle f,\psi_n\rangle|^2 \\ &\le (1+\sqrt{\lambda})^2\sum_{n\in\mathbb{Z}}|\langle f,\phi_n\rangle|^2, \ f\in H.\end{aligned}$$

Hence

$$\begin{aligned}(1-\sqrt{\lambda})^2 A\,\|f\|^2 &\le \sum_{n\in\mathbb{Z}}|\langle f,\psi_n\rangle|^2 \\ &\le (1+\sqrt{\lambda})^2 B\,\|f\|^2, \ f\in H\end{aligned}$$

so that $\{\psi_n\}_{n\in\mathbb{Z}}$ is also a frame of H with bounds $\left((1-\sqrt{\lambda})^2A, (1+\sqrt{\lambda})^2B\right)$. □

Note that the frame part of Lemma 3.2(c) is a consequence of Lemma 4.1 since when $\{\psi_n - \phi_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence in H with a bound $M < A$, (12) holds with $\lambda = \frac{M}{A} < 1$. Lemma 4.1 was essentially proved in [22], where authors handled the case $H = V(\phi)$, $\{\phi_n\}_{n\in\mathbb{Z}} = \{q(t,n)\}_{n\in\mathbb{Z}}$ and $\{\psi_n\}_{n\in\mathbb{Z}} = \{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$. The above proof of Lemma 4.1 is much simpler than the one for Theorem 3 in [22].

Theorem 4.2. *If*

$$\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|\phi(l+\delta_n) - \phi(l)|^2 < \alpha^2, \tag{13}$$

then there is a frame $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which

$$f(t) = \sum_{n\in\mathbb{Z}} f(n+\delta_n)S_n(t), \ f\in V(\phi). \tag{14}$$

Proof. For any $f(t) = (\mathbf{c} * \phi)(t)$, $\mathbf{c} \in l^2$, we have by (4)

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|\langle f(t), q(t,n)\rangle|^2 &= \sum_{n\in\mathbb{Z}}|f(n)|^2 = \|\mathbf{c} * \mathbf{d}\|^2 \\ &= \frac{1}{2\pi}\left\|\widehat{\mathbf{c}}(\xi)\widehat{\phi}^*(\xi)\right\|^2_{L^2[0,2\pi]} \\ &\ge \alpha^2\,\|\mathbf{c}\|_2^2,\end{aligned} \tag{15}$$

where $\mathbf{d} = \{\phi(n)\}_{n\in\mathbb{Z}}$. On the other hand,

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}|\langle f(t), q(t,n+\delta_n) - q(t,n)\rangle|^2 \\ &\quad= \sum_{n\in\mathbb{Z}}|f(n+\delta_n) - f(n)|^2\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} c(k)\left(\phi(n+\delta_n-k)-\phi(n-k)\right) \right|^2$$
$$\leq \|\mathbf{c}\|_2^2 \sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} |\phi(l+\delta_n)-\phi(l)|^2$$
$$\leq \left\{ \frac{1}{\alpha^2} \sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} |\phi(l+\delta_n)-\phi(l)|^2 \right\} \cdot \sum_{n\in\mathbb{Z}} |\langle f(t), q(t,n)\rangle|^2 ,$$

where we use (15) for the last inequality.

Assume that the condition (13) holds. Then $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$ is also a frame of $V(\phi)$ by Lemma 4.1 since $\{q(t,n)\}_{n\in\mathbb{Z}}$ is a frame of $V(\phi)$. Hence we have (14), where $\{S_n(t)\}_{n\in\mathbb{Z}}$ is the canonical dual frame of $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$. □

We now give the frame versions of Theorems 3.3 and 3.4.

