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New approach to twisted q -Bernoulli polynomials

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Abstract

By using the theory of basic hypergeometric series, we present some formulas for q -consecutive integers, and we find certain new identities for twisted q -Bernoulli polynomials and q -consecutive integers (Simsek in *Adv. Stud. Contemp. Math.* 16(2):251-278, 2008).

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1 Introduction

The classical *Bernoulli polynomials* $B_n(x)$ and the *Euler polynomials* $E_n(x)$ are usually defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi \quad \text{and}$$

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi,$$

respectively. In addition, the Bernoulli numbers are given by $B_n := B_n(0)$ for $n \geq 1$. Recently, the Bernoulli polynomials and Bernoulli numbers have gained considerable significance in the fields of physics and mathematics [1–4]. For example, Kim [3] defined a new q -analogue of the Bernoulli polynomials and Bernoulli numbers, and he deduced some important relations between them. Moreover, q -analogues have been investigated in the study of quantum groups and q -deformed superalgebras [1]. The connection here is similar, in that much the string theory is set in the language of Riemann surfaces, resulting in connections with elliptic curves, which in turn relate to q -series. A q -analogue is an identity for a q -series that returns a known result in the ‘bosonic’ limit (in contrast to the conventional ‘fermionic’ limit $q \rightarrow -1$) as $q \rightarrow 1$ (from inside the complex unit circle in most situations). In addition to the widely used q -series, we have q -numbers, q -factorials, and q -binomial coefficients. A q -number is obtained by observing $\lim_{q \rightarrow 1} \frac{1-q^n}{1-q} = n$. Thus, we define a q -number as $[n]_q = \frac{1-q^n}{1-q}$. Accordingly, one can define the q -analogue of the factorial, namely, q -factorial, as

$$\begin{aligned} [n]_q! &= [1]_q [2]_q \cdots [n-1]_q [n]_q \\ &= \frac{1-q}{1-q} \frac{1-q^2}{1-q} \cdots \frac{1-q^n}{1-q} = 1 \cdot (1+q) \cdots (1+q+\cdots+q^{n-2})(1+q+\cdots+q^{n-1}). \end{aligned}$$

Using this notation, we can define the q -binomial coefficients, also known as Gaussian coefficients, by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}.$$

Furthermore, the q -Bernoulli polynomials $\beta_{n,q}(x)$ and the q -Bernoulli numbers $\beta_{n,q}$ can be defined in terms of the generating function $F_t(x, q)$ as follows [5]:

$$F_t(x, q) = e^{\frac{1}{1-t}t} \frac{t-1}{\log t} - q \sum_{n=0}^{\infty} t^{n+x} e^{[x+n]_t q} = \sum_{n=0}^{\infty} \frac{\beta_{n,t}(x)}{n!} q^n, \quad |q| < 1, |t| < 1.$$

Kim [6] established an interesting relation between Bernoulli numbers and q -integers, that is,

$$\int_0^n \beta_{l,q} d[x]_q = \frac{1}{l+1} (\beta_{l+1,q}(k) - \beta_{l+1,q}).$$

In addition, Kim [7, Theorem 1], Kim and Lee [8, Lemma 2.1] derived the relations between the Euler polynomials $E_n^{(r)}(x)$ of order r using the alternating sum of powers of consecutive integers $T_k(n)$. Here, $T_k(n) = \sum_{i=0}^n (-1)^i l^k$ and $E_n^{(r)}(x)$ is defined as

$$\left(\frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.$$

Simsek constructed twisted Bernoulli polynomials together with twisted Bernoulli numbers and obtained analytic properties of twisted L -functions [9, 10]. Further, he defined generating functions of the twisted q -Bernoulli numbers and polynomials [9]. In a complex case, the generating function of twisted q -Bernoulli numbers $f_{q,\omega}(t)$ and a q -analogue of the Hurwitz zeta function $f_{\omega}(t, x, q)$ are given by

$$f_{q,\omega}(t) = \sum_{l=0}^{\infty} B_{l,\omega}^*(q) \frac{t^l}{l!} \quad \text{and}$$

$$f_{\omega}(t, x, q) = e^{-t[x]_q} f_{q,\omega}(t) = \sum_{l=0}^{\infty} B_{l,\omega}^*(q, x) \frac{t^l}{l!},$$

where $q \in \mathbb{C}$ with $|q| < 1$, and ω is the r th root of 1. In a complex case, the generating function of twisted q -Bernoulli numbers $f_{q,\omega}(t)$ and a q -analogue of the Hurwitz zeta function $f_{\omega}(t, x, q)$ are given by

$$f_{q,\omega}(t) = \sum_{l=0}^{\infty} B_{l,\omega}^*(q) \frac{t^l}{l!} \quad \text{and}$$

$$f_{\omega}(t, x, q) = e^{-t[x]_q} f_{q,\omega}(t) = \sum_{l=0}^{\infty} B_{l,\omega}^*(q, x) \frac{t^l}{l!},$$

where $q \in \mathbb{C}$ with $|q| < 1$, and ω is the r th root of 1. Simsek [9] then derived the identities

$$B_{l,\omega}^*(q) = (-1)^l l \sum_{n=0}^{\infty} \omega^{-n} q^{-ln} [n]_q^{l-1}, \tag{S1}$$

$$\zeta_{\omega,q}(1-l) = \frac{(-1)^{l+1}}{l} B_{l,\omega}^*(q), \tag{S2}$$

$$B_{l,\omega}^*(x, q) = -(-1)^l l \sum_{n=1}^{\infty} \omega^{-n} q^{-ln} [n+x]_q^{l-1}, \tag{S3}$$

$$\zeta_{\omega,q}(1-l, x) = \frac{(-1)^{l+1}}{l} B_{l,\omega}^*(x, q), \tag{S4}$$

where $\zeta_{\omega,q}(s) = \sum_{n=1}^{\infty} \frac{\omega^{-n} q^{-n}}{(q^{-n} [n]_q)^s}$, $\zeta_{\omega,q}(s, x) = \sum_{n=0}^{\infty} \frac{\omega^{-n} q^{-n}}{(q^{-n} [n+x]_q)^s}$, and x is a natural number. In this paper, we first study relations among q -consecutive integers, q -Bernoulli numbers, and q -Euler numbers.

In 1631, Faulhaber [11] evaluated the sums of powers of consecutive integers $1^k + \dots + n^k$ up to $k = 17$. Further, in 1993, Knuth [12] presented an insightful alternative account of Faulhaber's work. Several mathematicians further considered the problems of q -analogues of such sums of powers [7–9, 13]. On the basis of Bernoulli's concept, Kim derived a q -analogue of the sums of powers of consecutive integers, by setting

$$T_{l,q^h}(n) = \sum_{k=0}^{n-1} [k]_q^l q^{hk}, \tag{1.1}$$

$$T_{l,t} = \sum_{k=0}^{\infty} [k]_q^l t^{k-1} \tag{1.2}$$

and

$$\begin{aligned} S_{l,n}(q) &= \sum_{k=1}^n \frac{1-q^{2k}}{1-q^2} \left(\frac{1-q^k}{1-q} \right)^{l-1} q^{\frac{(l+1)(n-k)}{2}} \\ &= \sum_{k=1}^n [k]_q^l \frac{1+q^k}{1+q} q^{\frac{(l+1)(n-k)}{2}} \\ &= \sum_{k=0}^{n-1} [k+1]_q^l \frac{1+q^{k+1}}{1+q} q^{\frac{(l+1)(n-k-1)}{2}} \end{aligned}$$

with $s \in \mathbb{C}$ and $|s| < 1$.

In Section 2, we recall some necessary identities for basic hypergeometric series [14]. Further, we obtain a generalization of Proposition 2.1, and accordingly, we obtain q -consecutive integers for $\sum_{k=0}^{n-1} [k]_q q^k$. These new results are similar to the ones presented in some other studies [7–9] and [13].

