Performance Analysis of Incremental Redundancy Type Hybrid ARQ for Finite-length Packets in AWGN Channel

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***Abstract*—We evaluate a hybrid automatic repeat request (HARQ) scheme adopting incremental redundancy (IR) type under a finite-length codeword condition in AWGN channels. In the IR-type HARQ scheme, a long codeword is divided into L blocks, so that L becomes the maximum number of HARQ rounds. Although a finite-length codeword has a transmission rate loss from the channel capacity, IR-type HARQ schemes can significantly reduce the loss due to an early-termination effect. We find the sub-optimal coding rate of a codeword for given parameters such as signal-to-noise ratio (SNR), error probability constraint, and L. In addition, we scale the gap between the channel capacity and the average transmission rate of the IR-type HARQ and show that the gap decreases in the order of $1/L$ when a specific condition is satisfied while the gap of the non-HARQ case decreases in the order of $1/\sqrt{L}$.**

***Index Terms*—Incremental redundancy, AWGN, finite-length packet**

I. Introduction

Information-theoretic channel capacity has been used to measure the maximum coding rate with an arbitrarily low error probability in the regime of infinite block-length [1]. In reality, however, the block-length must be finite. Moreover, real-time applications such as voice over IP and gaming traffic usually generate small-sized packets. Therefore, information-theoretic channel capacity may mislead in such an environment. In the regime of finite block-length, block error probability (BEP) is non-zero even under a coding rate lower than the capacity and it should be evaluated. The upper bound of the block error probability for given block-length and coding rate was studied by Feinstein [2] and Shannon [3]. Recently, Polyanskiy et al. [4] found lower and upper bounds of the maximal coding rate achievable at given block-length and error probability. These bounds also provide tighter BEP bounds for given coding rate and block-length than the conventional bounds. For a finite-length packet, the mutual information is recognized as a random variable and the above error probability bound depends on the distribution of the mutual information. Buckingham et al. [5] defined the information outage probability (IOP) as the probability that the mutual information is lower than the coding rate and showed that the IOP well estimates the BEP achieved by the actual codes.

In order to reduce the error probability or increase the coding rate, we need to increase the block-length. However, if we use acknowledgement (ACK) or non-ACK(NACK) feedback in automatic repeat and request (ARQ) schemes, error probability performance or the achievable coding rate may be improved. To obtain this gain, we consider an incremental redundancy type hybrid ARQ (IR-HARQ) scheme which yields higher spectral efficiency than other ARQ schemes [6], [7]. In the IR-HARQ scheme, a transmitter makes a long codeword composed of several blocks (e.g., L blocks) and sends the first block in the initial HARQ round. If the receiver feeds back NACK, additional redundancy blocks are transmitted and this process repeats until the whole blocks of a codeword are transmitted. Polyanskiy et al. [8] showed that the maximal achievable rate can be significantly improved by using variable-length coding and feedback in general discrete, memoryless channels (DMCs), and provided numerical examples only for binary symmetric channel (BSC) and binary erasure channel (BEC). In their variable-length coding, the receiver attempts to decode a codeword at every symbol reception and requests the transmitter not to send another symbol right after the decoding is successful. Moreover, the maximum number of symbols is not limited. In a practical scheme, however, a unit of transmission and reception is a block composed of typically several tens of symbols. Therefore, a block retransmission scheme based on ACK/NACK feedback and the limitation on the number of blocks should be considered together. Moreover, performance analysis in an AWGN environment is required in reality. Williamson et al. [9] considered the transmission of a group of symbols and showed that the capacity can be approached with small block lengths by using rate compatible sphere-packing (RCSP) analysis and a numerical optimization. However, they did not intuitively present how the external parameters such as L affect the performance.

