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Brief paper

# Reduced order disturbance observer for discrete-time linear systems[☆]

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ABSTRACT

In the paper, an output-based disturbance observer of reduced order is presented for a class of discrete-time linear systems. First, a general form of a disturbance observer is proposed when full states are available. Then, by combining a state function estimator of minimal order, an output-based disturbance observer is derived. The existence condition will be formulated in the form of a static output feedback. Through examples, the effectiveness and advantages of the proposed approach will be demonstrated. A servo control problem in practice is addressed to show the validity of the approach. Furthermore, it will be shown that the proposed approach does provide a smaller order of disturbance observer than that of conventional approaches, while maintaining satisfactory performances.



## 1. Introduction

One of the major methodologies for motion control problems subjected to external disturbances is to use disturbance observers. In literature, several types of disturbance observers can be found in various applications such as the positioning table in Kempf and Kobayashi (1999), a linear stage control in Yoon, Jung, and Sul (2010), the high precision control of a CNC machining center in Yeh and Hsu (2004), the track-following control of hard-disk drive system (Kang, Kim, Lee, & Chung, 2011; Ryoo, Jin, Moon, & Chung, 2003; Teoh, Du, Guo, & Xie, 2008), the servo control of optical data storage system (Kim, 2005; Kim, Lee, & Chung, 2011), hysteresis compensation of piezo-actuators in Yi, Chang, and Shen (2009), a robot manipulator in Katsura, Matsumoto, and Ohnishi (2007), etc. Moreover, disturbance observer approaches have been effectively adopted for fault-detection (Patton & Chen, 1997; Zhang & Ding, 2007).

Many of the disturbance observer approaches are based on the inversion of the transfer function and the inclusion of *Q-filters* (Kim & Chung, 2003; Ryoo et al., 2003; Yi et al., 2009). This so-called *Q-filter* approach has merits in that (i) the concept behind the system inversion is very straight-forward and simple, and (ii) the analysis and synthesis can be carried on with ease in the transfer function framework which has been standard in practice. Moreover, to a certain extent, the robustness of the closed loop stability can be achieved by properly choosing the $Q$-filters (Shim & Jo, 2009). On the other hand, the $Q$-filter approach does not allow transient performance analysis in the time-domain and may not be applicable to the non-minimum phase systems. There are several approaches to overcome the latter weakness in literature (Back & Shim, 2008; Yeh & Hsu, 2004).

Different from the $Q$-filter approaches, other important methodologies for disturbance estimation are the state–space approaches in which the states and unknown inputs are jointly estimated. The unknown inputs may be reconstructed by making use of differentiation of the outputs or a full state observer incorporating the *output error-based* correction term (Corless and Tu (1998) and references therein). The correction term may be adopted in various forms to generate the disturbance estimate: for example, statically proportional form (Corless & Tu, 1998; Gillijns & Moor, 2007), an integral form (Chang, 2006; Orjuela, Marx, Ragot, & Maquin, 2009; Zhang, Jiang, & Shi, 2010), a filtered form (She, Fang, & Ohyama, 2008), etc. It is noted that, in Chang (2006), a proportional integral observer is proposed to simultaneously estimate the unknown states and unknown inputs for a certain class of non-minimum phase systems. The state–space approaches are advantageous in that the full states are obtained, which can be used for full state feedback, and the error dynamics may be analyzed in the time domain. Moreover, they can be effectively extended for handling a class of nonlinear systems (Chen, Su, & Fukuda, 2004; Ha & Trinh, 2004) or for a fault detection method (Zhang et al., 2010).

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As a matter of fact, the study has started from the practical considerations for applying a disturbance observer to motion control systems. In many cases, the disturbance observer-based compensator is often a so-called *add-on* controller which enhances further the disturbance rejection performance at a low frequency range while not affecting the existing control loop (Kim, 2005; Kim & Chung, 2003; Ryoo et al., 2003; Shim & Jo, 2009). To this end, we may need the features:

- An apparent transient behavior of the disturbance observer.
- Low computational load for a low cost digital signal processor.

Toward these, attention is paid to the reduced order disturbance observer approach, in which unknown inputs and a state function are estimated instead of the whole state estimation. In Xiong and Saif (2003), two types of disturbance observer of reduced order are proposed based on the state function estimation. By relaxing the constraint that the observer states should follow the system states, error dynamics can be notably simplified. Motivated by this generic advantage, this paper aims at proposing a novel reduced order disturbance observer which may result in the minimal dynamic order.

The basic idea of the paper starts from a friction observer by Friedland and Park (1992), in which the Coulomb friction (of constant magnitude) is estimated. The modified algorithm from it was proposed by Kim (2002) and successfully applied to the compensation of the eccentric disturbance in the tracking servo system of an optical data storage system. The disturbance observer perfectly estimates the constant disturbance with the exponential convergence. Also, it turned out that the disturbance observer of Kim (2002) can be viewed as a part of feedback controller which increases the control gain at low frequency range (Kim, 2005). Since it does not distort the phase of the loop transfer function nominally designed, the two-stage control scheme does improve the robustness of the closed loop performance against unknown disturbances in the low frequency band. More recently, in the continuous-time system, a general form of the "constant" disturbance observer in Kim (2002, 2005) was proposed by Kim, Rew, and Kim (2010) under full state measurements. It was shown that the constant disturbance observer can be extended to cope with a disturbance of higher order in time series expansion.

In this paper, we extend the results of Kim et al. (2010) to a class of discrete-time linear systems and newly propose a disturbance observer of reduced order relying on partial measurements. To achieve these, in Section 2, a full state discrete-time disturbance observer is newly introduced, which reveals an exponential convergence to an unknown disturbance. Then, to relax the full state availability, a state functional observer is devised. Interestingly, it will be shown that designing the proposed disturbance observer of reduced order is equivalently expressed by a static output feedback problem. In Section 3, through a practical example of an optical data storage system in Kim (2005) and the examples of nonminimum phase introduced in Chang (2006) and Xiong and Saif (2003), the effectiveness of the proposed method is demonstrated. The conclusion follows in Section 4.