In Section 3, we derive a formula for $S_{2,n}$ and $T_{2,q}(n)$ by using a property of basic hypergeometric series, such as

$$\sum_{k=0}^{\infty} \frac{(aq; q)_k (cq; q)_k}{(q; q)_k (bq; q)_k} t^k = \frac{(aq; q)_{\infty} (ctq; q)_{\infty}}{(bq; q)_{\infty} (t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(b/a; q)_k (t; q)_k}{(q; q)_k (ctq; q)_k} a^k q^k.$$

The q -analogue Eulerian numbers are defined as [15]:

$$E_{m,q} = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{l=0}^{p^N-1} [l]_q^m (-q)^l \quad \text{and}$$

$$E_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} q^{lx} E_{l,q}[x]_q^{n-l}.$$

For these, we establish certain new identities by utilizing basic hypergeometric series, which differ from Bernoulli numbers and polynomials constructed by Kim *et al.* [16, 17] as follows:

$$(-1)^{n+1} q^n E_{1,q}(n) + E_{1,q} = \begin{cases} [n]_q - [2]_q^2 \frac{[2n]_q}{[4]_q} & \text{if } n \text{ is even,} \\ [\infty]_q \left(\frac{[2n]_q}{[n]_q} - \frac{[2]_q^2 [4n]_q}{[4]_q [2n]_q} \right) & \text{otherwise} \end{cases}$$

and

$$(-1)^{n+1} q^n E_{2,q}(n) + E_{2,q} = \begin{cases} [\infty]_q \left([n]_q - 2 \frac{[2]_q^2 [2n]_q}{[4]_q} + \frac{[2]_q [3]_q [3n]_q}{[6]_q} \right) & \text{if } n \text{ is even,} \\ [\infty]_q^2 \left(\frac{[2n]_q}{[n]_q} - 2 \frac{[2]_q^2 [4n]_q}{[4]_q [2n]_q} + \frac{[2]_q [3]_q [6n]_q}{[6]_q [3n]_q} \right) & \text{otherwise.} \end{cases}$$

Here, we note that these are related to $T_{l,q^h}(n)$.

In Section 4, we deduce recursive formulas from Lemma 4.1 for basic hypergeometric series. More precisely, let $S_l(t) = \sum_{k=0}^{\infty} \frac{(q^2; q)_k^l}{(q; q)_k^l} t^k$. Then, we derive the recursive formulas

$$S_{l+1}(t) = \frac{1}{1-q} (S_l(t) - qS_l(tq))$$

$$= \frac{1}{(1-q)^l} \sum_{k=0}^{l+1} \binom{l+1}{k} S_0(tq^k) (-q)^k.$$

Using these identities, we obtain a formula for $\sum_{k=0}^{n-1} [k]_q^l q^k$, and we present relations between q -Bernoulli numbers and q -consecutive integers, which are related to (S1)-(S4). Lastly, the *rank of partition* is defined as the difference between its largest part and the number of its parts. The number of partitions of n with the rank r would be denoted by $P_r(n)$. We use the convention $P_0(0) = 1$, $P_r(n) = 0$ for $r \neq 0$, $n \leq 0$ and $r = 0$, $n < 0$. Here, for the sake of convenience, we define

$$C_1(t; q) = 1,$$

$$C_l(t; q) = \frac{1}{1-q} (C_{l-1}(t; q)(1 - tq^l) - qC_{l-1}(tq; q)(1 - t)).$$

Then, these are related to $P_r(n)$ by the following identity (Remark 4.13):

$$\sum_{l=0}^{\infty} \frac{T_{l,q}}{C_l(u^{-1}q; q)} u^l q^{l+1} = \sum_{r=-\infty}^{\infty} \sum_{n=1}^{\infty} P_r(n) u^r q^n \tag{1.3}$$

with $|q|^2 < |u| < 1$. Finally, we shall relate through Theorem 4.7 and Remark 4.14, q -Bernoulli polynomials with the third-order mock theta functions introduced by Ramanujan.

Throughout this paper, we adopt the following notations:

- $[k]_t = \frac{1-t^k}{1-t}$.
- $[\infty]_t = \frac{1}{1-t}$.
- $S_{m,n}(q) = \sum_{k=1}^n \frac{1-q^{2k}}{1-q^2} \left(\frac{1-q^k}{1-q}\right)^{m-1} q^{\frac{m+1}{2}(n-k)}$.
- $T_{l,q^h}(n) = \sum_{k=0}^{n-1} [k]_q^l q^{hk}$.
- $(a; q)_n = (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1})$.
- $(a; q)_\infty = \prod_{n=0}^{\infty} (1-aq^n)$.
- $(a; q)_0 = 1$.
- $\begin{bmatrix} m \\ k \end{bmatrix}_q = \begin{cases} \frac{(q; q)_m}{(q; q)_k (q; q)_{m-k}} & \text{if } 0 \leq k \leq m, \\ 0 & \text{otherwise.} \end{cases}$
- ω : the r th root of unity.
- $F(a, b; t; q) := F(a, b; t; q) = 1 + \sum_{n=1}^{\infty} \frac{(1-aq)(1-aq^2) \cdots (1-aq^n)}{(1-bq)(1-bq^2) \cdots (1-bq^n)} t^n = \sum_{n=0}^{\infty} \frac{(aq; q)_n}{(bq; q)_n} t^n$.

2 Identities of basic hypergeometric series and $\sum_{k=0}^{n-1} [k]_q q^k$

In this section, we investigate some identities of basic hypergeometric series. To this end, we refer to [14]. Now, we consider the series defined by

$$F(a, b; t; q) = 1 + \sum_{n=1}^{\infty} \frac{(1-aq)(1-aq^2) \cdots (1-aq^n)}{(1-bq)(1-bq^2) \cdots (1-bq^n)} t^n. \tag{2.1}$$

Fine presented many interesting properties in his book; the following identity represents one such property:

$$F(a, b; t; q) = \frac{1-b}{1-t} + \frac{b-atq}{1-t} F(a, b; tq; q). \tag{2.2}$$

Throughout this paper, q denotes a fixed complex number of absolute value less than 1, so that we may write $q = \exp(\pi i \tau)$, where τ is a complex number with a positive imaginary part. We use q^c to denote $\exp(c\pi i \tau)$. The partial product $(aq; q)_n$ converges for all values of a , as may be easily seen from the absolute convergence of $\sum q^n$. Hence, if b is not one of the values q^{-1}, q^{-2}, \dots , the coefficients $\frac{(aq; q)_n}{(bq; q)_n}$ are bounded, and the series (2.1) converges for all t inside the unit circle, and represents an analytic function therein. Hence, the function on the right-hand of (2.2) is regular in the domain $|t| < |q|^{-1}$, except for a simple pole at $t = 1$. Therefore, we obtain the continuation of F to a larger circle. Then, it is easy to apply (2.2) again to the continuation of F to the circle $|t| < |q|^{-2}$, and thus, we conclude that for $b \neq q^{-n}$, $n > 1$, the only possible singularities of F occur at the points $t = q^{-n}$ ($n \geq 0$), which are simple poles in general. As a function of b , F is regular, except possibly at the simple poles $b = q^{-n}$ ($g \geq 1$), provided that b and t do not have one of the singular values mentioned above. First, we derive Theorem 2.2 by generalizing the following proposition.

Proposition 2.1 *For the complex number q and t with $|q| < 1$, we have*

$$\sum_{m=0}^{\infty} \frac{(aq; q)_{2m} (bq; q)_m}{(aq; q)_m (q; q)_m} t^m = \frac{(btq; q)_\infty}{(t; q)_\infty} \sum_{k=0}^{\infty} \frac{(bq; q)_k (t; q)_k}{(q; q)_k (btq; q)_{2k}} (-at)^k q^{\frac{3k^2+k}{2}}.$$

Proof Equation (25.96) in [14]. □

Theorem 2.2 For complex numbers q, t with $|q| < 1$ and an integer $l \geq 0$, we get

$$\sum_{m=0}^{\infty} \frac{(aq; q)_{(l+1)m} (bq; q)_m}{(aq; q)_{lm} (q; q)_m} t^m = \frac{(btq; q)_{\infty}}{(t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(bq; q)_k (t; q)_{lk}}{(q; q)_k (btq; q)_{(l+1)k}} (-at)^k q^{\frac{(2l+1)k^2+k}{2}}.$$

To prove this, we need some identities from [14].

