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Stability Analysis of Discrete Time Delay Control for Nonlinear Systems

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***Abstract*—This paper presents a solution to the long standing problem of the stability of Time Delay Control (TDC) for nonlinear systems. Ever since it was first introduced, TDC has rapidly drawn attentions owing to its unusually robust performance and yet its extraordinarily compact form. The existing stability analyses have been made based on the assumption that the TDC is *continuous* and *time delay* $L \to 0$. The assumption, however, not only fails to reflect the reality that TDC is usually implemented in a digital processor, but also leads to a stability criterion in which important parameters, such as L, that play crucial roles are absent. In this paper, therefore, we present our theoretical investigation on the stability of TDC with the premise that TDC is *discrete* and L is *nonzero and finite*. Specifically, stability criteria based on the premise are derived, so that one may clearly grasp *which parameters affect stability and how*. For the analysis of the closed-loop stability, we have first derived its *approximate discrete model* (approximate discrete plant model with the discrete TDC). Then by using the model and the concepts of *consistency* and *Lyapunov stability,* we have analyzed the stability of the *exact discrete model of* closed loop systems. The analysis results in a stability criteria consisting of L and other parameters that affect the performance of TDC. The suggested stability analysis has been verified by simulation results.**

I. INTRODUCTION

THIS paper presents a solution to the long standing problem of the stability of Time Delay Control (TDC), proposing a new stability analysis for nonlinear systems. Provided below are the context and background associated with the issue at hand.

TDC is a control technique that utilizes time-delayed signals of some system variables to estimate and compensate system uncertainty, such as unmodeled dynamics, parameter variations and disturbances [1]-[5]. Thanks to the effectiveness and efficiency of the Time-Delayed Estimation (TDE), TDC displays particularly robust performance despite its unusually compact structure and simple gain selection procedure.

Ever since it was first introduced, TDC has been successfully applied to various nonlinear electromechanical systems including robots: position control, force control and impedance control for robots [5]-[9]; hybrid position/force control for robots [11]-[12]; electro-hydraulic excavator and telescopic handler, well-known heavy duty nonlinear systems [14]-[15]; and brushless DC motor [13], frictionless positioning device [16] and PM synchronous motor [17]. Unquestionably, TDC has proven itself as a truly effective practical control method.

In order for its value to be fully appreciated, however, TDC leaves a serious issue yet to be resolved: its stability. And it is the rationale for this paper. Until now, the stability analysis of TDC has been done only in continuous domain, which becomes difficult due to the time delay terms in the closed-loop dynamics. And, it becomes even more difficult when the plant happens to be a nonlinear multivariable system. In [2], a necessary and sufficient stability criterion for Linear Time Invariant (LTI) plants has been presented based on Nyquist stability criterion, whereas in [3] a sufficient criterion has been derived from Nyquist stability criterion and Kharitonov method. These analyses, though complete, are limited to LTI plants, and even in these simple cases, the analyses tend to be quite complicated.

For nonlinear multivariable systems, on the other hand, stability analysis has been made by Youcef-Toumi and Wu [4]. The analysis is based on a set of assumptions: TDC is in continuous domain; time delay $L \to 0$. The analysis results in a compact sufficient criterion of $\| \mathbf{I} - \mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1} \| < 1$ where $\mathbf{I}$ denotes an identity matrix, $\mathbf{B}(\mathbf{x})$, input distribution matrix resulting from input/output linearization and $\bar{\mathbf{B}}$ a constant matrix of TDC – which is relatively easy to determine for controller design. In [10], stability analysis of TDC for robot manipulators is reported using Popov's hyperstability theory based on infinitesimal time delay, with a resulting criterion essentially identical to $\| \mathbf{I} - \mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1} \| < 1$. And other research works including ours [11]-[15] also incorporated the same assumption of infinitesimal time delay in continuous time domain.

The assumption of *continuous* TDC, however, fails to represent reality: To our knowledge, TDC has never been implemented in analog devices, yet. Instead, it has usually been implemented in digital processors, where time delay is set to its sampling period(s), which obviously cannot be made infinitesimal, owing to hardware capacity. That is, TDC is *discrete*, L is *finite,* and the resulting closed-loop dynamics becomes a *sampled-data system*.

In addition to discrepancies from reality, the assumption that $L \to 0$ gives rise to two important problems: firstly, the assumption becomes self-contradictory in that TDC with $L \to 0$ (no time-delay) is not TDC any more; secondly, it causes L and some control gains, crucial parameters in determining stability, to

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disappear from stability criteria – notice that L and other control gains except for $\bar{\mathbf{B}}$ are missing in $\|\mathbf{I}-\mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1}\|<1$. For example, for L as small as 0.001sec, one can find a $\bar{\mathbf{B}}$ that satisfies $\|\mathbf{I}-\mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1}\|<1$ and yet drives a system unstable. (See the counter-example in II.B). In short, the assumption causes the criterion to exclude a crucial parameter, seriously restricting its utility in practice.