Corollary 4.3. (a) *Assume further that there exists $\phi'(t)$ on $\mathbb{R}\backslash\mathbb{Z}$, $\mathbf{u} \in l^2$ with $\mathbf{u}$ as in Theorem 3.3, and $|\delta_n| < 1$, $n \in \mathbb{Z}$. If $\|\delta\|_2 < \frac{\alpha}{\|\mathbf{u}\|_2}$, then there is a frame $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which (14) holds.*
(b) *Assume further that $\phi(t) \in AC_{loc}(\mathbb{R})$ and $C_{\phi'}(t) \in L^\infty(\mathbb{R})$. If $\|\delta\|_2 < \frac{\alpha}{\sqrt{\|C_{\phi'}(t)\|_{L^\infty(\mathbb{R})}}}$, then there is a frame $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which (14) holds.*

Proof. As in the proofs of Theorems 3.3 and 3.4, we have

$$\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} |\phi(l+\delta_n)-\phi(l)|^2 \leq \begin{cases} \|\delta\|_2^2 \, \|\mathbf{u}\|_2^2 & \text{for the case (a)} \\ \|\delta\|_2^2 \, \left\|C_{\phi'}(t)\right\|_{L^\infty(\mathbb{R})} & \text{for the case (b).} \end{cases}$$

Hence both (a) and (b) follow from Theorem 4.2. □

Corollary 4.3(b) improves Theorem 4 in [18], where $\phi(t) \in AC_{loc}(\mathbb{R})$ and $\phi'(t) = O((1+|t|)^{-r})$, $r > \frac{1}{2}$ so that $C_{\phi'}(t) \in L^\infty(\mathbb{R})$. Sun and Zhou [18] then obtained a weaker l^2-bound: $\|\delta\|_2 < \sqrt{\frac{A}{B}}\frac{\alpha}{\sqrt{\|C_{\phi'}(t)\|_{L^\infty(\mathbb{R})}}}$ by applying Lemma 3.2(b).

As a frame version of Theorem 3.9, we have:

Theorem 4.4. *Assume further that $\phi(t) \in AC_{loc}(\mathbb{R})$, $\phi'(t) \in L^2(\mathbb{R})$ and $Z_{\phi'}(t,\xi) \in L^\infty(\mathbb{R}^2)$. If $\|\delta\|_\infty < \frac{\alpha}{\sqrt{2}\|Z_{\phi'}(t,\xi)\|_{L^\infty(\mathbb{R})}}$, then there is a frame $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which (14) holds.*

Proof. As in the proof of Theorem 3.9, we have for any $f(t) = (\mathbf{c} * \phi)(t)$ with $\mathbf{c} \in l^2$,

$$\sum_{n\in\mathbb{Z}} |\langle f(t), q(t, n+\delta_n) - q(t,n)\rangle|^2 \leq \|\delta\|_\infty \int_{-\|\delta\|_\infty}^{\|\delta\|_\infty} \sum_{n\in\mathbb{Z}} \left| f'(t+n)\right|^2 dt \leq 2\,\|\delta\|_\infty^2 \left\|Z_{\phi'}(t,\xi)\right\|_{L^\infty(\mathbb{R})}^2 \|\mathbf{c}\|_2^2 .$$

Then we have together with (15) for any $f \in V(\phi)$,

$$\sum_{n\in\mathbb{Z}} |\langle f(t), q(t, n+\delta_n) - q(t,n)\rangle|^2 \leq \frac{2}{\alpha^2}\|\delta\|_\infty^2 \left\|Z_{\phi'}(t,\xi)\right\|_{L^\infty(\mathbb{R})}^2 \sum_{n\in\mathbb{Z}} |\langle f(t), q(t,n)\rangle|^2 ,$$

from which the conclusion follows since $\{q(t, n+\delta_n)\}_{n\in\mathbb{Z}}$ is also a frame of $V(\phi)$ when $\|\delta\|_\infty < \frac{\alpha}{\sqrt{2}\|Z_{\phi'}(t,\xi)\|_{L^\infty(\mathbb{R})}}$. □

When all δ_n's have the same sign in Theorem 4.4, (14) holds on $V(\phi)$ for any $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ with $\|\delta\|_\infty < \frac{\alpha}{\|Z_{\phi'}(t,\xi)\|_{L^\infty(\mathbb{R}^2)}}$. Theorem 4.4 was first obtained by Sun and Zhou [18], where $\phi(t) \in AC_{loc}(\mathbb{R})$ is such that $\phi'(t) = O((1+|t|)^{-r})$, $r > 1$ so that $\sum_{n\in\mathbb{Z}} |\phi'(t+n)| \in L^\infty(\mathbb{R})$ and so $Z_{\phi'}(t,\xi) \in L^\infty(\mathbb{R}^2)$. They obtained a weaker l^∞-bound by applying Lemma 3.2(c)(see Theorem 5 in [18]).