The notations in the paper are fairly standard. For instance, $(\cdot)^+$ denotes the Moore–Penrose pseudo-inverse of the argument matrix.

## 2. Main results

Consider a discrete-time linear system

$$\begin{cases} x_{k+1} = \Phi x_k + \Gamma u_k + G d_k, x_0 = x(0), \\ y_k = C x_k, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$ and $y \in \mathbb{R}^l$ are the state variable, the control input, the disturbance, and the measurement output, respectively. The sampling period is $T$ and $G$ is of full column-rank, *i.e.*, $\mathrm{rank}(G) = q$. The disturbance is assumed to be unknown but slowly time-varying in the following.

**Assumption 1.** $d_k \triangleq [d_k^1, \dots, d_k^q]^T$ is slowly time-varying such that, for some constants $\mu_i$'s,

$$\left|\Delta d_k^i\right| \le T \cdot \mu_i, \quad \forall k \ge 1, (i = 1, \dots, q),$$

where $\Delta d_k^i = d_k^i - d_{k-1}^i$.

Now, let us propose a discrete-time disturbance observer as follows.

**Theorem 1** (*DOB0*). *Suppose that $C = I_n$, which allows the availability of the full state vector. Given a matrix $K \in \mathbb{R}^{q\times n}$, consider a disturbance observer*

$$\begin{cases} \hat{d}_k = K x_k - z_k \\ z_{k+1} = z_k + K\left\{(\Phi - I_n)x_k + \Gamma u_k + G\hat{d}_k\right\} \end{cases} \tag{2}$$

*where $\hat{d}_k \in \mathbb{R}^q$ is the disturbance estimate, $z_k \in \mathbb{R}^q$ is the state variable. Then, the state estimation error, $e_k \triangleq d_k - \hat{d}_k$, has the dynamics*

$$e_{k+1} = (I_q - KG)e_k + \Delta d_{k+1} \tag{3}$$

*where $\Delta d_{k+1} = d_{k+1} - d_k$.*

**Proof.** To prove the stability of the disturbance observer, one may show that, for $e_k = d_k - \hat{d}_k$,

$$\begin{aligned} e_{k+1} &= d_{k+1} - \hat{d}_{k+1} \\ &= d_{k+1} - (K x_{k+1} - z_{k+1}) \\ &= d_{k+1} - K(\Phi x_k + \Gamma u_k + G d_k) \\ &\quad + z_k + K\left\{(\Phi - I_n)x_k + \Gamma u_k + G\hat{d}_k\right\} \\ &= d_{k+1} - (K x_k - z_k) - KG(d_k - \hat{d}_k) \\ &= d_{k+1} \underline{- d_k + d_k} - \underbrace{(K x_k - z_k)}_{\hat{d}_k} - KG(d_k - \hat{d}_k) \\ &= \Delta d_{k+1} + (I_q - KG)e_k \end{aligned}$$

which completes the proof. □

In fact, the stability of DOB0 can be achieved if the pair $(I_q, G)$ is observable. It is evident that the pair $(I_q, G)$ is observable since $\mathrm{rank}(G) = q$. Hence, it is always possible to estimate the (slow) disturbance within a bound when the full state is available. One of the easiest choices for $K$ is provided in the following lemma.

**Lemma 1.** *Given any matrix $M$ satisfying $MG$ left-invertible (i.e., $(MG)^+MG = I_q$), suppose that $K = (I_q - \Lambda)(MG)^+M$ for a matrix $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_q\}$ with $|\lambda_i| < 1$ ($i = 1, \dots, q$). This results in each error dynamics as follows:*

$$e_{k+1}^i = \lambda_i e_k^i + \Delta d_{k+1}^i \tag{4}$$

*where $\Delta d_{k+1}^i = d_{k+1}^i - d_k^i$. Then, the estimation error converges to (or is confined within) a bound such that*

$$|e_\infty^i| \le \frac{T\mu_i}{1 - |\lambda_i|} \tag{5}$$

*for $i = 1, \dots, q$.*

The proof is omitted for saving the space. It is noted that the error dynamics (4) are exponentially stable with an accuracy bound in the order of $\mathcal{O}(T)$.

Overall, the advantages of DOB0 are as follows:

- The exponential stability of the estimation error dynamics can be easily assigned (i.e., by the scalar design parameters, $\lambda_i$).

- The order of the disturbance observer dynamics is *minimal* (i.e., its order is merely the number of disturbances, $q$).

**Remark 1.** DOB0 may be viewed as a discrete-time counterpart to a continuous-time disturbance observer presented in Kim et al. (2010) if $M = I_n$. However, the introduction of $M$ provides a further design freedom. For example, when the matrix $CG$ is left-invertible, one may choose as $M = C$ (i.e., $K = (I_q - \Lambda)(CG)^+C$) so that, in (2), the output equation of DOB0 can be computed only with the measurements, that is, $\hat{d}_k = (I_q - \Lambda)(CG)^+ y_k - z_k$.

**Remark 2.** To prohibit the undesirable transient output of the disturbance observer, one may consider two cases for setting the initial value of the state variable, $z_0$, in practice. When the level of disturbance is roughly known as $\tilde{d}$, it is desirable to set it as $z_0 = Kx_0 - \tilde{d}$, with which DOB0 starts estimation from $\tilde{d}$. If the disturbance is fully unknown, it is recommended to set it as $z_0 = Kx_0$, which will result in $\hat{d}_0 = 0$. In practice, the latter is advantageous in terms of smooth transition of control input (from zero to a certain value) when the DOB0-based compensation is initiated in the feedback loop.