For this reason, we are going to present new a stability analysis of the closed-loop system under discrete TDC with finite L. The purpose of the analysis is to derive new stability criteria, similar to $\|\mathbf{I}-\mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1}\|<1$, yet including important parameters, so that one can understand how these parameters affect the closed-loop stability under the discrete TDC. It is another principal concern, of course, to see how the new criteria compare with the existing one in terms of predicting stability regions.

Since the closed-loop system is a sampled-data system, we analyze the stability of the closed-loop system by using second approach shown in [24] as follows:

- Stability analysis based on the *exact* discrete-time plant model and discrete controller ignoring inter-sample behavior.

From [24], since this approach deals with the issue of sampling naturally and effectively, enabling direct theoretical investigation of the effect of sampling (on the key system theoretic properties), we use this approach to analyze the stability of the closed-loop system. Therefore the closed-loop system that we deal with becomes discrete.

However, since it is almost impossible to find exact discrete plant model of continuous nonlinear plant [23]-[24], we first derive an approximate discrete plant model, and then derive approximate discrete model of closed loop system using discrete TDC. Finally, after establishing the ***consistency*** ([23], [24]) of this approximate discrete model with the exact discrete model, we analyze the stability of the *exact discrete model of* closed loop system by using ***Lyapunov stability concept.***

This paper is organized as follows. In Section II, we briefly review TDC, its stability and the problem of the previous stability criteria of TDC. Section III presents discrete TDC, the approximate discrete model of plant, and approximate discrete model of the closed loop system. Also we show that the approximate discrete model is *consistent* with exact discrete model. In Section IV we are going to present our stability analysis of exact discrete model of closed loop systems using approximate discrete model derived from Section III by the concepts of *consistency* and *Lyapunov stability*. In Section V, we show simulation results to verify the proposed stability analysis. Finally, Section VI summarizes the results and draws conclusions.

II. TDC and Existing Stability Criteria

A. Time Delay Control(TDC) and Existing Stability Criteria

Consider a general plant with p inputs, m outputs, and n states, as described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \tag{1}$$

$$\mathbf{y} = \mathbf{c}(\mathbf{x}) \tag{2}$$

where $\mathbf{x}\in\Re^n$ denotes the state vector, $\mathbf{u}\in\Re^p$ the input vector, and $\mathbf{y}\in\Re^m$ the output vector. In (1) and (2), $\mathbf{f}:\Re^n\to\Re^n$, $\mathbf{g}:\Re^n\to\Re^{n\times p}$, $\mathbf{c}:\Re^n\to\Re^m$ are assumed to be smooth functions of state vector $\mathbf{x}$. Also consider that $\mathbf{x}$, $\mathbf{f}$ and $\mathbf{g}$ are expressed in phase variable form as follows:

$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_l\\ \mathbf{x}_p\end{bmatrix};\ \mathbf{f}(\mathbf{x}) = \begin{bmatrix}\mathbf{x}_s\\ \mathbf{f}_p(\mathbf{x})\end{bmatrix};\ \mathbf{g}(\mathbf{x}) = \begin{bmatrix}\mathbf{0}\\ \mathbf{g}_p(\mathbf{x})\end{bmatrix} \tag{3}$$

where $\mathbf{x}_l=[x_1,\ldots,x_{n-p}]^T$, $\mathbf{0}$, and $\mathbf{x}_s=[x_{p+1},\ldots,x_n]^T$ denote $(n-p)\times 1$ vectors; both $\mathbf{x}_p$ and $\mathbf{f}_p$ denote $p\times 1$ vectors; and $\mathbf{g}_p$ denotes an $p\times p$ nonsingular matrix.