Lemma 2.3 (1) For a nonnegative integer N ,

$$(tq; q)_N = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_q (-t)^k q^{\frac{k^2+k}{2}}.$$

It is an analogue of the binomial series, to which one can reduce termwise with $q = 1$.

(2)

$$F(a, 1; t : q) = \sum_{n=0}^{\infty} \frac{(aq; q)_n}{(q; q)_n} t^n = \frac{(atq; q)_{\infty}}{(t; q)_{\infty}}.$$

(3)

$$(aq; q)_{\infty} = \sum_{k=0}^{\infty} \frac{(-a)^k}{(q; q)_k} q^{\frac{k^2+k}{2}}.$$

Proof Equations (6.23), (6.2), and (12.44) in [14], respectively. □

Proof of Theorem 2.2 We start with the left-hand side in our assertion:

$$\sum_{m=0}^{\infty} \frac{(aq; q)_{(l+1)m} (bq; q)_m}{(aq; q)_{lm} (q; q)_m} t^m = \sum_{m=0}^{\infty} (aq^{lm+1}; q)_m \frac{(bq; q)_m}{(q; q)_m} t^m.$$

Replacing t by aq^{lm} in Lemma 2.3(1), we claim that

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{(bq; q)_m}{(q; q)_m} t^m \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-aq^{lm})^k q^{\frac{k^2+k}{2}} \\ &= \sum_{k=0}^{\infty} (-a)^k q^{\frac{k^2+k}{2}} \sum_{m=k}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q t^m q^{klm} \frac{(bq; q)_m}{(q; q)_m} \\ &= \sum_{k=0}^{\infty} (-a)^k q^{\frac{k^2+k}{2}} \sum_{n=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q t^{n+k} q^{kl(n+k)} \frac{(bq; q)_{n+k}}{(q; q)_{n+k}} \\ &= \sum_{k=0}^{\infty} (-a)^k q^{\frac{k^2+k}{2}} \sum_{n=0}^{\infty} \frac{(q; q)_{n+k}}{(q; q)_k (q; q)_n} t^{n+k} q^{kl(n+k)} \frac{(bq; q)_{n+k}}{(q; q)_{n+k}} \\ &= \sum_{k=0}^{\infty} (-at)^k q^{\frac{(2l+1)k^2+k}{2}} \frac{(bq; q)_k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(bq^{k+1}; q)_n}{(q; q)_n} t^n q^{kln}. \end{aligned}$$

By substituting bq^k and tq^{kl} for a and t , respectively, in Lemma 2.3(2), we derive

$$\sum_{m=0}^{\infty} \frac{(aq; q)_{(l+1)m} (bq; q)_m}{(aq; q)_{lm} (q; q)_m} t^m = \sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (-at)^k q^{\frac{(2l+1)k^2+k}{2}} \frac{(btq^{(l+1)k+1}; q)_{\infty}}{(tq^{lk}; q)_{\infty}}.$$

Since $(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}$, it follows that the last can be written as

$$\frac{(btq; q)_{\infty}}{(t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(bq; q)_k (t; q)_{lk}}{(q; q)_k (btq; q)_{(l+1)k}} (-at)^k q^{\frac{(2l+1)k^2+k}{2}}.$$

Thus, we deduce the identity as desired. □

Next, we present alternative proofs of the following results of Kim [6] as an application of Theorem 2.2.

Corollary 2.4 (1)

$$\sum_{k=0}^{n-1} [k]_q t^k = \frac{1}{1-q} \sum_{k=0}^{n-1} (1-q^k) t^k = \frac{1}{1-q} \left(\frac{1-t^n}{1-t} - \frac{1-t^n q^n}{1-tq} \right).$$

(2)

$$T_{1,q} = \sum_{k=0}^{n-1} [k]_q q^k = q \begin{bmatrix} n \\ 2 \end{bmatrix}_q = \frac{1}{2} \left([n]_q^2 - \frac{[2n]_q}{[2]_q} \right).$$

Note that it is exactly the same as (1.1).

(3)

$$\sum_{k=1}^n [k]_q q^{k-1} = \sum_{k=1}^n [k]_q q^{2n-2k} = \sum_{k=1}^n [k]_q \frac{1+q^k}{1+q} q^{n-k} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q.$$

These are the q -analogues of $\sum_{k=1}^n k = \binom{n+1}{2}$.

Proof (1) Replacing both b and l by 0 in Theorem 2.2, we see that

$$\sum_{k=0}^{\infty} \frac{(aq; q)_k}{(q; q)_k} t^k = \frac{1}{(t; q)_{\infty}} \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} (-at)^k q^{\frac{k^2+k}{2}}.$$

After substituting at for a in Lemma 2.3(3), if we apply it to the above, we get

$$\sum_{k=0}^{\infty} \frac{(aq; q)_k}{(q; q)_k} t^k = \frac{(atq; q)_{\infty}}{(t; q)_{\infty}}.$$

Putting $a = q$ in the above, we get

$$\sum_{k=0}^{\infty} \frac{(q^2; q)_k}{(q; q)_k} t^k = \frac{1}{(1-t)(1-tq)}.$$

However, by using the notation defined in Section 1, the left-hand side of the above can be written as

$$\sum_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q} t^k = \sum_{k=0}^{\infty} [k+1]_q t^k.$$

Thus, from the calculations above and the fact that $[0]_q = 1$, we derive

$$\begin{aligned} \sum_{k=0}^{\infty} [k]_q t^k &= \sum_{k=0}^{\infty} [k+1]_q t^{k+1} = \frac{t}{(1-t)(1-tq)} \\ &= \frac{1}{1-q} \left(\frac{1}{1-t} - \frac{1}{1-tq} \right) = \frac{1}{1-q} \left(\sum_{k=0}^{\infty} (1-q^k) t^k \right). \end{aligned}$$

By considering the exponent of t , we conclude that

$$\sum_{k=0}^{n-1} [k]_q t^k = \frac{1}{1-q} \sum_{k=0}^{n-1} (1-q^k) t^k = \frac{1}{1-q} \left(\frac{1-t^n}{1-t} - \frac{1-t^n q^n}{1-tq} \right).$$

(2) If we put $t = q$ in (1), we have the first equality

$$\begin{aligned} T_{1,q} &= \sum_{k=0}^{n-1} [k]_q q^k = \frac{1}{1-q} \left(\frac{1-q^n}{1-q} - \frac{1-q^{2n}}{1-q^2} \right) \\ &= \frac{q(1-q^{n-1})(1-q^n)}{(1-q)(1-q^2)} = q \begin{bmatrix} n \\ 2 \end{bmatrix}_q. \end{aligned}$$

On the other hand, a direct calculation gives

$$\frac{1}{1-q} \left(\frac{1-q^n}{1-q} - \frac{1-q^{2n}}{1-q^2} \right) = [\infty]_q \left([n]_q - \frac{[2n]_q}{[2]_q} \right) = \frac{1}{2} \left([n]_q^2 - \frac{[2n]_q}{[2]_q} \right).$$

Therefore, we establish the claim.