Now, to relax the requirement for the full state availability in DOB0, let us rewrite (2) as follows:

$$\begin{cases} \hat{d}_k = KC^+ y_k - z_k + \underline{K\mathcal{N}_C x_k} \\ z_{k+1} = z_k + K\left\{(\Phi - I_n)C^+ y_k + \Gamma u_k + G\hat{d}_k\right\} \\ +\underline{K(\Phi - I_n)\mathcal{N}_C x_k} \end{cases} \tag{6}$$

where $\mathcal{N}_C = I_n - C^+C$. This clearly shows that the underlined vectors above are needed for constructing DOB0 for estimating a disturbance. To identify the states minimally required, consider the minimal rank decomposition such that

$$H_e \triangleq \begin{bmatrix} K\mathcal{N}_C \\ K(\Phi - I_n)\mathcal{N}_C \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} V^T, \tag{7}$$

where $H_1 \in \mathbb{R}^{q\times h}$, $H_2 \in \mathbb{R}^{q\times h}$, $V^T \in \mathbb{R}^{h\times n}$ for $h = \mathrm{rank}(H_e)$. Then, with a state function vector

$$\eta_k \triangleq V^T x_k \in \mathbb{R}^h, \tag{8}$$

one may have an alternative expression of DOB0 as follows: from (6),

$$\begin{cases} \hat{d}_k = KC^+ y_k - z_k + H_1\eta_k \\ z_{k+1} = z_k + K\left\{(\Phi - I_n)C^+ y_k + \Gamma u_k + G\hat{d}_k\right\} + H_2\eta_k. \end{cases} \tag{9}$$

It should be stressed that the state $\eta_k \in \mathbb{R}^h$ is the *unmeasurable* state function of *minimal order* for constructing DOB0. Thus, when an estimator for $\eta_k$ is combined with DOB0 in (9), the order of the disturbance observer would be $q + h$.

**Remark 3.** Note that the order of the state estimator is determined by the rank of $H_e$ (i.e., $h$). In the case of $CG$ left-invertible, one may select the matrix as $K = (I_q - \Lambda)(CG)^+C$ (i.e., with $M = C$). Therefore, it can be seen that $H_1 = 0$ and $\mathrm{rank}(H_e) = \mathrm{rank}\left((CG)^+C(\Phi - I_n)\mathcal{N}_C\right)$, which would reduce the rank number.

**Theorem 2** (*DOB1*). *Given a system in* (1), *consider a disturbance observer as follows:*

$$\begin{cases} \hat{d}_k = KC^+ y_k - z_k + H_1\hat{\eta}_k \\ z_{k+1} = z_k + K\left\{(\Phi - I_n)C^+ y_k + \Gamma u_k + G\hat{d}_k\right\} + H_2\hat{\eta}_k \\ \hat{\eta}_k = \xi_k + Qy_k \\ \xi_{k+1} = R\xi_k + Sy_k + W_u u_k + W_d\hat{d}_k \end{cases} \tag{10}$$

*where* $z_k \in \mathbb{R}^q$ *and* $\xi_k \in \mathbb{R}^h$,

$$W_u \triangleq (V^T - QC)\Gamma, \qquad W_d \triangleq (V^T - QC)G$$

*for the matrices* $S$, $Q$, *and* $R$ *satisfying*

$$(V^T - QC)\Phi - R(V^T - QC) - SC = 0. \tag{11}$$

*Then, it follows that*

$$\begin{pmatrix} e_{k+1} \\ \epsilon_{k+1} \end{pmatrix} = A_e \begin{pmatrix} e_k \\ \epsilon_k \end{pmatrix} + \begin{pmatrix} \Delta d_{k+1} \\ 0_{h\times 1} \end{pmatrix} \tag{12}$$

*where* $e_k \triangleq d_k - \hat{d}_k$, $\epsilon_k \triangleq \eta_k - \hat{\eta}_k$ *and*

$$A_e \triangleq \begin{bmatrix} I_q - KG + H_1(V^T - QC)G & H_1R - H_1 - H_2 \\ (V^T - QC)G & R \end{bmatrix}.$$

**Proof.** First, using that $\eta_k = V^T x_k$ and, thus, $\hat{\eta}_k = \eta_k - \epsilon_k = V^T x_k - \epsilon_k$, one may have

$$\begin{aligned} \hat{d}_k &= KC^+ y_k - z_k + H_1\hat{\eta}_k \\ &= Kx_k - z_k - H_1\epsilon_k. \end{aligned} \tag{13}$$

And,

$$\begin{aligned} z_{k+1} &= z_k + K\left\{(\Phi - I_n)C^+ y_k + \Gamma u_k + G\hat{d}_k\right\} + H_2\hat{\eta}_k \\ &= z_k + K\left\{(\Phi - I_n)x_k + \Gamma u_k + G\hat{d}_k\right\} - H_2\epsilon_k. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} e_{k+1} &= d_{k+1} - \hat{d}_{k+1} \\ &= d_{k+1} - (Kx_{k+1} - z_{k+1}) + H_1\epsilon_{k+1} \\ &= d_{k+1} - K(\Phi x_k + \Gamma u_k + Gd_k) + z_k \\ &\quad + K\left\{(\Phi - I_n)x_k + \Gamma u_k + G\hat{d}_k\right\} - H_2\epsilon_k + H_1\epsilon_{k+1} \\ &= d_{k+1} \underbrace{-Kx_k + z_k}_{-\hat{d}_k - H_1\epsilon_k} - KGe_k - H_2\epsilon_k + H_1\epsilon_{k+1} \\ &= \Delta d_{k+1} + (I_q - KG)e_k - (H_1 + H_2)\epsilon_k + H_1\epsilon_{k+1}. \end{aligned} \tag{14}$$

Second, by routine manipulations, it is straightforward to show that

$$\begin{aligned} \epsilon_{k+1} &= \eta_{k+1} - \hat{\eta}_{k+1} \\ &= V^T x_{k+1} - (\xi_{k+1} + Qy_{k+1}) \\ &= (V^T - QC)x_{k+1} - \xi_{k+1} \\ &= (V^T - QC)(\Phi x_k + \Gamma u_k + Gd_k) \\ &\quad - (R\xi_k + SCx_k + W_u u_k + W_d\hat{d}_k). \end{aligned} \tag{15}$$