In this paper, we evaluate the performance gain of the IR-HARQ over the non-HARQ scheme under a finite block-length regime in AWGN channels by scaling the gap between the capacity and the performance of the IR-HARQ according to L. We measure a long-term average transmission rate (LATR) defined as the total number of information-bits transmitted over the total number of symbols used to send the information-bits. The coding rate of the IR-HARQ is optimized to maximize the

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LATR while keeping the error probability not exceeding the a given limit. A random encoding and sup-optimal decoding scheme is used in our IR-HARQ protocol. Moreover, by using a sub-optimal rate selection scheme, we scale the gap between the LATR performance of the IR-HARQ and the channel capacity with parameter L. Through scaling, we show that the performance gap to the capacity has two different tendencies: one scaling law indicates that the gap decreases approximately on the order of $1/L$ when the size of the block-length is small, and L is large. The other scaling law indicates that the gap decreases on the order of $1/\sqrt{L}$ for a large block-length, and the small L. We show the criterion distinguishing these two scaling laws.

II. Error probability of a finite-length packet

A. System model

We consider the following communication model. A set $\mathsf{M} := \{1, 2, \cdots, M\}$ represents a message set and messages are chosen equiprobably. A and B represent the input and output sets, respectively. A mapping $\mathcal{F} : \mathsf{M} \mapsto \mathsf{A}$ represents an encoder. A mapping $\mathcal{G} : \mathsf{B} \mapsto \mathsf{M}$ represents a decoder. A channel follows a conditional probability, $p_{Y^n|X^n} : \mathsf{A} \mapsto \mathsf{B}$, where X^n and Y^n represent the length-n random variables in A and B, respectively. If the error probability of every message is less than or equal to φ, i.e., $\Pr[\mathcal{G}(Y^n) \neq m | X^n = \mathcal{F}(m)] \leq \varphi, \forall m \in \mathsf{M}$, then the codebook is called (n, M, φ)-code. X^n and Y^n follow the marginal probabilities of $p_{X^n}(x^n)$ and $p_{Y^n}(y^n)$, respectively, and a joint probability of $p_{X^n,Y^n}(x^n, y^n)$. The set A can be an n-fold Cartesian product of input alphabet $\mathbb{A}$. If we consider a constraint on the cost of codewords such that $\mathsf{F} = \{x^n : c(x^n) \leq P\}$ where $c(x^n)$ is the cost function and P is a constraint, then the distribution p_{X^n} is defined only on F. We consider the random coding in which we randomly choose M codewords from F by $p_{X^n}(x^n)$.

B. Lower bound of coding rate in AWGN channels

We briefly introduce to find the lower bound of the achievable coding rate of (n, M, φ)-code. Coding rate is defined as $R \triangleq \frac{\log_2 M}{n}$. A sub-optimal decoding method is introduced to find a lower bound (achievability) of coding rate. For a given codebook $\mathsf{C} = (c_1, \cdots, c_M)$, M likelihood testers are operated in parallel such as

$$\mathcal{Z}_{c_i}(y^n) = 1\left\{\frac{p_{Y|X=c_i}(y^n)}{p_Y(y^n)} \geq \gamma\right\}, \tag{1}$$

where the optimal γ value is defined in [4]. The decoder returns

$$\min j \quad \text{s.t.} \quad \{j | Z_{c_j}(y^n) = 1, \forall j \in \mathsf{M}\}. \tag{2}$$

We consider a real-valued AWGN channel where $p_{Y|X=x^n} = \mathcal{N}(x^n, \mathbf{I_n})$ which denotes a real Gaussian random vector with mean vector x^n and covariance matrix $\mathbf{I_n}$. We consider an equal-power constraint on the input set where, i.e., $\mathsf{F}_n \triangleq \{x^n : \|x^n\|^2 = nP\} \subset \mathbb{R}^n$. And we choose a zero-mean, circularly symmetric, real, Gaussian distribution as an output distribution, $p_Y(y^n) = \mathcal{N}(0, \sigma_Y^2 \mathbf{I}_n)$. In this environment, the lower bound of coding rate was derived by Polyanskiy et al. in Theorem 67 of [4] as follows:

Theorem 1 (Polyanskiy's bounds in AWGN): For the AWGN channel with SNR P and the equal-power constraint, the coding rate of (n, M, φ)-code is lower-bounded by

$$R \geq C - \sqrt{\frac{V}{n}} Q^{-1}(\varphi) + \frac{O(1)}{n}, \tag{3}$$

where $C = \frac{1}{2}\log(1+P)$ is the channel capacity and $V = \frac{P}{2}\frac{P+2}{(P+1)^2}(\log e)^2$ is the channel dispersion. For a sufficiently large $n \geq 100$, the last term of the right-hand side (RHS) in (3) rapidly decreases, compared with other terms. Using Theorem 1, $\varphi(R)$ is upper-bounded as

$$\varphi(R) \leq Q\left(\frac{C - (R - O(1/n))}{\sqrt{V/n}}\right). \tag{4}$$

(4) implies that we can achieve a rate R with at least $\varphi(R) = Q\left(\frac{C-(R-O(1/n))}{\sqrt{V/n}}\right)$. For a sufficiently large $n > 100$, we can approximate the upper bound of the error probability of (n, M, φ)-code as follows:

$$\varphi(R) \leq Q\left(\frac{C - (R - O(1/n))}{\sqrt{V/n}}\right) \approx Q\left(\frac{C - R}{\sqrt{V/n}}\right) = \widehat{\varphi}(R). \tag{5}$$

III. Optimization of the IR-HARQ

A. Error probability on the kth HARQ round

We consider the following IR-HARQ protocol in AWGN channel. An encoder maps a message into a length-n codeword. A chosen codeword x^n is divided into L blocks of n' symbols, i.e., $x^n = (x^{(1)}, \cdots, x^{(L)})$ where $x^{(k)} = (x_{n'(k-1)+1}, \cdots, x_{n'k})$. The first block is transmitted on the first HARQ round. If an ACK is fed back, the transmitter sends a new message. Otherwise, the transmitter sends the next block. It repeats until the Lth block is sent. Let $\mathcal{A}_k$ ($\overline{\mathcal{A}}_k$) denotes an event that the decoder declares the ACK (NACK) on the kth HARQ round. We define $R_\text{f} \triangleq \frac{\log_2(M)}{n'}$ which is the initial transmission rate, i.e. the transmission rate when the first block is transmitted. We use R_f as a control parameter.

We evaluate the performance of the IR-HARQ based on the lower bound of the coding rates on the kth HARQ round similar to *Theorem 1*. However, we need to modify some conditions to apply the theorem to the IR-HARQ case. The equal power constraint is set on each HARQ round as $\left\|x^{(k)}\right\|^2 = n'P$, for $1 \leq k \leq L$. Then, a codebook C is constructed by randomly selecting M codewords from $\mathsf{F}_n \triangleq \{x^n : \left\|x^{(k)}\right\|^2 = n'P, \text{ for } 1 \leq k \leq L\} \subset \mathbb{R}^n$. After the kth HARQ round, the decoder has an output sequence $y^{n'k} = (y^{(1)}, \cdots, y^{(k)})$. The decoder operates the M likelihood testers for $y^{n'k}$:

$$\mathcal{Z}_{c_i}(y^{n'k}) = 1\left\{\frac{p_{Y|X=c_i}(y^{n'k})}{p_Y(y^{n'k})} \geq \gamma\right\}. \tag{6}$$

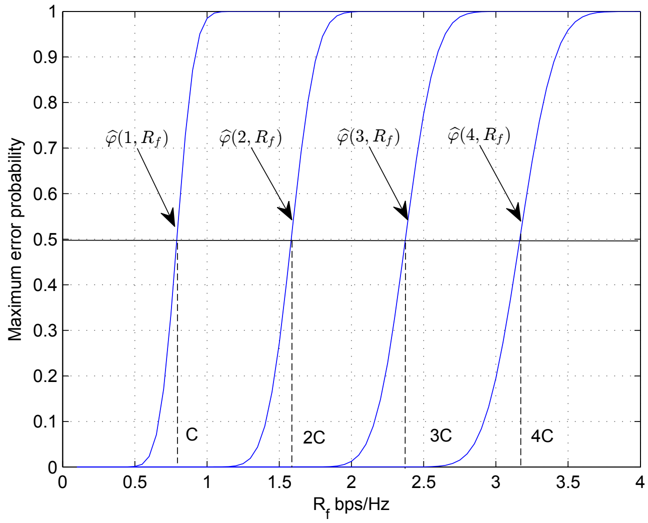


Fig. 1. Approximated maximum error probability at the kth HARQ round for varying R_{f} when $P = 3$ dB and $n' = 100$.