(3) It follows from (2) that

$$\sum_{k=1}^n [k]_q q^{k-1} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q. \tag{2.3}$$

Moreover, if we replace $n-1$ by n in (1) and multiply both sides by q^{2n} , we obtain

$$q^{2n} \sum_{k=1}^n [k]_q t^k = \frac{q^{2n}}{1-q} ([n+1]_t - [n+1]_{tq}).$$

Observe that the identity above with $t = q^{-2}$ turns out to be a Warnaar's identity [13]:

$$\sum_{k=1}^n [k]_q q^{2n-2k} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q. \tag{2.4}$$

Finally, on the basis of geometric series, we get

$$q^n \sum_{k=0}^n \frac{1 - q^{2k}}{1 - q^2} t^k = \frac{q^n}{1 - q^2} \left(\frac{1 - t^{n+1}}{1 - t} - \frac{1 - (tq^2)^{n+1}}{1 - tq^2} \right).$$

When $t = q^{-1}$ in the above, we have

$$\begin{aligned} \sum_{k=0}^n \frac{1 - q^{2k}}{1 - q^2} q^{n-k} &= \frac{q^n}{1 - q^2} \left(\frac{1 - q^{-n-1}}{1 - q^{-1}} - \frac{1 - q^{n+1}}{1 - q} \right) \\ &= \frac{(1 - q^n)(1 - q^{n+1})}{(1 - q)(1 - q^2)} = \begin{bmatrix} n + 1 \\ 2 \end{bmatrix}_q. \end{aligned}$$

Note that this formula was also derived by several mathematicians such as Schlosser [17] and Warnaar [13]. Since $[0]_q = 0$,

$$\sum_{k=1}^n \frac{1 + q^k}{1 + q} [k]_q q^{n-k} = \sum_{k=0}^n \frac{(1 - q^k)(1 + q^k)}{(1 - q)(1 + q)} q^{n-k} = \begin{bmatrix} n + 1 \\ 2 \end{bmatrix}_q. \tag{2.5}$$

Thus, by combining (2.3), (2.4), and (2.5), we reach the conclusion. \square

3 q -Consecutive and q -analogue of Eulerian numbers $E_{1,q}$ and $E_{2,q}$

We have studied the infinite sum with linear coefficients for q -numbers in the previous section. In this section, we consider the sum with quadratic coefficients, *i.e.*, the following equation.

Lemma 3.1

$$\begin{aligned} \sum_{k=0}^{n-1} [k]_q^2 t^k &= \frac{1}{(1 - q)^2} \sum_{k=0}^{n-1} (1 - 2q^k + q^{2k}) t^k \\ &= \frac{1}{(1 - q)^2} \left(\frac{1 - t^n}{1 - t} - 2 \frac{1 - t^n q^n}{1 - tq} + \frac{1 - t^n q^{2n}}{1 - tq^2} \right). \end{aligned}$$

Theorem 3.2 *As a finite sum for q , we get*

$$\begin{aligned} \sum_{k=0}^{n-1} [k]_q^2 q^{k+1} &= [\infty]_q ([\infty]_q - [1]_q) \left([n]_q - 2 \frac{[2n]_q}{[2]_q} + \frac{[3n]_q}{[3]_q} \right) \\ &= \frac{1}{3} \left([n]_q^3 - \frac{[3n]_q}{[3]_q} \right) - \frac{1}{2} \left([n]_q^2 - \frac{[2n]_q}{[2]_q} \right). \end{aligned}$$

Proof Putting $t = q$ in Lemma 3.1, we derive

$$\begin{aligned} \sum_{k=0}^{n-1} [k]_q^2 q^k &= \frac{1}{(1 - q)^2} \left(\frac{1 - q^n}{1 - q} - 2 \frac{1 - q^{2n}}{1 - q^2} + \frac{1 - q^{3n}}{1 - q^3} \right) \\ &= [\infty]_q^2 \left([n]_q - 2 \frac{[2n]_q}{[2]_q} + \frac{[3n]_q}{[3]_q} \right). \end{aligned}$$

Since $[\infty]_q - [1]_q = \frac{1}{1 - q} - \frac{1 - q}{1 - q} = \frac{q}{1 - q}$, we get the first equality.

By routine calculations, we have

$$\begin{aligned} & \frac{q}{(1-q)^2} \left(\frac{1-q^n}{1-q} - 2 \frac{1-q^{2n}}{1-q^2} + \frac{1-q^{3n}}{1-q^3} \right) \\ &= \frac{q^2(1-q^{n-1})(1-q^n)}{(1-q)(1-q^2)(1-q^3)} \{q^2(1-q^{n-1}) + (1-q^n)\} \\ &= \frac{q^2[n-1]_q[n]_q}{[2]_q[3]_q} (q^2[n-1]_q + [n]_q). \end{aligned}$$

Moreover, $\frac{1}{3}([n]_q^3 - \frac{[3n]_q}{[3]_q}) - \frac{1}{2}([n]_q^2 - \frac{[2n]_q}{[2]_q})$ is also equal to

$$\frac{q^2[n-1]_q[n]_q}{[2]_q[3]_q} (q^2[n-1]_q^2 + [n]_q)$$

by the definition of $[n]_q$. Therefore, the last equality follows. □

Theorem 3.3 Let $S_l(t) = \sum_{k=0}^{\infty} \frac{(q^2; q)_k^l}{(q; q)_k^l} t^k$. For $l = 2$, we obtain

$$S_{2,n}(q) = \frac{(1-q^{n+\frac{1}{2}})(1-q^n)(1-q^{n+1})}{(1-q^{\frac{3}{2}})(1-q)(1-q^2)}.$$

The other cases $l > 2$ will be studied in greater detail in the next section. This was previously proved by Schlosser [17] by using Bailey’s terminating very-well-poised balanced $_{10}\phi_9$ transformation.

Proof By definition in Section 1, we see that

$$\begin{aligned} S_{2,n}(q) &= \sum_{k=1}^n \frac{1-q^{2k}}{1-q} \left(\frac{1-q^k}{1-q} \right) q^{\frac{3n-3k}{2}} \\ &= \sum_{k=1}^n \frac{1+q^k}{1+q} \left(\frac{1-q^k}{1-q} \right)^2 q^{\frac{3n-3k}{2}} \\ &= \frac{q^{\frac{3}{2}n}}{1+q} \left(\sum_{k=1}^n [k]_q^2 q^{-\frac{3}{2}k} + \sum_{k=1}^n [k]_q^2 q^{-\frac{1}{2}k} \right). \end{aligned}$$

From Lemma 3.1, we know that

$$\sum_{k=0}^n [k]_q^2 t^k = \frac{1}{(1-q)^2} \left(\frac{1-t^{n+1}}{1-t} - 2 \frac{1-t^{n+1}q^{n+1}}{1-tq} + \frac{1-t^{n+1}q^{2n+2}}{1-tq^2} \right).$$

Then, the sum of formulas after setting $t = q^{-\frac{3}{2}}$ and $t = q^{-\frac{1}{2}}$ shows that our corollary is true. □

Carlitz [18] constructed a q -analogue of Eulerian numbers. On the other hand, Kim considered the following functions [19]:

$$H_q(t) = [2]_q e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{(-1)^j}{1+q^{j+1}} \left(\frac{1}{1-q} \right)^j \frac{t^j}{j!} = \sum_{k=0}^{\infty} E_{k,q} \frac{t^k}{k!}$$

and

$$H_q(x, t) = [2]_q e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{(-1)^j q^{jx}}{1 + q^{j+1}} \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!} = \sum_{k=0}^{\infty} E_{k,q}(x) \frac{t^k}{k!}.$$

For $m, n \in \mathbb{N}$, he showed that [19, Proposition 2]

$$\sum_{k=0}^{n-1} (-1)^k [k]_q! q^k = \frac{1}{[2]_q} ((-1)^{n+1} q^n E_{l,q}(n) + E_{l,q}). \tag{3.1}$$

A similar result is in [8, Lemma 2.1]. Thus, we get the results for $l = 1$ and 2 as follows.