Noting that $\xi_k = \hat{\eta}_k - QCx_k = (V^T - QC)x_k - \epsilon_k$ and $\hat{d}_k = d_k - e_k$, (15) leads to

$$\begin{aligned} \epsilon_{k+1} &= W_d e_k + R\epsilon_k + \left\{(V^T - QC)\Gamma - W_u\right\} u_k \\ &\quad + \left\{(V^T - QC)\Phi - R(V^T - QC) - SC\right\} x_k \\ &\quad + \left\{(V^T - QC)G - W_d\right\} d_k \\ &= (V^T - QC)Ge_k + R\epsilon_k \end{aligned} \tag{16}$$

by the definitions of matrices $W_u, W_d, Q, R$ and $S$. Also, by combining (16) with (14), it holds that

$$\begin{aligned} e_{k+1} &= \left\{I_q - KG + H_1(V^T - QC)G\right\} e_k \\ &\quad + (H_1R - H_1 - H_2)\epsilon_k + \Delta d_{k+1}. \end{aligned} \tag{17}$$

These leads to (12), which completes the proof. □

**Remark 4.** The initial condition of DOB1 may be chosen such that $\xi_0 = -Qy_0$ and $z_0 = KC^+ y_0$ in order to start the disturbance estimation from zero. In fact, this does improve the transient response just after initiating the estimation in practice.

If the design parameters $Q, R, S$ are chosen to satisfy (11) and the stability of $A_e$, it is apparent that $\hat{\eta}_k \to \eta_k$ and $\hat{d}_k \to d_k$, as $k \to \infty$, within the accuracy of $\mathcal{O}(T)$ since $\|\Delta d_{k+1}\|_2 \le T \cdot \sqrt{q}\max\{\mu_1, \dots, \mu_q\}$ from Assumption 1.

Now, we are concerned with the solvability of the condition (11) in the following.

**Lemma 2.** *There exist some $Q$, $R$ and $S$ satisfying (11) if and only if*

$$\text{rank}\left(\begin{bmatrix} Z_1 \\ V^T\Phi \end{bmatrix}\right) = \text{rank}(Z_1), \tag{18}$$

*where* $Z_1 \triangleq \begin{bmatrix} C \\ C\Phi \\ V^T \end{bmatrix} \in \mathbb{R}^{(2l+h)\times n}$.

**Proof.** Using a change of variable such that $S_1 \triangleq S - RQ$, (11) can be rewritten as

$$V^T\Phi - [S_1\ Q\ R]Z_1 = 0 \tag{19}$$

which is a linear equation for the augmented matrix $[S_1\ Q\ R]$. The solvability of it is immediately given by (18). □

Thanks to Lemma 2 (and Eq. (19) in the proof), it is straightforward to express the general solution to the equality in (11) as follows: for any $\pi \in \mathbb{R}^{h\times(2l+h-n_{Z1})}$,

$$[S_1\ Q\ R] = V^T\Phi Z_1^+ + \pi U_2^T \tag{20}$$

where $n_{Z1} = \text{rank}(Z_1)$ and $U_2^T \in \mathbb{R}^{(2l+h-n_{Z1})\times(2l+h)}$ is the matrix spanning the left-null space of $Z_1$. Therefore, with appropriate dimensions, one may parameterize all the feasible solutions as follows:

$$\begin{cases} S_1 = X_S + \pi U_{2S}^T \in \mathbb{R}^{h\times l} \\ Q = X_Q + \pi U_{2Q}^T \in \mathbb{R}^{h\times l} \\ R = X_R + \pi U_{2R}^T \in \mathbb{R}^{h\times h} \end{cases} \tag{21}$$

where $[X_S, X_Q, X_R] \triangleq V^T\Phi Z_1^+$ and $[U_{2S}^T, U_{2Q}^T, U_{2R}^T] \triangleq U_2^T$. In fact, the above results eliminate the equality constraint (11) and, instead, a search problem can be formulated for a matrix variable $\pi$ that satisfies the stability of $A_e$. That is, with $Q$ and $R$ in (21), observe that

$$A_e = \bar{A}_e - L_1\pi L_2 \tag{22}$$

where

$$\bar{A}_e = \begin{bmatrix} I_q - KG + H_1(V^T - X_Q C)G & H_1X_R - H_1 - H_2 \\ (V^T - X_Q C)G & X_R \end{bmatrix},$$

$$L_1 = \begin{bmatrix} H_1 \\ I_h \end{bmatrix}, \qquad L_2 = [U_{2Q}^T CG, -U_{2R}^T].$$

Note that, given a matrix $K$, the design of DOB1 is to solve a *static output feedback* problem for a system pair $(\bar{A}_e, L_1, L_2)$. Even though the general solvability of the static output feedback is not known yet, there have been many researches numerically applicable (e.g., see Henrion & Lasserre, 2006 and the references therein).

As a summary, the numerical solution procedures can be summarized as follows:

(1) Choose a matrix $\Lambda$, which is the stability matrix of DOB0, considering the full state disturbance observer dynamics, DOB0.
(2) Choose a matrix $M$ for $K = (I_q - \Lambda)(MG)^+M$. Since the order of a state function (8), $h = \text{rank}(H_e)$, may vary with $M$, the matrix $M$ is chosen to give a minimal order.
(3) Check if the rank condition in (18) is satisfied. Otherwise, go to the step (1) (or, DOB1 may not exist).
(4) Given the matrix $K$, solve the static output feedback problem in (22) to find $\pi = \pi^*$ with which the matrix $A_e$ is stable (i.e., all the eigenvalues are in a unit circle).
(5) For $\pi = \pi^*$, compute $S_1$, $Q$ and $R$ in (21), and $S = S_1 + RQ$ as well.
(6) Compute the matrices such that $W_u = (V^T - QC)\Gamma$ and $W_d = (V^T - QC)G$.