For the plant, (1)-(2), there are two types of TDC in [1] and [4], according to the forms of desired dynamics to be achieved, each of which is summarized as follows:

1) TDC in [1] is expressed as

$$\begin{aligned}&\mathbf{u}(t)=\mathbf{u}(t-L)\\ &\quad+\bar{\mathbf{g}}^{+}[-\dot{\mathbf{x}}(t-L)+\mathbf{A}_m\mathbf{x}(t)+\mathbf{B}_m\mathbf{R}(t)-\mathbf{K}\left(\mathbf{x}_m(t)-\mathbf{x}(t)\right)]\end{aligned} \tag{4}$$

where L denotes the time delay, which is usually set to a sampling period, $\bar{\mathbf{g}}^{+}$ a pseudoinverse matrix of $\bar{\mathbf{g}}=[\mathbf{0}\quad \bar{\mathbf{g}}_p]^T$ in which $\bar{\mathbf{g}}_p$ denotes a $p\times p$ constant matrix representing the known range of $\mathbf{g}(\mathbf{x})$, $\mathbf{K}$ $n\times n$ error feedback matrix, and $\mathbf{x}_m\in\Re^n$ the state vector of a reference model given by

$$\dot{\mathbf{x}}_m = \mathbf{A}_m\mathbf{x}_m + \mathbf{B}_m\cdot\mathbf{R} \tag{5}$$

where $\mathbf{A}_m$ denotes an $n\times n$ constant stable matrix by which desired performance is specified, $\mathbf{B}_m$ an $n\times p$ command distribution matrix, and $\mathbf{R}$ a $p\times 1$ command vector.

The stability analysis of closed-loop system using (4) has been done *only* for LTI systems in continuous domain for both $L\neq 0$ and $L\to 0$ [2]. Note that there is no stability analysis for the case when (4) is applied to nonlinear plants.

2) TDC in [4] is based on input-output linearization of (1)-(2) with $p=m$. The input-output linearization renders (1)-(2) into the following form:

$$\mathbf{D}\mathbf{y} = \mathbf{a}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{6}$$

where $\mathbf{D}\equiv diag(d^{r_i}/dt^{r_i})$, $\mathbf{a}(\mathbf{x})$ denotes vector function and $\mathbf{B}(\mathbf{x})$ matrix function, whose forms are shown in [4]. Here, r_i denotes the smallest integer such that for at least one of $\mathbf{g}_j(\mathbf{x})$ the following holds:

$$L_{\mathbf{g}_j}(L_{\mathbf{f}}^{k}(c_i(\mathbf{x})))=0,\qquad \forall\mathbf{x}\in\Re^n,\ k=0,1,\ldots,r_i-2$$

$$L_{\mathbf{g}_j}(L_{\mathbf{f}}^{r_i-1}(c_i(\mathbf{x})))\neq 0,\quad \forall\mathbf{x}\in\Re^n$$

where $\mathbf{g}_j(\mathbf{x})$ denotes the j-th column of $\mathbf{g}(\mathbf{x})$, c_j the i-th component of $\mathbf{c}$ in (2); $L_{\mathbf{f}}(\varphi(\mathbf{x}))$ and $L_{\mathbf{g}_j}(\varphi(\mathbf{x}))$ stand for the Lie derivative of $\varphi(\mathbf{x})$, an arbitrary function, with respect to $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_j(\mathbf{x})$ respectively.

For (6), TDC has been presented as

$$\mathbf{u}(t)=\mathbf{u}(t-L)+\bar{\mathbf{B}}^{-1}\left(-\mathbf{y}^{r}(t-L)+\mathbf{v}(t)\right) \tag{7}$$

where $\bar{\mathbf{B}}$ denotes an $m\times m$ nonzero constant matrix approximating $\mathbf{B}(\mathbf{x})$, $\mathbf{y}^r(t-L)\equiv[y_1^{r_1}(t-L),\ldots,y_m^{r_m}(t-L)]^T$ denotes the r-th derivative of y at time $t-L$, and $\mathbf{v}\in\Re^m$ stands for the new input vector the i-th component of which is given by

$$v_i = y_{d_i}^{(r_i)} + \gamma_{1i}e_i^{(r_i-1)} + \cdots + \gamma_{r_i i}e_i \tag{8}$$

where y_{d_i} denotes the i-th component of desired trajectory vector $\mathbf{y}_d$, and e_i the i-th component of error vector defined as $e_i \triangleq y_{d_i}-y_i$. The parameters $\gamma_{1i},\cdots,\gamma_{r_i i}$ are chosen so that the following characteristic equation becomes Hurwitz.

$$s^{r_i} + \gamma_{1i} s^{r_i - 1} + \cdots + \gamma_{r_i i} s = 0 \tag{9}$$

The sufficient criterion for stability of (7) for $L \rightarrow 0$ is given by

$$\| \mathbf{I}_m - \mathbf{B(x)}\overline{\mathbf{B}}^{-1} \| < 1 \tag{10}$$

where $\mathbf{I}_m$ denotes an $m \times m$ identity matrix. Note that, in order to benefit from the criterion (10), one needs to have accurate information of $\mathbf{B}(\mathbf{x})$, which is often difficult to obtain. For example, $\mathbf{B}(\mathbf{x})$ for a robot becomes the inverse of its inertia matrix, the accurate estimation of which is known to be quite involved and difficult. Nevertheless, (10) is still compact and practical in most applications.