When $\phi(t)$ is a differentiable frame generator satisfying $C_\phi(t) \in L^\infty(\mathbb{R})$ and $G_{\phi'}(\xi) \in L^\infty(\mathbb{R})$ (so that $\{\phi'(t-n)\}_{n\in\mathbb{Z}}$ is a Bessel sequence), one can obtain another l^∞-bound on δ for (14) holds. See Theorem 2.4 in [19] and Theorem 5 in [22] for more details.

5. Examples

Now, we consider some illustrative examples.

Example 5.1. Let

$$\phi_1(t) = (\phi_0 * \phi_0)(t) = t\chi_{[0,1)}(t) + (2-t)\chi_{[1,2)}(t)$$

be the B-spline of degree 1. Then $G_{\phi_1}(\xi) = \frac{1}{3} + \frac{2}{3}\cos^2\frac{\xi}{2}$ ([9]) so that $\phi_1(t)$ is a continuous Riesz generator with the optimal bounds $(A, B) = \left(\frac{1}{3}, 1\right)$ and $\sup_{\mathbb{R}} C_{\phi_1}(t) < \infty$ since $\phi_1(t)$ has the compact support. On the other hand, $\widehat{\phi_1^*}(\xi) = \sum_{n\in\mathbb{Z}} \phi_1(n)e^{-in\xi} = e^{i\xi}$ so that $\alpha = \beta = 1$ (cf. (6)). Note also that $\phi_1(t) \in AC_{loc}(\mathbb{R})$. It is easy to see that $\left\|C_{\phi_1'}(t)\right\|_{L^\infty(\mathbb{R})} = \left\|Z_{\phi_1'}(t,\xi)\right\|_{L^\infty(\mathbb{R}^2)} = 2$. Hence there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi_1)$ for which the irregular sampling expansion (8) holds on $V(\phi_1)$ if $\|\delta\|_2 < \frac{1}{\sqrt{6}}$ by Theorem 3.4 or $\|\delta\|_\infty < \frac{1}{2\sqrt{6}}$ by Theorem 3.9. These l^2- and l^∞-bounds on δ improve the l^1-bound $\|\delta\|_1 < \frac{1}{3\sqrt{3}}$ in [6] (see Remark 3.3 in [6]) and the l^∞-bound $\|\delta\|_\infty < \frac{1}{3\sqrt{3}}$ in [5] (see Example 3 in [5]) respectively.

Since any Riesz basis is also a frame (in fact, an exact frame), we may apply results in Sect. 4 to $\phi_1(t)$. Then there is a frame $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi_1)$ for which (14) holds if $\|\delta\|_2 < \frac{1}{\sqrt{2}}$ by Corollary 4.3(b) or $\|\delta\|_\infty < \frac{1}{2\sqrt{2}}$ by Theorem 4.4. The latter l^∞-bound improves the l^∞-bound $\|\delta\|_\infty < \frac{1}{3}$ obtained in [22](see Example 2 in [22]).