## 3. Numerical examples

### 3.1. Eccentricity compensation of optical data storage device

The proposed disturbance observer, DOB0, can be effectively applied to a servo control system subjected to a sinusoidal disturbance. A typical example is the tracking control of optical data storage systems. The eccentricity of a disk causes the disturbance synchronized with the rotational speed of the spindle motor. To validate the effectiveness of the constant DOB (i.e., DOB0 in Theorem 1), let us consider a track-following system, from Kim (2005), given by

$$\ddot{e}_{tr} + 2\zeta\omega_n\dot{e}_{tr} + \omega_n^2 e_{tr} = -K_o\beta u(t) + K_o d$$

where $e_{tr}$ is the tracking error, which represents the deviation of a beam spot from a track center, and, $\zeta = 0.05$, $\omega_n = 315$ (rad/s), the optical gain $K_o = 1100$ (V/mm), the motor driver gain $\beta = 0.45$ (mm/V). And, the disturbance may be modeled by, at the steady state,

$$d = \ddot{x}_r + 2\zeta\omega_n\dot{x}_r + \omega_n^2 x_r$$

where $x_r = \epsilon\cos(2\pi f_{op}t)$ for an eccentricity $\epsilon$ and the rotational speed of a disk $f_{op}$. It is assumed that $\epsilon = 0.140$ (mm) and $f_{op} = 100$ (Hz). Also, only the tracking error is available for feedback and measured with the uniformly distributed random noise of 0.07 (V).

The sampled data system with $T_s = 1 \times 10^{-4}$ (s) is given by, with $x_k = [e_{tr,k}, \dot{e}_{tr,k}]^T$,

$$\Phi = \begin{bmatrix} 0.9995 & 9.983\times 10^{-5} \\ -9.905 & 0.9964 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} -0.2328 \\ -4653 \end{bmatrix},$$

$$G = \begin{bmatrix} -5.494\times 10^{-6} \\ -0.1098 \end{bmatrix}, \qquad C = [1\ 0].$$

Indeed, the aims of the feedback controller are to achieve a good convergence of the track error from an initial track deviation and maintain the track error within a small bound at the steady state under eccentric disturbance. To this end, a lead-lag compensator ($C_{nom}(z)$) is designed and implemented to have the nominal performance. It is noted that the initial position error rapidly converges to zero within 1 (ms) when there is no disturbance as shown in Fig. 1. Under an eccentric disturbance, a steady state fluctuation can be observed in Fig. 2.

To suppress the track deviation further, it may be needed to increase the controller gain around 100 (Hz). However, it turns out that increasing the low frequency gain would result in a phase loss around the cross-over frequency (e.g., 1.5 (kHz) in this example) so that the phase margin would decrease. Also, the phase lag deteriorates the initial convergence behavior. To resolve the difficulty, we compose a control input with an add-on compensator such that, by using the disturbance estimate of DOB0,

$$u_k = \underbrace{u_{nom,k}}_{\text{by } C_{nom}(z)} - \underbrace{\Gamma^+ G\hat{d}_k}_{\text{by DOB0}}, \tag{23}$$

where $u_{nom,k}$ is the control input by the nominal controller, $C_{nom}(z)$. This leads to the closed loop dynamics

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_{nom,k} + (I_2 - \Gamma\Gamma^+)Gd_k + \Gamma\Gamma^+ Ge_k \\ &= \Phi x_k + \Gamma u_{nom,k} + Ge_k \end{aligned}$$

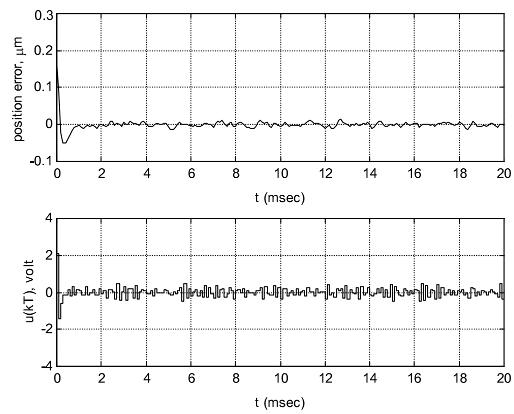



**Fig. 1.** Control performance with $C_{nom}$ when $d(t) = 0$.

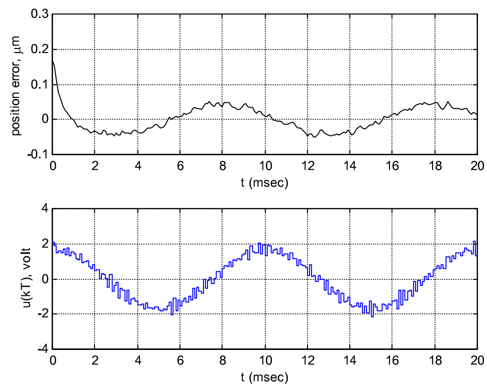



**Fig. 2.** Control performance with $C_{nom}$ when $d(t)$ exists with $\epsilon = 0.140$ mm.

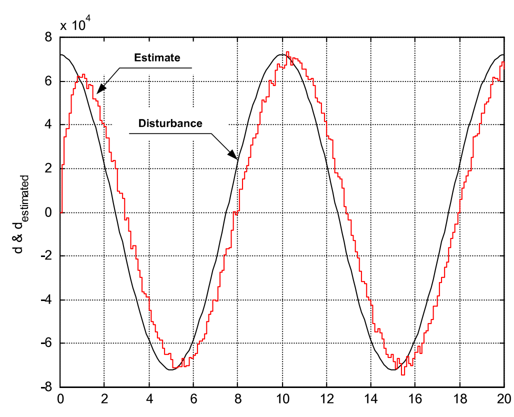



**Fig. 3.** Disturbance estimation performance with DOB0.

since $(I_2 - \Gamma\Gamma^+)G = 0$ and $\Gamma\Gamma^+G = G$ for the example. It is evident that the closed loop system would become robust against to the disturbance as the estimation error gets small.