Since we use the same sub-optimal decoder and the equal power constraint for the length-$n'k$ codeword, the achievable coding rate of the $(n'k, M, \varphi)$-code is also achieved by decoding on the kth HARQ round[1]:

$$R_{\mathrm{f}}/k = \frac{\log_2 M}{n'k} \geq C - \sqrt{\frac{V}{n'k}} Q^{-1}(\varphi) + \frac{O(1)}{n'k}, \quad (7)$$

Then, the maximum error probability after the kth HARQ round is approximated as

$$\varphi(k, R_f) \approx \widehat{\varphi}(k, R_f) = Q\left(\frac{kC - R_{\mathrm{f}}}{\sqrt{kV/n'}}\right). \quad (8)$$

For a fixed n', R_{f} depends only on M and is assumed to be a real number. Fig. 1 shows the $\widehat{\varphi}(k, R_{\mathrm{f}})$ for varying R_{f} for $k = 1, \cdots, 4$ when $P = 3$ dB, and $n' = 100$. We can observe that $\varphi(k, kC) = 0.5$ where $C = \log_2(1 + P)$. And the slope of $\varphi(k, R_{\mathrm{f}})$ becomes lower as k increases.

B. Expected number of HARQ rounds

Since the number of HARQ rounds used for each message is a random variable, the spectral efficiency of the IR-HARQ scheme should be observed for a very long time. S_i denotes the number of HARQ rounds used for transmission of the ith message. The average transmission rate [2] over K messages becomes $\frac{K \log_2 M}{\sum_{i=1}^{K} n' S_i} = \frac{R_{\mathrm{f}}}{\frac{1}{K}\sum_{i=1}^{K} S_i}$. If K goes to infinity, we obtain the long-term average transmission rate (LATR) as $\frac{R_{\mathrm{f}}}{\mathbb{E}[S]}$ [bps/Hz]. Now, we derive $\mathbb{E}[S]$.

The probability of $S = k$ can be written as

$$\mathbb{P}[S = k] = \begin{cases} \mathbb{P}[\overline{\mathcal{A}}_1, \cdots, \overline{\mathcal{A}}_{k-1}] - \mathbb{P}[\overline{\mathcal{A}}_1, \cdots, \overline{\mathcal{A}}_k], & \text{for } k < L \\ \mathbb{P}[\overline{\mathcal{A}}_1, \cdots, \overline{\mathcal{A}}_{L-1}], & \text{for } k = L \end{cases} \quad (9)$$

[1]Rigorous proof will be presented at a journal version and is omitted in this version.

[2][the total number of information-bits attempted to be transmitted]/[the total number of symbols used]

The expected number of HARQ rounds per message is derived as

$$\mathbb{E}[S] = \sum_{k=1}^{L} k\mathbb{P}[S = k] = 1 + \sum_{k=1}^{L-1} \mathbb{P}[\overline{\mathcal{A}}_1, \cdots, \overline{\mathcal{A}}_k], \quad (10)$$

where $\mathbb{P}[\overline{\mathcal{A}}_0]$ is 1 since NACK always occurs if there is no transmission. And it is upper-bound and approximated as

$$\mathbb{E}[S] \leq 1 + \sum_{k=1}^{L-1} \mathbb{P}[\overline{\mathcal{A}}_k] \quad (11)$$

$$\approx 1 + \sum_{k=1}^{L-1} \varphi(k, R_{\mathrm{f}}) \quad (12)$$

$$\approx 1 + \sum_{k=1}^{L-1} Q\left(\frac{kC - R_{\mathrm{f}}}{\sqrt{kV/n'}}\right), \quad (13)$$

where (12) follows from the assumption that the NACK event is identical to the error event[3].