Example 5.2. Consider the Meyer scaling function ([16]) $\phi(t)$ with

$$\hat{\phi}(\xi) = \begin{cases} 1, & |\xi| < \dfrac{2\pi}{3} \\ \cos\left[\dfrac{\pi}{2}\upsilon(\dfrac{3}{2\pi}|\xi|-1)\right], & \dfrac{2\pi}{3} \leq |\xi| \leq \dfrac{4\pi}{3} \\ 0, & |\xi| > \dfrac{4\pi}{3}, \end{cases}$$

where $\upsilon(\xi) \in C^\infty(\mathbb{R})$ is such that $0 \le \upsilon(\xi) \le 1$, $\upsilon(\xi) = 1$ for $\xi \ge 1$, $\upsilon(\xi) = 0$ for $\xi \le 0$, and $\upsilon(\xi) + \upsilon(1-\xi) = 1$. Then $G_\phi(\xi) \equiv 1$ on $\mathbb{R}$ so that $\phi(t)$ is an orthonormal generator, i.e., $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $V(\phi)$. We also have by the Poisson summation formula

$$\widehat{\phi}^*(\xi) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi + 2n\pi) = \begin{cases} 1, & 0 \le \xi \le \dfrac{2\pi}{3} \\ \cos\dfrac{\pi}{2}\theta + \sin\dfrac{\pi}{2}\theta, & \dfrac{2\pi}{3} \le \xi \le \dfrac{4\pi}{3}, \end{cases}$$

where $\theta = \upsilon(\frac{3}{2\pi}\xi - 1) \in [0,1]$ so that $\alpha = \left\|\widehat{\phi}^*(\xi)\right\|_0 = 1$ and $\beta = \left\|\widehat{\phi}^*(\xi)\right\|_\infty = \sqrt{2}$. On the other hand,

$$H_{\phi'}(\xi) = \sum_{n\in\mathbb{Z}} \left|\widehat{\phi'}(\xi + 2n\pi)\right| = \begin{cases} \xi, & 0 \le \xi \le \dfrac{2\pi}{3} \\ \xi\cos\dfrac{\pi}{2}\theta + (2\pi - \xi)\sin\dfrac{\pi}{2}\theta, & \dfrac{2\pi}{3} \le \xi \le \dfrac{4\pi}{3}. \end{cases}$$

Hence, $H_{\phi'}(\xi) \le \frac{2\sqrt{5}}{3}\pi$ on $\mathbb{R}$ so that by (10)

$$C_{\phi'}(t) \le \frac{1}{2\pi}\left\|H_{\phi'}(\xi)\right\|^2_{L^2[0,2\pi]} \le \frac{20}{9}\pi^2$$

and

$$\left|Z_{\phi'}(t,\xi)\right| \le \left\|H_{\phi'}(\xi)\right\|_{L^\infty(\mathbb{R})} \le \frac{2\sqrt{5}}{3}\pi.$$

Hence, there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which the irregular sampling expansion (8) holds on $V(\phi)$ if $\|\delta\|_2 < \frac{3}{2\sqrt{5}\pi}$ by Theorem 3.4 or $\|\delta\|_\infty < \frac{3}{2\sqrt{10}\pi}$ by Theorem 3.9. Note that in [4], [5], authors obtained only an implicit l^∞-bound $\|\delta\|_\infty < \left(\sum_{n\in\mathbb{Z}} \sup_{[n,n+1]} |\phi'(t)|\right)^{-1}$ (see Example 2 in [5]).

Example 5.3. Let $\phi(t) = (2a)\mathrm{sinc}(2at) = \frac{\sin(2a\pi t)}{\pi t}$, $0 < a < \frac{1}{2}$. Then $\widehat{\phi}(\xi) = \chi_{[-2a\pi,2a\pi]}(\xi)$ and so

$$G_\phi(\xi) = \sum_{n\in\mathbb{Z}} \left|\widehat{\phi}(\xi + 2n\pi)\right|^2 = \widehat{\phi}(\xi) + \widehat{\phi}(\xi - 2\pi) = \chi_{[0,2a\pi]\cup[2\pi-2a\pi,2\pi]}(\xi) \text{ on } [0,2\pi].$$