Theorem 3.4 (1)

$$(-1)^{n+1} q^n E_{1,q}(n) + E_{1,q} = \begin{cases} [n]_q - [2]_q^2 \frac{[2n]_q}{[4]_q} & \text{if } n \text{ is even,} \\ [\infty]_q \left(\frac{[2n]_q}{[n]_q} - \frac{[2]_q^2 [4n]_q}{[4]_q [2n]_q} \right) & \text{otherwise.} \end{cases}$$

(2)

$$(-1)^{n+1} q^n E_{2,q}(n) + E_{2,q} = \begin{cases} [\infty]_q \left([n]_q - 2 \frac{[2]_q^2 [2n]_q}{[4]_q} + \frac{[2]_q [3]_q [3n]_q}{[6]_q} \right) & \text{if } n \text{ is even,} \\ [\infty]_q^2 \left(\frac{[2n]_q}{[n]_q} - 2 \frac{[2]_q^2 [4n]_q}{[4]_q [2n]_q} + \frac{[2]_q [3]_q [6n]_q}{[6]_q [3n]_q} \right) & \text{otherwise.} \end{cases}$$

Proof Replacing t by $-t$ in Corollary 2.4(1), we see that

$$\sum_{k=0}^{n-1} (-1)^k [k]_q t^k = \frac{1}{1-q} \left(\frac{1 - (-t)^n}{1+t} - \frac{1 - (-tq)^n}{1+tq} \right).$$

If we let $t = q$, it becomes

$$\sum_{k=0}^{n-1} (-1)^k [k]_q q^k = \frac{1}{1-q} \left(\frac{1 - (-q)^n}{1+q} - \frac{1 - (-1)^n q^{2n}}{1+q^2} \right). \tag{3.2}$$

Therefore,

$$\sum_{k=0}^{n-1} (-1)^k [k]_q q^k = \begin{cases} \frac{[n]_q}{[2]_q} - \frac{[2n]_q [2]_q}{[4]_q} & \text{if } n \text{ is even,} \\ [\infty]_q \left(\frac{[2n]_q}{[2]_q [n]_q} - \frac{[2]_q [4n]_q}{[4]_q [2n]_q} \right) & \text{otherwise.} \end{cases}$$

Comparing this with (3.1), we can prove (1).

As for (2), if we substitute $-q$ with t in Lemma 3.1, it turns out that

$$\sum_{k=0}^{n-1} (-1)^k [k]_q^2 q^k = \frac{1}{(1-q)^2} \left(\frac{1 - (-q)^n}{1+q} - 2 \frac{1 - (-q^2)^n}{1+q^2} + \frac{1 - (-q^3)^n}{1+q^3} \right).$$

For an even integer n , the above becomes

$$\begin{aligned} & \frac{1}{1-q} \left(\frac{1-q^n}{1-q^2} - 2 \frac{(1-q^{2n})(1-q^2)}{(1-q)(1-q^4)} + \frac{(1-q^{3n})(1-q^3)}{(1-q)(1-q^6)} \right) \\ &= [\infty]_q \left(\frac{[n]_q}{[2]_q} - 2 \frac{[2n]_q [2]_q}{[4]_q} + \frac{[3n]_q [3]_q}{[6]_q} \right). \end{aligned}$$

Similarly, for an odd integer n , we get the result

$$[\infty]_q^2 \left(\frac{[2n]_q}{[2]_q [n]_q} - 2 \frac{[2]_q [4n]_q}{[4]_q [2n]_q} + \frac{[3]_q [6n]_q}{[6]_q [3n]_q} \right)$$

as desired. □

Remark 3.5 In the proof above, as in the case of (3.2), we can obtain an equation by plugging $-q$ into t :

$$\sum_{k=0}^{n-1} [k]_q q^k = \frac{1}{1-q} \left(\frac{1-q^n}{1-q} - \frac{1-q^{2n}}{1-q^2} \right). \tag{3.3}$$

Further, we have $2 \sum_{k=0}^{n-1} [2k]_q q^{2k}$ by adding (3.2) and (3.3). Indeed,

$$\begin{aligned} & \sum_{k=0}^{2n-1} (-1)^k [k]_q q^k + \sum_{k=0}^{2n-1} [k]_q q^k \\ &= \frac{1}{1-q} \left(\frac{1-q^{2n}}{1+q} - \frac{1-q^{4n}}{1+q^2} \right) + \frac{1}{1-q} \left(\frac{1-q^{2n}}{1-q} - \frac{1-q^{4n}}{1-q^2} \right) \\ &= \frac{2}{1-q} \left(\frac{1-q^{2n}}{1-q^2} - \frac{1-q^{4n}}{1-q^4} \right). \end{aligned}$$

Note that this can be written as $2[\infty]_q \left(\frac{[2n]_q}{[2]_q} - \frac{[4n]_q}{[4]_q} \right)$ in terms of q -number notation. Alternatively, it may be factorized and expressed as

$$\frac{2q^2(1-q^{2n-2})(1-q^{2n})}{(1-q)(1-q^4)} = \frac{2q^2[2n-2]_q [2n]_q}{[4]_q},$$

from which we derive

$$\sum_{k=0}^{n-1} [2k]_q q^{2k} = [\infty]_q \left(\frac{[2n]_q}{[2]_q} - \frac{[4n]_q}{[4]_q} \right) = \frac{q^2[2n-2]_q [2n]_q}{[4]_q}.$$

Remark 3.6 In [15, Theorem 1], Kim derived a summation formula for $E_{m,q}$,

$$E_{m,q} = [2]_q \left(\frac{1}{1-q} \right)^m \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{1}{1+q^{k+1}}.$$

We also see one for $E_{k,q}(x)$,

$$E_{k,q}(x) = [2]_q \sum_{m=1}^{\infty} (-1)^m q^m [n+x]_q^k$$

for any positive integer k from [20, Proposition 1]. The proof of Theorem 3.4 is obtained without the help of the summations above.

4 Difference equation and q -consecutive integer

As mentioned in Theorem 3.3, in this section, we study $\sum_{k=0}^{\infty} \frac{(q^2; q)_k^l}{(q; q)_k} t^k$ for more general cases l and its similar sum with q -binomial coefficients. In addition, we show the relations between these and twisted q -Bernoulli numbers. To this end, we need the following lemma.

Lemma 4.1 *Given a sequence A_k ($k \geq 0$) for which $g(t) = \sum_{k=0}^{\infty} A_k t^k$ converges,*

$$\sum_{k=0}^{\infty} \frac{(aq; q)_k}{(bq; q)_k} A_k t^k = \frac{(aq; q)_{\infty}}{(bq; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(b/a; q)_k}{(q; q)_k} a^k q^k g(tq^k).$$

Proof See Section 20, [14]. □

Using this lemma, we generalize the identities considered in the previous two sections.

Proposition 4.2 *For a positive integer l , we have the identities*

$$\begin{aligned} \sum_{k=0}^{n-1} [k]_q^l q^k &= \frac{q}{(1-q)^l} \left(\frac{1-q^{n-1}}{1-q} - lq \frac{1-q^{2(n-1)}}{1-q^2} + \dots + (-1)^l q^l \frac{1-q^{(l+1)(n-1)}}{1-q^{l+1}} \right) \\ &= \frac{q}{(1-q)^l} \sum_{k=0}^l \binom{l}{k} \frac{1-q^{(k+1)(n-1)}}{1-q^{k+1}} (-q)^k. \end{aligned}$$

If l is 1 (respectively, 2), this would be the result of Corollary 2.4(1) (respectively, Lemma 3.1).