C. Optimization Problem

We find the maximum R_{f} to satisfy that the maximum error probability is less than or equal to ϵ for given n' and L. Parameter n' can be given from resource allocation rules in a specification and L can be given from a delay constraint. If we obtain the optimal R_{f}, $M = \left\lfloor 2^{n' R_{\mathrm{f}}} \right\rfloor$ is selected as the optimal M. We optimize R_{f} to maximize the LATR for given n', ϵ and L as follows:

$$R_{\mathrm{f}}^* = \arg\max_{R_{\mathrm{f}}} \quad \frac{R_{\mathrm{f}}}{\mathbb{E}[S|n', R_{\mathrm{f}}, L]} \quad \text{s.t. } \widehat{\varphi}(L, R_{\mathrm{f}}) \leq \epsilon. \quad (14)$$

$\mathcal{T}(R_{\mathrm{f}})$ denotes the objective function of (14). $\mathcal{T}(R_{\mathrm{f}})$ has multiple local maximum points. Therefore, we have to use a numerical search to find the optimal solution, R_{f}^*.

IV. Scaling Performance

We scale the LATR performance in both non-HARQ and IR-HARQ cases and compare two schemes using scaling law equations in terms of L.

In the non-HARQ case, the only way to increase the achievable rate is to increase the codeword length n, as shown in (3). We assume that n' is the minimum number of symbols assigned to one transmission. Then, the codeword length increases as a multiple of n', such as $n = n'L$. If L goes to infinity, the achievable rate converges to the channel capacity. However, the maximum allowable HARQ rounds, L, is limited by the delay or buffer limitation in reality. We can easily observe how fast the achievable rate goes to the channel capacity by scaling the gap between the achievable rate and the channel capacity according to L such as

$$\Delta_{\mathrm{NH}} = C - R^{\mathrm{NH}} \approx \sqrt{\frac{V}{Ln'}} Q^{-1}(\epsilon), \quad (15)$$

[3]This assumption helps to make the analysis tractable. The study on the effect of mismatch between the NACK event and the error event remains for future work.

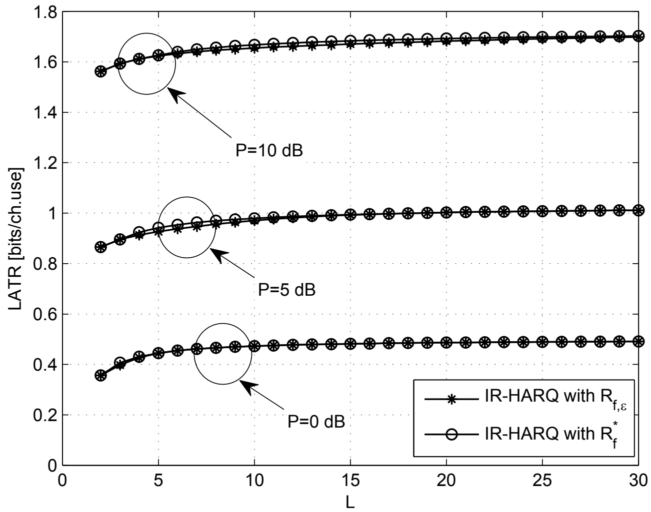


Fig. 2. LATR versus L for IR-HARQ with $R_{\rm f}^*$ or $R_{{\rm f},\epsilon}$ for $P = 0$, 5 and 10 dB when $n' = 100$, and $\epsilon = 10^{-2}$.

where $R^{\rm NH}$ is the achievable coding rate of the non-HARQ case as indicated by (3). From (15), we can observe that the gap decreases on the order of $1/\sqrt{L}$ as L increases.

Now, we scale the gap between the LATR of IR-HARQ and the channel capacity. A difficulty in scaling the LATR of the IR-HARQ occurs due to a complicated form of the objective function in (14) and numerical searches for $R_{\rm f}^*$. We define $R_{{\rm f},\epsilon}$ as the maximum initial transmission rate satisfying the error probability constraint after the Lth HARQ round. Then, from (8), $R_{{\rm f},\epsilon}$ is expressed as

$$R_{{\rm f},\epsilon} = LC - \sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon). \tag{16}$$