As in Lemma 1, it was chosen that $K = [1.139 \times 10^{-4}, 2.277]$ for $\Lambda = 0.75$ and $M = I_2$. Also, the derivative of the track error, which is required for computing the DOB0, is approximately calculated such that $\dot{e}_{tr,k} \approx LPF\left(\frac{y_k - y_{k-1}}{T}\right)$, where $LPF(\cdot)$ is a first order low pass filter with a cutoff frequency of 5 (kHz). As shown in Fig. 3, the DOB0 gives a satisfactory disturbance estimate while having a certain delay. The exponential convergence at the beginning phase can be observed, which is a notable advantage of the proposed approach in practice. The time delay in the estimation

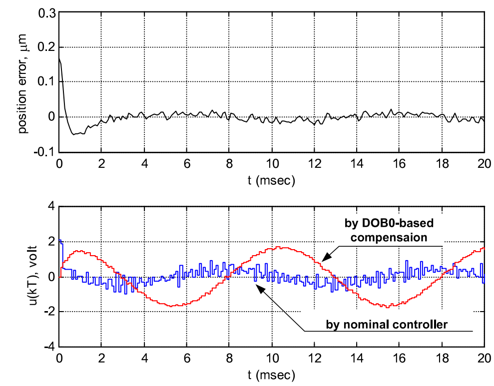



**Fig. 4.** Control performance with $C_{nom}$ and DOB0-based compensation.

can be further decreased by selecting $\Lambda$ smaller, which would increase the gain $K$. This results in more noise contamination in the estimated disturbance. Hence, designing the gain $K$ should be a trade-off between the estimation performance and the noise effect in practice. Now, incorporating the DOB0-based compensation into the control, it can be seen that, in Fig. 4, the control performance is significantly enhanced, compared with that in Fig. 2. Interestingly, as can be seen in Fig. 4, the DOB0-based compensation takes the role to suppress the low frequency disturbance while the nominal lead-lag controller produces only the limited control input against the disturbance.

### 3.2. Case study: a nonminimum phase system

For comparison, an example of a nonminimum phase system is chosen from Chang (2006), in which a proportional integral observer (PIO) of full order was proposed. The system is of third order and two measurements are available. The disturbance is given by

$$d_k = \begin{bmatrix} 0.3\sin(0.1k) + 0.5\cos(0.03k) \\ 0.2\cos(0.05k) + 2 \end{bmatrix}.$$

For the system, we chose as $M = I_3$ and $\Lambda = \text{diag}\{0.7, 0.85\}$ that give

$$K = \begin{bmatrix} 0.9417 & -0.5975 & 1.385 \\ -0.05384 & 1.185 & -0.003718 \end{bmatrix}.$$

With $K$, we obtained that

$$H_e = \begin{bmatrix} 1.164 & 0 & 1.164 \\ -0.02878 & 0 & -0.02878 \\ -0.1533 & 0 & -0.1533 \\ 0.07743 & 0 & 0.07743 \end{bmatrix},$$

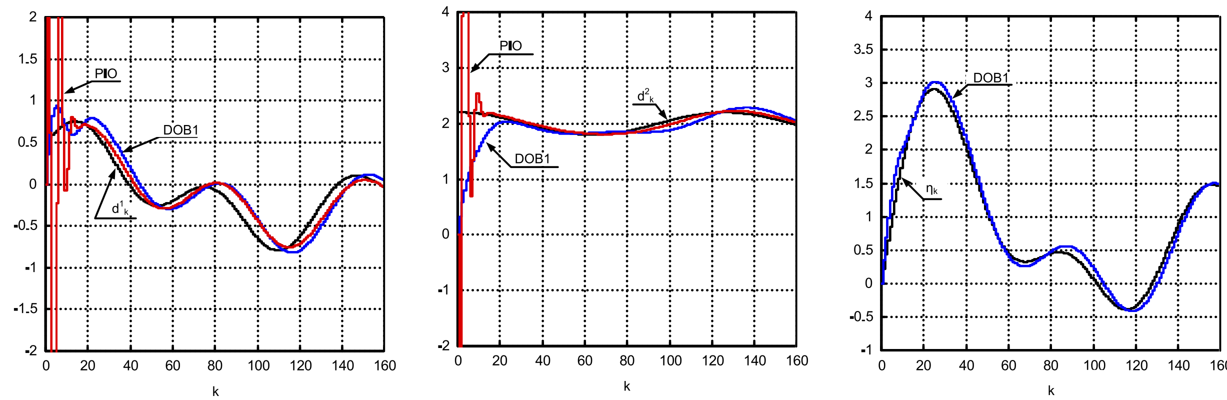
which gives $h = \text{rank}(H_e) = 1$ and the minimal rank decomposition such that

$$H_1 = \begin{bmatrix} 1.646 \\ -0.04070 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} -0.2167 \\ 0.1095 \end{bmatrix},$$

$$V = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \end{bmatrix}.$$

The above implies that only an estimation of the single state function is needed for the disturbance estimation.

Now, as in (18), it was confirmed that the rank condition is met with $\text{rank}(Z_1) = 3$.




(a) $d_k^1$ and its estimates. (b) $d_k^2$ and its estimates. (c) $\eta_k$ and its estimate.

**Fig. 5.** Simulation results. Signals of the plant in black, estimates by DOB1 in blue, and those by the PIO of Chang (2006) in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To set up a static output feedback in (22), we calculated the matrices as follows:

$$V^T \Phi Z_1^+ = [\underbrace{-4.640, 22.66}_{X_S}, \underbrace{40.38, 18.09}_{X_Q}, \underbrace{910.6}_{X_R}] \times 10^{-3}$$

$$U_2 = \begin{bmatrix} U_{2S} \\ U_{2Q} \\ U_{2R} \end{bmatrix} = \begin{bmatrix} 0.3256 & -0.6224 \\ -0.5796 & -0.2735 \\ \hline -0.2902 & 0.6471 \\ 0.6878 & 0.3398 \\ \hline 0.03021 & -0.06036 \end{bmatrix}.$$

Note that $U_2$ is obtained by the singular value decomposition of $Z_1$ (i.e., $Z_1 = [U_1, U_2] \cdot \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$ computed by a MATLAB function—svd(·)). Thus, one may have the following results.