Hence $G_\phi(\xi) \equiv 1$ on $\sigma(V) = [0,2a\pi] \cup [2\pi - 2a\pi, 2\pi]$ so that $\phi(t)$ is a smooth(C^∞-) frame generator with the optimal bounds $(A,B) = (1,1)$. On the other hand,

$$\widehat{\phi}^*(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi} = \sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi + 2n\pi) = \widehat{\phi}(\xi) + \widehat{\phi}(\xi - 2\pi) \text{ on } [0,2\pi]$$

so that $\alpha = \beta = 1$. We also have by (10)

$$C_{\phi'}(t) \le \frac{1}{2\pi}\left\|H_{\phi'}(\xi)\right\|^2_{L^2[0,2\pi]} = \frac{1}{2\pi}\left\||\xi|\widehat{\phi}(\xi) + |\xi - 2\pi|\widehat{\phi}(\xi - 2\pi)\right\|^2_{L^2[0,2\pi]} = \frac{8a^3\pi^2}{3}$$

and $\left|Z_{\phi'}(t,\xi)\right| \le \left\|H_{\phi'}(\xi)\right\|_{L^\infty(\mathbb{R})} \le 2a\pi$. Hence, there is a frame $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which the irregular sampling expansion (14) holds on $V(\phi)$ if $\|\delta\|_2 < \frac{1}{\pi}\sqrt{\frac{3}{8a^3}}$ by Corollary 4.3(b) or $\|\delta\|_\infty < \frac{1}{2\sqrt{2}a\pi}$ by Theorem 4.4.

The latter l^∞-bound improves the l^∞-bound: $\|\delta\|_\infty \lceil 2\|\delta\|_\infty \rceil < \frac{1}{(2a\pi)^2}$, which can be obtained by applying Theorem 5 in [22]. Here $\lceil x \rceil$ is the smallest integer which is larger than or equal to x. See Example 4 in [22], where $\|\delta\|_\infty < \frac{1}{(2a\pi)^2}$ must be replaced by $\|\delta\|_\infty \lceil 2\|\delta\|_\infty \rceil < \frac{1}{(2a\pi)^2}$.

Example 5.4. Let

$$\phi(t) = \begin{cases} \sqrt{3}\left(t + \dfrac{1}{2}\right), & -\dfrac{1}{2} \le t < 0 \\ \sqrt{3}\,(1-t), & 0 \le t < \dfrac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$, $\|\sum_{n\in\mathbb{Z}} c(n)\phi(t-n)\|^2 = \|\mathbf{c}\|^2$. Hence $\phi(t)$ is an orthonormal generator. On the other hand, $\widehat{\phi}^*(\xi) = \sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi} = \sqrt{3}$ so that $\alpha = \beta = \sqrt{3}$. For any $\delta = \{\delta_n\}_{n\in\mathbb{Z}}$ with $\|\delta\|_\infty < \frac{1}{2}$, we have

$$M \le \sqrt{3}\left(\frac{1}{2} + \|\delta\|_\infty\right) < \sqrt{3} \text{ and}$$

$$\sup_{t\le\|\delta\|_\infty} \Phi(t) \le \sqrt{3}\left(\frac{1}{2} + \|\delta\|_\infty\right) < \sqrt{3},$$

where $M = \sup_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\phi(k + \delta_{k+n}) - \phi(k)|$ and $\Phi(t) = \sum_{j\in\mathbb{Z}} |\phi(j+t) - \phi(j)|$. Hence, there is a Riesz basis $\{S_n(t)\}_{n\in\mathbb{Z}}$ of $V(\phi)$ for which the irregular sampling expansion (8) holds on $V(\phi)$ if $\|\delta\|_\infty < \frac{1}{2}$ by Theorem 3.5.

Acknowledgments

The authors thank the referees for their valuable comments by which the paper can be improved. This work was partially supported by Korea Research Foundation (Grant No. 2009-0084583).

References

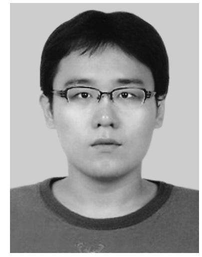
[1] A. Aldroubi and K. Grochenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," SIAM Rev., vol.43, no.4, pp.585–620, 2001.

[2] J.J. Benedetto, C. Heil, and D.F. Walnut, "Differentiation and the Balian-Low theorem," J. Fourier Anal. Appl., vol.1, no.4, pp.355–402, 1994.