We use $R_{{\rm f},\epsilon}$ as a sub-optimal solution instead of $R_{\rm f}^*$. As shown in Fig. 2, $\mathcal{T}(R_{{\rm f},\epsilon})$ approaches $\mathcal{T}(R_{\rm f}^*)$ as L increases especially at low P. The difference between the channel capacity and the LATR of IR-HARQ with $R_{{\rm f},\epsilon}$ is derived as

$$\Delta_{\rm IR} = C - \frac{R_{{\rm f},\epsilon}}{E[S]} = \frac{C}{E[S]}\left(E[S] - \left(\frac{R_{{\rm f},\epsilon}}{C}\right)\right). \tag{17}$$

We approximate the upper bound of $\mathbb{E}[S]$ using (13), which approximately yields the worst case of $\mathcal{T}(R_{{\rm f},\epsilon})$, as follows (see Appendix A for the detail calculation)

$$\mathbb{E}\left[S|R_{f,e}\right] \approx \min\left(L - \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon) + 0.5(1-\epsilon), L\right). \tag{18}$$

Note that the sign of $0.5(1-\epsilon) - \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)$ determines the value of $\mathbb{E}\left[S|R_{f,e}\right]$. We set $\mathcal{B} = 1$ if $0.5(1-\epsilon) < \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)$ and $\mathcal{B} = 0$, otherwise. If we substitute (18) into (17), the gap is approximated as

$$\Delta_{\rm IR} \approx \begin{cases} \frac{C0.5(1-\epsilon)}{L - \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)}, & \text{if } \mathcal{B} = 1, \\ \sqrt{\frac{V}{Ln'}}Q^{-1}(\epsilon), & \text{if } \mathcal{B} = 0. \end{cases} \tag{19}$$

In the case of $\mathcal{B} = 1$, the scaling law can be further approximated for a large L as

$$\frac{C0.5(1-\epsilon)}{L - \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)} \approx \frac{C0.5(1-\epsilon)}{L}. \tag{20}$$

From (20), we can observe that the gap decreases inversely with L if $\mathcal{B} = 1$. Otherwise, the gap decreases inversely with $\sqrt{L}$ which is the same scaling with the non-HARQ case. As n' is smaller, ϵ is smaller, or L is larger, the probability that $\mathcal{B}$ becomes 1 increases.

A. Numerical Results

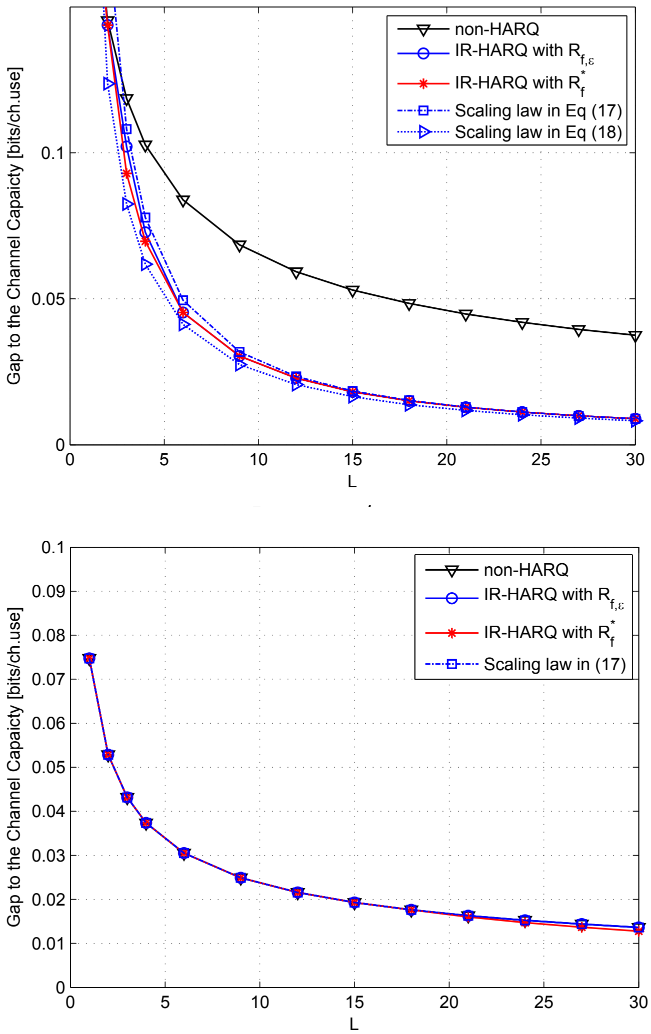
Fig. 3(a) shows the gap to the channel capacity as L increases for $P = 0$ dB, $n' = 100$, and $\epsilon = 10^{-2}$. Since these parameters satisfy $0.5(1-\epsilon) < \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)$, the first law in (19) is applied. We can observe that the two gaps of the IR-HARQ schemes with the optimal rate selection, $R_{\rm f}^*$, and with the sub-optimal rate selection, $R_{{\rm f},\epsilon}$ agree well with the scaling law (19) especially for L larger than 8. The approximated scaling law (20) is also close to the two gaps especially for L larger than 14.