$$\bar{A}_e = \begin{bmatrix} 0.9933 & 0.06893 & 0.06959 \\ -0.007255 & 0.8483 & -0.1059 \\ 0.1783 & 0.04189 & 0.9106 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 1.646 \\ -0.04070 \\ 1 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0.05370 & 0.1067 & -0.03021 \\ -0.01409 & 0.03747 & 0.06036 \end{bmatrix}.$$

With the matrices $\bar{A}_e$, $L_1$ and $L_2$, in (22), we found a matrix

$$\pi^* = [-1.582, 9.575],$$

which assigns the eigenvalues of $A_e = \bar{A}_e - L_1 \pi^* L_2$ at $z = \{0.8773, 0.8092 \pm 0.2848j\}$. Therefore, it is immediate to have the design matrices as follows: from (21), for $\pi = \pi^*$,

$$S_1 = [-6.521, -1.679], \qquad Q = [6.695, 2.184],$$
$$R = 0.2849,$$
$$S = S_1 + RQ = [-4.614, -1.057].$$

Also, $W_u = 0$ and $W_d = [3.982, -0.1481]$.

Note that the proposed DOB1 is of 3rd order (i.e., 1st order for estimation of $\eta_k \in \mathbb{R}$ and 2nd order for estimation of $d_k \in \mathbb{R}^2$). For comparison, the PIO of 5th order (3rd order for estimation of $x_k \in \mathbb{R}^3$ and 2nd order for disturbance estimation) in Chang (2006) was also simulated as shown in Fig. 5. With the reduced order dynamics, the proposed DOB1 shows the satisfactory disturbance estimation performance in the steady state more or less same with that of the PIO in Fig. 5(a)–(b). In particular, the performance of DOB1 is significantly enhanced in the transient response. Also, observe that, in Fig. 5(c), the single order state function, $\eta_k$, is well estimated.

In this typical example, the full state vector can be reconstructed as a *byproduct*. That is, considering that $y_k = Cx_k$ and $\eta_k = V^T x_k$, one may see that

$$\hat{x}_k = \begin{bmatrix} C \\ V^T \end{bmatrix}^{-1} \begin{pmatrix} y_k \\ \hat{\eta}_k \end{pmatrix}$$

since the matrix $\begin{bmatrix} C \\ V^T \end{bmatrix}$ is invertible.

### 3.3. Case study: a double-effect pilot plant evaporator

In order to show the effectiveness of the proposed DOB1 (in view of the order of dynamics), an example is chosen from Xiong and Saif (2003), in which an unknown input observer (UIO) of reduced order was presented in the continuous time. The plant is a double-effect pilot plant evaporator of 5th order in the continuous time domain and measured by two outputs. The external disturbance is given by $d(t) = [5\sin(5t), 2\sin(0.5t)]^T$.

To design the DOB1, the plant is discretized with $T_s = 0.02$ (s) as follows:

$$\Phi = \begin{bmatrix} 1 & 0 & -3.381 \times 10^{-5} & 0 & 0 \\ 0 & 0.9996 & 1.292 \times 10^{-5} & 0 & 0 \\ 0 & 0 & 0.9886 & 0 & 0 \\ 0 & 0 & -3.579 \times 10^{-5} & 1 & 0 \\ 0 & 9.396 \times 10^{-4} & 5.667 \times 10^{-5} & 0 & 0.9995 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} -0.01 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9.426 \times 10^{-3} \\ 9.160 \times 10^{-3} & -0.01 & 0 \\ -5.979 \times 10^{-3} & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0.01 \\ 6.199 \times 10^{-4} & -1.320 \times 10^{-3} \\ 0 & -0.07148 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Considering that $d^1$ is faster than $d^2$, we chose as $\Lambda = \{0.5, 0.75\}$. With $M = C$, we have

$$K = \begin{bmatrix} 106.5 & 806.5 & 0 & 0 & 0 \\ 25 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

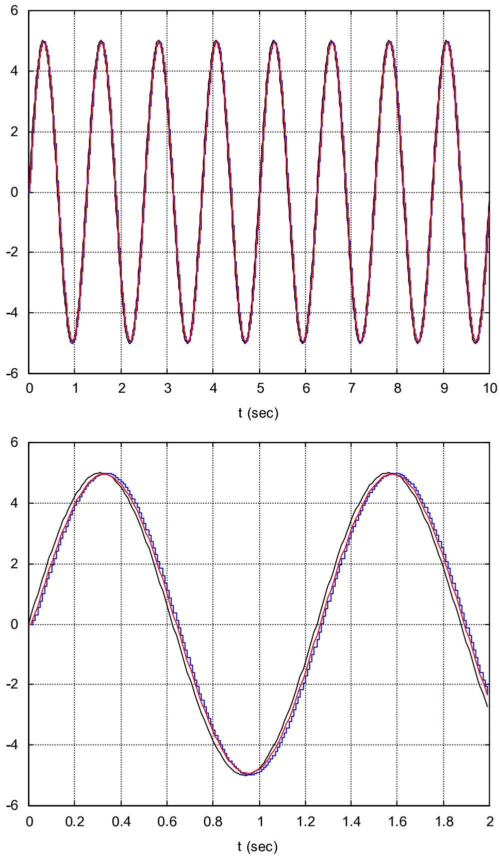



**Fig. 6.** Simulation results. $d^1$ and its estimates for 10 s (top), and for 2 s (bottom). $d^1(t)$ in black, its estimate by DOB1 in blue and, by the UIO of Xiong and Saif (2003) in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Then, one may see that

$$H_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.871\times10^{-3} & 0 & 0 \\ 0 & 0 & -7.5\times10^{-4} & 0 & 0 \end{bmatrix},$$

which gives $H_1 = [0, 0]^T$, $H_2 = [4.871 \times 10^{-3}, -7.5 \times 10^{-4}]^T$ and $V = [0, 0, 1, 0, 0]^T$ for

$$h = \text{rank}(H_e) = 1.$$

That is, only the single order state function (i.e., $\eta_k \in \mathbb{R}$) is required for the disturbance estimation instead of estimating 3 other state variables. It is noted that $h = \text{rank}(H_e) = 2$ if the matrix is chosen as $M = I_5$ in this example. This clearly shows that the order of DOB1 can be reduced by choosing the matrix $M$ appropriately.