[3] W. Chen and S. Itoh, "A sampling theorem for shift-invariant subspace," IEEE Trans. Signal Process., vol.46, no.10, pp.2822–2824, 1998.

[4] W. Chen and S. Itoh, "On irregular sampling in wavelet subspaces," IEICE Trans. Fundamentals, vol.E80-A, no.7, pp.1299–1307, July 1997.

[5] W. Chen, S. Itoh, and J. Shiki, "Irregular sampling theorems for wavelet subspaces," IEEE Trans. Inf. Theory, vol.44, no.3, pp.1131–1142, 1998.
[6] W. Chen, S. Itoh, and J. Shiki, "Wavelet sampling theorems for irregularly sampled signals," Electron. Commun. Jpn. Part 3 Fundam. Electron. Sci., vol.82, no.5, pp.65–71, 1999.
[7] O. Christensen, "A paley-wiener theorem for frames," Proc. Am. Math. Soc., vol.123, no.7, pp.2199–2201, 1995.
[8] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[9] C.K. Chui, An Introduction to Wavelets, Academic Press, N.Y., 1992.
[10] S.J. Favier and R.A. Zalik, "On the stability of frames and Riesz bases," Appl. Comput. Harmon. Anal., vol.2, pp.160–173, 1995.
[11] G. Friedlander and M. Joshi, Introduction to the Theory of Distributions, Cambridge Univ. Press, Cambridge, 1998.
[12] J.R. Higgins, Sampling theory in Fourier and Signal Analysis: Foundations, Oxford Univ. Press, Oxford, 1996.
[13] A. Janssen, "The Zak transform and sampling theorems for wavelet subspaces," IEEE Trans. Signal Process., vol.41, no.12, pp.3360–3364, 1993.
[14] J.M. Kim and K.H. Kwon, "Sampling expansion in shift invariant spaces," Int. J. Wavelets Multiresolutions Inf. Process., vol.6, no.2, pp.223–248, 2008.
[15] Y.M. Liu and G.G. Walter, "Irregular sampling in wavelet subspaces," J. Fourier Anal. Appl., vol.2, no.2, pp.181–189, 1995.
[16] Y. Meyer, Wavelets and Operators, Cambridge Univ. Press, Cambridge, 1992.
[17] C.E. Shannon, "Communication in the presence of noise," Proc. IRE, vol.37, no.1, pp.10–21, 1949.
[18] W. Sun and X. Zhou, "Frames and sampling theorem," Sci. China, vol.41, no.6, pp.606–612, 1998.
[19] W. Sun and X. Zhou, "Sampling theorem for wavelet subspaces: Error estimate and irregular sampling," IEEE Trans. Signal Process., vol.48, no.1, pp.223–226, 2000.
[20] G.G. Walter, "A sampling theorem for wavelet subspaces," IEEE Trans. Inf. Theory, vol.38, no.2, pp.881–884, 1992.
[21] R.M. Young, An introduction to nonharmonic Fourier series, Academic Press, N.Y., 1980.
[22] P. Zhao, C. Zhao, and P.G. Casazza, "Perturbation of regular sampling in shift-invariant spaces for frames," IEEE Trans. Inf. Theory, vol.52, no.10, pp.4643–4648, 2006.
[23] X. Zhou and W. Sun, "On the sampling theorem for wavelet subspaces," J. Fourier Anal. Appl., vol.5, no.4, pp.347–354, 1999.



Jaekyu Lee received the B.S. degree in applied mathematics in 2006 and the M.S. degree in mathematical sciences in 2008, both from KAIST, S. Korea. Now he is a Ph.D. candidate in Department of Mathematical Sciences, KAIST, S. Korea.

Kil Hyun Kwon received the B.S. degree in applied mathematics in 1974 and the M.S. degree in mathematics in 1976, both from Seoul National University, S. Korea. He received the Ph.D. degree in mathematics from Rutgers University, U.S.A., in 1983. Currently, he is a professor at Department of Mathematical Sciences, KAIST, S. Korea. His research interests include Fourier and wavelet analysis, and sampling theory.