The LATR gap between the IR-HARQ and non-HARQ is approximately 0.04 bits per channel use (BPCU) at $L = 10$. In order to obtain 0.05 bps/Hz gap from the channel capacity, the non-HARQ scheme requires only $L = 18$ while the IR-HARQ scheme requires $L = 6$. Since the scaling law in (20) well agree with the LATR of the IR-HARQ with the optimal solution, we can know that the gap decreases on the order of $1/L$.

Fig. 3(b) shows the gap to the channel capacity as L increases for $P = 10$ dB, $n' = 1000$, and $\epsilon = 10^{-2}$. Since these parameters satisfy $0.5(1-\epsilon) > \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)$, the second law in (19) is applied. We can observe that the gap between the LATR and the channel capacity decreases on the similar order with that of the non-HARQ scheme. Therefore, if the condition $0.5(1-\epsilon) > \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)$ is satisfied, we do not need to use the IR-HARQ schemes.

V. Conclusions

The performance of the IR-type HARQ scheme was evaluated under a finite-length codeword condition in AWGN channel. We formulated an optimization problem and proposed a numerical searching method in order to optimize the coding rate of a codeword for given parameters such as average SNR, error probability constraint, the size of a block, and L. We scaled the gap between the channel capacity and the average transmission rate of the IR-type HARQ and showed that the gap decreases in the order of $1/L$ if the condition $0.5(1-\epsilon) < \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)$ is satisfied while the gap of the non-HARQ case decreases in the order of $1/\sqrt{L}$.



(b) $P = 10$ dB, $n' = 1000$

Fig. 3. The LATR gap to the channel capacity versus L at $\epsilon = 10^{-2}$.

Appendix A
Approximation of $\mathbb{E}[S]$

Note that we use $R_{\mathrm{f},\epsilon}$ instead of $R_{\mathrm{f},*}$. By substituting $R_{\mathrm{f},\epsilon}$ in (16) into $\widehat{\varphi}(k, R_{\mathrm{f}})$, $\widehat{\varphi}(k, R_{\mathrm{f},\epsilon})$ can be expressed as

$$\widehat{\varphi}(k, R_{\mathrm{f},\epsilon}) = Q\left(\frac{(k-L)C + \sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)}{\sqrt{kV/n'}}\right). \quad \text{(A.1)}$$

From (13),

$$\mathbb{E}[S|R_{\mathrm{f},\epsilon}] \le 1 + \sum_{k=1}^{L-1} \widehat{\varphi}(k, R_{\mathrm{f},\epsilon}) \quad \text{(A.2)}$$

$$\approx 1 + \int_1^L \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx + \sum_{k=1}^{L-1}\frac{1}{2}\left[\widehat{\varphi}(k, R_{\mathrm{f},\epsilon}) - \widehat{\varphi}(k+1, R_{\mathrm{f},\epsilon})\right] \quad \text{(A.3)}$$

$$= 1 + \int_1^L \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx + 0.5\widehat{\varphi}(1, R_{\mathrm{f},\epsilon}) - 0.5\widehat{\varphi}(L, R_{\mathrm{f},\epsilon}) \quad \text{(A.4)}$$