Proof Let $S_l(t)$ be a series defined by $\sum_{k=0}^{\infty} \frac{(q^2; q)_k^l}{(q; q)_k} t^k$ for a nonnegative integer l . By setting $a = q$, $b = 1$, $A_k = \frac{(q^2; q)_k^l}{(q; q)_k}$, and $g(t) = S_l(t)$ in Lemma 4.1, we derive the following recursive formula:

$$\begin{aligned} S_{l+1}(t) &= \sum_{k=0}^{\infty} \frac{(q^2; q)_k}{(q; q)_k} \frac{(q^2; q)_k^l}{(q; q)_k} t^k = \frac{(q^2; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(1/q; q)_k}{(q; q)_k} q^{2k} S_l(tq^k) \\ &= \frac{1}{1-q} \left(S_l(t) + \frac{1-\frac{1}{q}}{1-q} q^2 S_l(tq) \right) = \frac{1}{1-q} (S_l(t) - q S_l(tq)). \end{aligned} \tag{4.1}$$

Multiplying both sides by t , we get

$$\begin{aligned} tS_1(t) &= \frac{t}{1-q} (S_0(t) - qS_0(tq)), \\ tS_2(t) &= \frac{t}{1-q} (S_1(t) - qS_1(tq)) = \frac{t}{(1-q)^2} (S_0(t) - 2qS_0(tq) + q^2S_0(tq^2)). \end{aligned}$$

Further, by induction,

$$tS_l(t) = \frac{t}{(1-q)^l} \sum_{k=0}^l \binom{l}{k} S_0(tq^k) (-q)^k.$$

Considering $tS_0(t) = t \sum_{k=0}^{\infty} t^k = \frac{t}{1-t}$, we are able to rewrite the above as

$$\begin{aligned} \sum_{k=0}^{\infty} [k]_q t^k &= \frac{t}{1-q} \left(\frac{1}{1-t} - \frac{q}{1-tq} \right), \\ \sum_{k=0}^{\infty} [k]_q^2 t^k &= \frac{t}{(1-q)^2} \left(\frac{1}{1-t} - \frac{2q}{1-tq} + \frac{q^2}{1-tq^2} \right), \\ &\dots, \\ \sum_{k=0}^{\infty} [k]_q^l t^k &= \frac{t}{(1-q)^l} \left(\frac{1}{1-t} - \frac{lq}{1-tq} + \dots + \frac{(-q)^l}{1-tq^l} \right) = \frac{t}{(1-q)^l} \sum_{k=0}^l \binom{l}{k} \frac{(-q)^k}{1-tq^k}. \end{aligned} \tag{4.2}$$

If we take a finite sum from the above, we get

$$\begin{aligned} \sum_{k=0}^{n-1} [k]_q^l t^k &= \frac{t}{(1-q)^l} \left(\sum_{k=0}^{n-2} t^k - lq \sum_{k=0}^{n-2} t^k q^k + \dots + (-q)^l \sum_{k=0}^{n-2} t^k q^{lk} \right) \\ &= \frac{t}{(1-q)^l} \left(\frac{1-t^{n-1}}{1-t} - lq \frac{1-t^{n-1}q^{n-1}}{1-tq} + \dots + (-q)^l \frac{1-t^{n-1}q^{l(n-1)}}{1-tq^l} \right) \\ &= \frac{t}{(1-q)^l} \sum_{k=0}^l \binom{l}{k} \frac{1-(tq^k)^{n-1}}{1-tq^k} (-q)^k. \end{aligned} \tag{4.3}$$

Therefore, if we let $t = q$, we are done, which amounts to recovering Corollary 2.4, Lemma 3.1, and Theorem 3.2. \square

Remark 4.3 In Section 1, we mentioned Kim’s relation about q -Bernoulli polynomials and q -consecutive integers, from which we obtain some identities for $\int_0^n \beta_{l,q} d[x]_q$, namely

$$\begin{aligned} \int_0^n \beta_{l,q} d[x]_q &= \frac{1}{l+1} (\beta_{l+1,q}(n) - \beta_{l+1,q}) = T_{l,q}(n) \\ &= \sum_{k=0}^{n-1} [k]_q^l q^k = \frac{1}{(1-q)^l} \sum_{k=0}^l (-1)^k \binom{l}{k} \frac{1-q^{(k+1)(n-1)}}{1-q^{k+1}} q^{k+1} \\ &= [\infty]_q^l \sum_{k=0}^l \binom{l}{k} (-1)^k [n-1]_{q^{k+1}} q^{k+1}. \end{aligned}$$

Next, we would like to consider the sum $T_{l,t}$ from (4.2) when $l = 1$,

$$T_{1,t} = \sum_{k=0}^{\infty} [k]_q t^{k-1} = \frac{1}{(1-t)(1-tq)}.$$

By the same argument as that in the proof of Proposition 4.2, we have more general identities for $T_{l,t}$:

$$T_{2,t} = \sum_{k=0}^{\infty} [k]_q^2 t^{k-1} = \frac{1+ tq}{(1-t)(1-tq)(1-tq^2)} =: g(t),$$

$$\begin{aligned}
 T_{3,t} &= \sum_{k=0}^{\infty} [k]_q^3 t^{k-1} = \frac{1}{1-q} \{g(t) - qg(tq)\} \\
 &= \frac{1}{1-q} \left(\frac{1+tq}{(1-t)(1-tq)(1-tq^2)} - \frac{q(1+tq^2)}{(1-tq)(1-tq^2)(1-tq^3)} \right) \\
 &= \frac{1+2tq+2tq^2+t^2q^3}{(1-t)(1-tq)(1-tq^2)(1-tq^3)} \quad \text{and} \\
 T_{4,t} &= \sum_{k=0}^{\infty} [k]_q^4 t^{k-1} = \frac{(1+tq^2)(1+3tq+4tq^2+3tq^3+t^2q^4)}{(1-t)(1-tq)(1-tq^2)(1-tq^3)(1-tq^4)}.
 \end{aligned}$$

All the denominators on the right-hand side are factorized as $l + 1$ terms. However, the numerators are somewhat complex. Therefore, we recursively define a sequence $C_l(t; q)$ with $l \geq 1$ as follows:

$$\begin{aligned}
 C_0(t; q) &= 1, \\
 C_l(t; q) &= \frac{1}{1-q} \{ (1-tq^l)C_{l-1}(t; q) - q(1-t)C_{l-1}(tq; q) \}.
 \end{aligned}$$

Then, we get the following theorem.

Theorem 4.4 (1) *The infinite sum $T_{l,t}$ is expressed as a quotient of $C_l(t; q)$ by $l + 1$ products, precisely speaking,*

$$T_{l,t} = \sum_{k=0}^{\infty} [k]_q^l t^{k-1} = \frac{C_l(t; q)}{(t; q)_{l+1}}.$$

(2) *Since $C_l(q; q) = C_{l-1}(q; q)[l + 1]_q - qC_{l-1}(q^2; q)$, we have*

$$\sum_{k=0}^{\infty} [k]_q^l q^{k-1} = \frac{C_l(q; q)}{(q; q)_{l+1}}.$$

Replacing t by $\frac{1}{t}$ in Theorem 4.4(1), we can deduce one of Simsek's relations [9, Proposition 3.1].