The rank condition in (18) is also satisfied, which implies that the search variables can be parameterized with the matrices

$$V^T \Phi Z_1^+ = [\underbrace{0, 0}_{X_S}, \underbrace{0, 0}_{X_Q}, \underbrace{0.9886}_{X_R}],$$

and

$$U_2 = \begin{bmatrix} U_{2S} \\ U_{2Q} \\ U_{2R} \end{bmatrix} = \begin{bmatrix} -0.4999 & -0.5001 \\ -0.5000 & 0.4998 \\ \hline 0.4999 & 0.5001 \\ 0.5002 & -0.5000 \\ \hline 9.995\times10^{-6} & 2\times10^{-5} \end{bmatrix}.$$

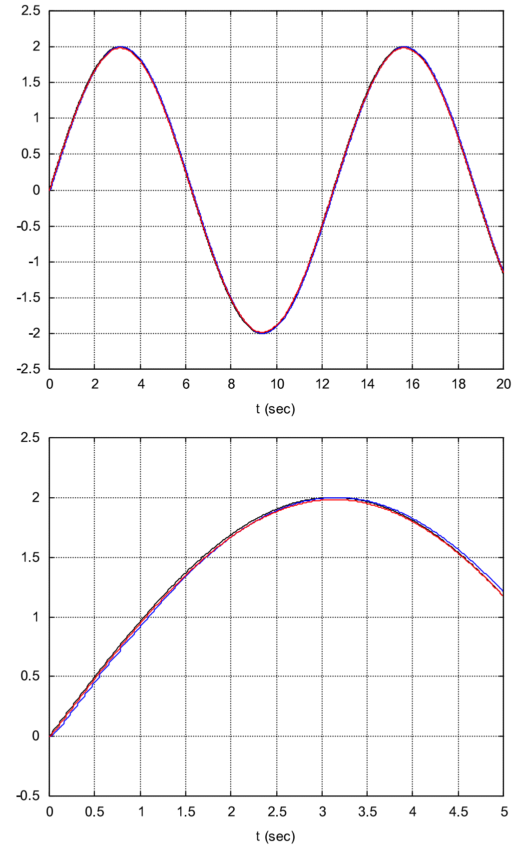



**Fig. 7.** Simulation results. $d^2$ and its estimates by DOB1 and the UIO of Xiong and Saif (2003) for 20 s (top), and for 5 s (bottom). All are nearly overlapped.

Hence, similar to the procedures in Section 3.3, it is immediate to have a static output feedback problem for the system pair $(\bar{A}_e, L_1, L_2)$ such that

$$\bar{A}_e = \begin{bmatrix} 0.5 & 0 & -4.871\times^{-3} \\ 0 & 0.75 & 7.5\times10^{-4} \\ 0 & -0.07148 & 0.98859 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 310.1 & 4339 & -9.995 \\ -310.0 & 5661 & -20.00 \end{bmatrix} \times 10^{-6}.$$

By routine computations, with $\pi^* = [-5.601, -8.381]$, it was obtained that the eigenvalues of $A_e = \bar{A}_e - L_1\pi^* L_2$ are located at $z = \{0.45, 0.75, 0.9884\}$. Thus, we have the design parameters for DOB1 as follows:

$$\begin{cases} S_1 = [6.991, -1.388], \\ Q = [-6.991, 1.389], \\ R = 0.9884, \\ S = S_1 + RQ = [0.08126, -0.01557], \\ W_u = [-69.91, 0, 9.430] \times 10^{-3}, \\ W_d = [-8.610, 2.651] \times 10^{-4}. \end{cases}$$

For comparison, we simulated the UIO (or the state function/input estimator) of 4th order presented in Xiong and Saif (2003). Note that the dynamic order of the proposed DOB1 is three, which is lower than that of the UIO. Given the disturbances, the simulation results are shown in Figs. 6–8. The disturbance estimation performances of both approaches are almost same. Also, it can be seen that the single order state function $\eta_k$ is well estimated by the proposed DOB1.

Overall, through two examples, it was shown that the proposed approach presents lower order disturbance observers

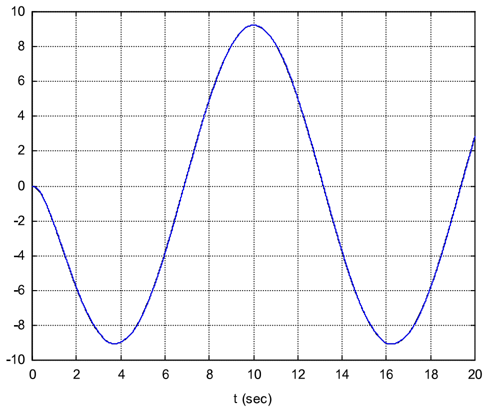



**Fig. 8.** Simulation results. $\eta(t)$ and $\hat{\eta}_k$ by DOB1.

than the existing approaches in literature do. Also, the transient performance is shown to be remarkably improved in the proposed method.

As a remark, it should be noted that the examples in two case studies have unstable zeros. This implies that the proposed method is also applicable to a certain class of nonminimum phase systems.

## 4. Concluding remarks

In this paper, an output-based disturbance observer of reduced order (named DOB1) was newly proposed in the discrete-time domain. Under the availability of the full state, first, an unknown input observer (so-called DOB0) was derived, which has the dynamic order as the number of disturbances. Then, DOB0 was combined with a state function estimator in order to rely on only the partial measurements. By defining a minimal set of states required for DOB1, the proposed DOB1 presents the lowest order dynamics, compared with the conventional approaches. It was shown that the solution procedures of DOB1 leads to a static output feedback problem. Through simulations, the effectiveness of the proposed approach was demonstrated.

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