$$\approx 1 + \int_1^L \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx + 0.5(1-\epsilon), \quad \text{(A.5)}$$

where (A.5) follows from the fact that $\widehat{\varphi}(1, R_{\mathrm{f},\epsilon}) = Q\left(\frac{(1-L)C + \sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)}{\sqrt{V/n'}}\right) \approx 1$ for a sufficiently large L and $\widehat{\varphi}(L, R_{\mathrm{f},\epsilon}) = \epsilon$. Using the fact $\varphi(L, R_{\mathrm{f},\epsilon}) = \epsilon$ and $1 - Q(\alpha) = Q(-\alpha)$, x satisfying $\widehat{\varphi}(x, R_{\mathrm{f},\epsilon}) = 1 - \epsilon$ can be found as $x_0 = \left(\sqrt{L} - \frac{\sqrt{V/n'}Q^{-1}(\epsilon)}{C}\right)^2$. Therefore, for a small ϵ (< 0.01), $\widehat{\varphi}(x, R_{\mathrm{f},\epsilon}) \ge 1 - \epsilon$ for $x \le x_0$. We are interested in a point $x_1 = L - \frac{2Q^{-1}(\epsilon)}{C}\sqrt{\frac{LV}{n'}}$ which is less than x_0 since $x_0 - x_1 = \left(\frac{Q^{-1}(\epsilon)}{C}\sqrt{\frac{V}{n'}}\right)^2$. By dividing the integral in (A.5), $\mathbb{E}[S|R_{\mathrm{f},\epsilon}]$ is approximated as

$$\mathbb{E}\left[S|R_{f,e}\right] \approx 1 + \int_1^{x_1} \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx + \int_{x_1}^L \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx + 0.5(1-\epsilon) \quad \text{(A.6)}$$

$$\le x_1 + \int_{x_1}^L \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx + 0.5(1-\epsilon). \quad \text{(A.7)}$$

The integral in (A.7) is approximated as

$$\int_{x_1}^L \widehat{\varphi}(x, R_{\mathrm{f},\epsilon})dx = \int_{x_1}^L Q\left(\frac{C}{\sqrt{V/n'}}\left[\frac{x - \left(L + \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon-)\right)}{\sqrt{x}}\right]\right)dx \quad \text{(A.8)}$$

$$= \int_{-\frac{Q^{-1}(\epsilon)}{C}\sqrt{\frac{LV}{n'}}}^{\frac{Q^{-1}(\epsilon)}{C}\sqrt{\frac{LV}{n'}}} Q\left(\frac{Cx\sqrt{n'/V}}{\sqrt{x + L + \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)}}\right)dx \quad \text{(A.9)}$$

$$\approx \int_{-\frac{Q^{-1}(\epsilon)}{C}\sqrt{\frac{LV}{n'}}}^{\frac{Q^{-1}(\epsilon)}{C}\sqrt{\frac{LV}{n'}}} Q\left(\frac{Cx}{\sqrt{\frac{V}{n'}}\sqrt{L + \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon)}}\right)dx \quad \text{(A.10)}$$

$$= \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon), \quad \text{(A.11)}$$

where $1 - Q\left(\lambda\left[\frac{x}{\sqrt{x+\alpha}}\right]\right)$ is regarded as the CDF of a Gaussian random variable with zero-mean and variance with $(x+\alpha)/\lambda^2$ at each point x and we simplify the Q-function by fixing the variance into $\frac{n'}{C^2 V}\left(L + \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\varepsilon)\right)$ in (A.10). By the symmetric property, $Q(\alpha) = 1 - Q(-\alpha)$, (A.11) is obtained.

Finally, $\mathbb{E}[S|R_{\mathrm{f},\epsilon}]$ is approximated as

$$\mathbb{E}\left[S|R_{f,e}\right] \approx \min\left(L - \frac{1}{C}\sqrt{\frac{LV}{n'}}Q^{-1}(\epsilon) + 0.5(1-\epsilon), L\right),$$

where $\min$ is used because $\mathbb{E}\left[S|R_{f,e}\right]$ should be less than or equal to L.

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