Theorem 4.5 *The generating function (complex cases) of twisted q -Bernoulli numbers is given by*

$$f_{q,\omega}(t) = \sum_{k=0}^{\infty} B_{k,\omega}^*(q) \frac{t^k}{k!},$$

where ω is the r th root of unity and

$$B_{l,\omega}^*(q) = \begin{cases} 0 & \text{if } l = 0, \\ \frac{\omega q}{1-\omega q} & \text{if } l = 1, \\ \frac{l\omega^{l-1}q^{\frac{l(l-1)}{2}}}{(\omega q; q)_l} C_{l-1}(\omega^{-1}q^{-l}; q) & \text{if } l \geq 2. \end{cases}$$

Proof If we recall (S1) from Section 1, we get

$$\begin{aligned}
 B_{0,\omega}^*(q) &= 0, \\
 B_{1,\omega}^*(q) &= -\sum_{k=0}^{\infty} \omega^{-k} q^{-k} = \frac{\omega q}{1 - \omega q}, \\
 B_{2,\omega}^*(q) &= 2 \sum_{k=0}^{\infty} \omega^{-k} q^{-2k} [k]_q = \frac{2}{\omega q^2} \sum_{k=0}^{\infty} [k]_q \omega^{-k+1} q^{-2k+2} \\
 &= \frac{2}{\omega q^2} \sum_{k=1}^{\infty} [k]_q \omega^{-k+1} q^{-2k+2} = \frac{2}{\omega q^2} \sum_{k=0}^{\infty} \frac{(q^2; q)_k}{(q; q)_k} \omega^{-k} q^{-2k} \\
 &= \frac{2}{\omega q^2} F(q, 1; \omega^{-1} q^{-2}; q).
 \end{aligned}$$

Since we know from [14, (6.2)] that

$$F(a, 1; t; q) = \frac{(atq; q)_{\infty}}{(t; q)_{\infty}},$$

by setting q and t to be a and $\omega^{-1} q^{-2}$, respectively, we obtain

$$B_{2,\omega}^*(q) = \frac{2\omega q}{(1 - \omega q)(1 - \omega q^2)}.$$

When l is greater than 2, we get

$$\begin{aligned}
 B_{l,\omega}^*(q) &= (-1)^l l \sum_{k=0}^{\infty} \omega^{-k} q^{-lk} [k]_q^{l-1} = \frac{(-1)^l l}{\omega q^l} T_{l-1, \omega^{-1} q^{-l}} \\
 &= \frac{(-1)^l l}{\omega q^l} \frac{C_{l-1}(\omega^{-1} q^{-l}; q)}{(\omega^{-1} q^{-l}; q)_l} = \frac{l \omega^{l-1} q^{\frac{l(l-1)}{2}}}{(\omega q; q)_l} C_{l-1}(\omega^{-1} q^{-l}; q).
 \end{aligned}$$

□

By (S2) and Theorem 4.5, we get a corollary.

Corollary 4.6 *If l is an integer greater than 1, we have*

$$\zeta_{\omega, q}(1 - l) = (-1)^{l+1} \frac{\omega^{l-1} q^{\frac{l(l-1)}{2}}}{(\omega q; q)_l} C_{l-1}(\omega^{-1} q^{-l}; q).$$

Theorem 4.7 *For any integer x ,*

$$B_{l,\omega}^*(x, q) = \begin{cases} \frac{-\omega q}{1 - \omega q} & \text{if } l = 1, \\ \frac{2(1 - \omega q - q^x + \omega q^{x+2})}{\omega q^2(1 - q)(1 - \omega q)(1 - \omega q^2)} & \text{if } l = 2, \\ \omega^x q^{lx} \left(-\frac{l \omega^l q^{\frac{l(l-1)}{2}}}{(\omega q; q)_l} C_l(\omega^{-1} q^{-l}; q) \right. \\ \quad \left. + \frac{(-1)^l l}{(1 - q)^{l-1}} \sum_{k=0}^{l-1} \binom{l-1}{k} [x]_{\omega q^{l-k}} \omega^{-x} q^{-lx} (-q^x)^k \right) & \text{if } l > 2. \end{cases}$$

Proof It follows from (3.6) in [9] that

$$\begin{aligned} B_{l,\omega}^*(x, q) &= (-1)^{l+1} l \sum_{k=1}^{\infty} \omega^{-k} q^{-kl} [k+x]_q^{l-1} \\ &= \omega^x q^{lx} \left((-1)^{l+1} l \sum_{k=1}^{\infty} \omega^{-k} q^{-kl} [k]_q^{l-1} - (-1)^{l+1} l \sum_{k=1}^x \omega^{-k} q^{-kl} [k]_q^{l-1} \right) \end{aligned}$$

with $x \in \mathbb{N}$. By direct calculation, we get the identities

$$\begin{aligned} B_{1,\omega}^*(x, q) &= \sum_{k=1}^{\infty} \omega^{-k} q^{-k} = -\frac{\omega q}{1-\omega q}, \\ B_{2,\omega}^*(x, q) &= -2 \sum_{k=1}^{\infty} \omega^{-k} q^{-2k} \frac{1-q^{k+x}}{1-q} = \frac{2(1-\omega q - q^x + \omega q^{x+2})}{\omega q^2(1-q)(1-\omega q)(1-\omega q^2)}, \end{aligned}$$

and when $l > 2$, we derive from Proposition 4.2 and Theorem 4.5 that

$$B_{l,\omega}^*(x, q) = \omega^x q^{lx} \left(-B_{l,\omega}^*(q) + (-1)^l l \sum_{k=1}^x \frac{1}{\omega^k q^{lk}} [k]_q^{l-1} \right).$$

Substituting $l-1, x+1$, and $\omega^{-1}q^{-l}$ for l, n , and t in (4.3), respectively, we establish the last identities. \square

As its immediate corollary, we have the following.

Corollary 4.8

$$\zeta_{\omega, q}(1-l, x) = \begin{cases} -\frac{\omega q}{1-\omega q} & \text{if } l = 1, \\ -\frac{1-\omega q - q^x + \omega q^{x+2}}{\omega q^2(1-q)(1-\omega q)(1-\omega q^2)} & \text{if } l = 2, \\ \omega^x q^{lx} \left((-1)^l \frac{\omega^l q^{\frac{l(l-1)}{2}}}{(\omega q; q)_l} C_l(\omega^{-1} q^{-l}; q) \right. \\ \left. - \frac{1}{(1-q)^{l-1}} \sum_{k=0}^{l-1} \binom{l-1}{k} [x]_{\omega q^{l-k}} \omega^{-x} q^{-lx} (-q^x)^k \right) & \text{if } l > 2. \end{cases}$$

Moreover, we can deduce the following corollary, which is analogous to Theorem 4.4.

Corollary 4.9

$$\sum_{k=1}^{\infty} [k]_q^l \frac{1+q^k}{1+q} t^{k-1} = \frac{1}{1+q} \left(\frac{C_l(t; q)}{(t; q)_{l+1}} + q \frac{C_l(tq; q)}{(tq; q)_{l+1}} \right).$$

Proposition 4.10 For a nonnegative integer n ,

$$(1) \quad \sum_{k=0}^{\infty} \begin{bmatrix} k+n \\ n \end{bmatrix}_q t^k = \frac{1}{(t; q)_{n+1}},$$

and when $n = 2$, we get the following by considering the summation from 0 to $n-1$ in the above:

$$(2) \quad \sum_{k=0}^{n-1} \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_q q^k = \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q.$$

Furthermore, we obtain

$$(3) \quad \sum_{k=0}^{\infty} \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_q t^k = \frac{1 + q^2 + 2q^3 + q^4 + q^6}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

Proof By Lemma 2.3(2), we get

$$\sum_{k=0}^{\infty} \begin{bmatrix} k+n \\ n \end{bmatrix}_q t^k = \sum_{k=0}^{\infty} \frac{(q^{n+1}; q)_k}{(q; q)_k} t^k = \frac{1}{(t; q)_{n+1}}.$$

As for the second, we set $n = 2$. Then, it follows from Lemma 4.1 that

$$\sum_{k=0}^{\infty} \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_q t^k = \frac{(q^3; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{-2}; q)_k}{(q; q)_k} q^{3k} \sum_{l=0}^{\infty} t^l q^{lk}.$$

Considering that the exponent of t is less than n only in the above, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_q t^k &= \frac{(q^3; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{-2}; q)_k}{(q; q)_k} q^{3k} \sum_{l=0}^{n-1} t^l q^{2l} \\ &= \frac{1}{(1-q)(1-q^2)} \left(\sum_{l=0}^{n-1} t^l + \frac{(1-q^{-2})}{1-q} q^3 \sum_{l=0}^{n-1} t^l q^l \right. \\ &\quad \left. + \frac{(1-q^{-2})(1-q^{-1})}{(1-q)(1-q^2)} q^6 \sum_{l=0}^{n-1} t^l q^{2l} \right). \end{aligned}$$

Then, putting $t = q$ we get

$$\begin{aligned} &\frac{1}{(1-q)(1-q^2)} \left(\frac{1-q^n}{1-q} - \frac{q(1+q)(1-q^{2n})}{1-q^2} + \frac{q^3(1-q^{3n})}{1-q^3} \right) \\ &= \frac{(1-q^n)(1-q^{n+1})(1-q^{n+2})}{(1-q)(1-q^2)(1-q^3)} = \begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q. \end{aligned}$$

In order to show (3), let $g(t) = \sum_{k=0}^{\infty} \frac{(q^3; q)_k}{(q; q)_k} t^k = \frac{1}{(t; q)_3}$. Then, we derive

$$\begin{aligned} \sum_{k=0}^{\infty} \begin{bmatrix} k+2 \\ 2 \end{bmatrix}_q t^k &= \sum_{k=0}^{\infty} \frac{(q^3; q)_k}{(q; q)_k} \frac{(q^3; q)_k}{(q; q)_k} t^k \\ &= \frac{(q^3; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{-2}; q)_k}{(q; q)_k} q^{3k} g(tq^k) \\ &= \frac{1}{(1-q)(1-q^2)} \left(g(t) + \frac{1-q^{-2}}{1-q} q^3 g(tq) + \frac{(1-q^{-2})(1-q^{-1})}{(1-q)(1-q^2)} q^6 g(tq^2) \right) \\ &= \frac{1}{(1-q)(1-q^2)} \left(\frac{1}{(t; q)_3} - \frac{q(1+q)}{(tq; q)_3} + \frac{q^3}{(tq^2; q)_3} \right) \\ &= \frac{1 + tq + 2tq^2 + tq^3 + t^2q^4}{(1-t)(1-tq)(1-tq^2)(1-tq^3)(1-tq^4)}. \end{aligned}$$

Thus, by substituting q for t , we conclude (3). □

Proposition 4.11 For a nonnegative integer l ,

$$(1) \quad \sum_{k=0}^{\infty} [k]_q^l t^{k-1} = C_l(t; q) \sum_{k=0}^{\infty} \begin{bmatrix} l+k \\ k \end{bmatrix}_q t^k,$$

$$(2) \quad \sum_{k=0}^{\infty} [k]_q^l q^{k-1} = C_l(q; q) \sum_{k=0}^{\infty} \begin{bmatrix} l+k \\ k \end{bmatrix}_q q^k.$$

Proof We see from (6.22) in [14] that $\sum_{k=0}^{\infty} \binom{l+k}{k}_q t^k = \frac{1}{(t; q)_{l+1}}$. Thus, the proposition follows from Theorem 4.4(2). \square

Henceforth, we concentrate on $S_{l,n}(t)$ introduced in Section 1.

Theorem 4.12 For a complex number s with $|s| < 1$ and positive integers m and n ,

$$S_{l,n}(q) = \frac{q^{\frac{(l+1)n}{2}}}{(1+q)(1-q)^l} \sum_{m=0}^l \binom{l}{m} (-q)^m \left(\sum_{k=0}^{n-1} q^{(m-\frac{l+1}{2})k} - q \sum_{k=0}^{n-1} (q^{(m+1-\frac{l+1}{2})k}) \right).$$

Here, we consider $\binom{0}{0}$ as 1.

Proof For fixed n , we consider $\sum_{k=0}^{n-1} [k+1]_q^l \frac{1+q^{k+1}}{1+q} t^k$, and we denote it by $g_l(t)$ so that $g_0(t) = \frac{1}{1+q} (\sum_{k=0}^{n-1} t^k - q \sum_{k=0}^{n-1} t^k q^k)$. By adopting the arguments used in Lemma 4.1, we obtain the following recursion

$$\begin{aligned} g_{l+1}(t) &= \sum_{k=0}^{n-1} \frac{1-q^{k+1}}{1-q} [k+1]_q^l \frac{1+q^{k+1}}{1+q} t^k \\ &= \sum_{k=0}^n \frac{(q^2; q)_k}{(q; q)_k} [k+1]_q^l \frac{1+q^{k+1}}{1+q} t^k \\ &= \frac{(q^2; q)_{\infty}}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(\frac{1}{q}; q)_k}{(q; q)_k} q^{2k} g_l(tq^k) \\ &= \frac{1}{1-q} (g_l(t) - qg_l(tq)). \end{aligned}$$

Since the above is true for all adjacent integers, we obtain

$$g_l(t) = \frac{1}{(1-q)^l} \sum_{k=0}^l \binom{l}{k} (-q)^k g_0(tq^k).$$

Multiplying both sides by $q^{\frac{(l+1)n}{2}}$ and replacing t by $q^{-\frac{l+1}{2}}$, we complete the proof. \square

Remark 4.13 In Section 1, we mentioned that the generating function for the rank of partition can be written as $T_{l,t}$ and $C_l(t; q)$:

$$\sum_{l=0}^{\infty} \frac{T_{l,t}}{C_l(u^{-1}q; q)} u^l q^{l+1} = \sum_{r=-\infty}^{\infty} \sum_{n=1}^{\infty} P_r(n) u^r q^n.$$

By letting $u = -1$ on the right-hand side, we get

$$\begin{aligned} \sum_{r=-\infty}^{\infty} \sum_{n=1}^{\infty} P_r(n)(-1)^r &= \sum_{n=1}^{\infty} \left(\sum_{r: \text{even}} P_r(n) - \sum_{r: \text{odd}} P_r(n) \right) q^n \\ &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}. \end{aligned}$$

Further, it gives rise to a third-order mock theta function $f(q) = 1 + \sum_{n=1}^{\infty} \alpha(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}$, where the explicit formula of $\alpha(n)$ was conjectured by both Andrews [21] and Dragonette [22], and later proved by Bringmann and Ono [23]. Let $N_e(n)$ (resp., $N_o(n)$) be the infinite sum $\sum_{r: \text{even}} P_r(n)$ (resp., $\sum_{r: \text{odd}} P_r(n)$). Then, we can find the coefficients of the infinite sum $\sum_{n=0}^{\infty} \frac{T_{n,-q}(-1)^n}{C_n(-q; q)} q^{n+1}$, because the formula for the partition function $p(n)$ is already known and $\alpha(n) = N_e(n) - N_o(n)$.

Remark 4.14 In Ramanujan’s lost notebook, there are 4 third-order mock theta functions [24, p.345]:

$$\begin{aligned} f(q) &= \sum_{k=0}^{\infty} \frac{q^{k^2}}{(-q; q)_k^2}, \\ \phi(q) &= 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \frac{q^9}{(1+q^2)(1+q^4)(1+q^6)} + \dots, \\ \psi(q) &= \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots, \\ \chi(q) &= 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \dots. \end{aligned}$$

By utilizing our notations, we interpret them as follows.

From Theorem 4.4, Proposition 4.10, Proposition 4.11, and the definition of the mock theta functions [14, pp.55-57], we are able to connect q -consecutive integers with these mock theta functions, namely

$$\begin{aligned} 1 + \sum_{l=0}^{\infty} \frac{T_{l,q}}{C_l(q; q)} q^{l+1} &= \frac{1}{(q; q)_{\infty}} = \sum_{k=0}^{\infty} p(k)q^k, \\ 1 + \sum_{l=0}^{\infty} \frac{T_{l,-q}(-1)^l}{C_l(-q; q)} q^{l+1} &= f(q), \\ 1 + \sum_{l=0}^{\infty} \frac{T_{l,-iq}}{C_l(-iq; q)} (i)^l q^{l+1} &= \phi(q), \\ 1 + \sum_{l=0}^{\infty} \frac{T_{l,-\omega q}}{C_l(-\omega q; q)} (-\omega^2)^l q^{l+1} &= \chi(q), \end{aligned}$$

with $i = e^{2\pi i/4}$ and $\omega = e^{2\pi i/3}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors worked on the results independently. All authors read and approved the final manuscript.

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