B. The Limitation of Existing Stability Criterion

As was mentioned, we have observed in simulations and experiments that the closed loop system with TDC goes unstable in spite of some $\overline{\mathbf{B}}$'s that satisfy (10). The following example is one of them.

Consider a 1st order plant and TDC given by

Plant: $\dot{x} = x^3 + \sin(x) + 5u$

TDC: $u(t) = u(t-L) + \overline{g}^{-1}\left(-\dot{x}(t-L) + \dot{x}_d(t) + k_P\left(x_d(t) - x(t)\right)\right)$

where x denotes the state, $x_d(t)$ the desired trajectory of x, u the input, $\overline{g}$ and k_p the parameters for TDC. For simulation, we set $L = 0.001\sec$ and $k_p = 40$. Here $x_d(t)$ is set a fifth order polynomial trajectory with final value 1 after $1\sec$.

According to the stability criterion (10), any $\overline{g}$ satisfying $\overline{g} > 2.5$ is supposed to make the closed-loop system stable. For $\overline{g} = 2.53$, however, the closed-loop system goes unstable, as is displayed in Fig. 1.

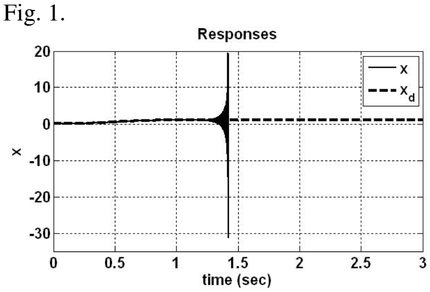


Fig. 1. Response of TDC at $\overline{g} = 2.53$

The discrepancy is conjectured as a direct consequence of the assumption in the derivation of (10) that time delay L is infinitesimal ($L \rightarrow 0$) in continuous time domain.

In practice, however, it is impossible to implement an *infinitesimal* time delay, since the smallest time delay L one can achieve in digital devices is the *sampling time interval*, which is finite.

Furthermore, (10) has been derived with the assumption that TDC in *continuous* form is used. In practice, however, almost all of TDC have been implemented in *discrete* form. Therefore existing stability condition of TDC (10) cannot be applied to discrete TDC case without further consideration.

III. DERIVATION OF DISCRETE MODEL OF CLOSED LOOP SYSTEM

With the use of discrete TDC as shown in Fig. 2, the closed-loop system consists of a discrete controller and a continuous plant, resulting in a *sampled data system*, for which stability analysis should be made.

As mentioned in Section I, we represent the continuous plant with *discrete model,* and hence the resulting model of closed-loop system becomes discrete. Since it is difficult to derive an exact discrete model of continuous nonlinear plant, we are going to obtain an approximate discrete plant model which can *legitimately* replace the exact discrete model. The legitimacy or fidelity of the approximate model is going to be established by the concept of *consistency*.

In the analysis from now on, we assume the plant is of the form in (1) with (3) and the TDC is of the type in (4). Again, L denotes sampling time interval of the digital devices in which TDC is implemented. $[\bullet]_{a,b}$ denotes the a-th row and the b-th column element of matrix $\bullet$.

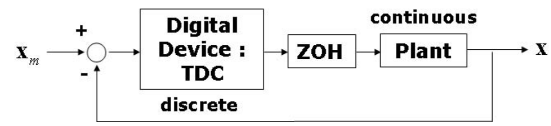


Fig. 2. The control system with TDC (ZOH: Zero Order Holder)

A. Discrete TDC

Discrete form of (4) at k -th sampling instant is given by

$$\begin{aligned}\mathbf{u}(k) = \mathbf{u}(k-1) + \overline{\mathbf{g}}^{+}\big(&-\dot{\mathbf{x}}(k-1) + \mathbf{A}_m\mathbf{x}(k-1) \\ &+ \mathbf{B}_m \cdot \mathbf{R}(k-1) - \mathbf{K}\left(\mathbf{x}_m(k-1) - \mathbf{x}(k-1)\right)\big)\end{aligned} \tag{11}$$

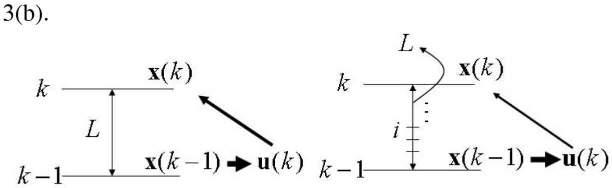
Without losing the essence of (11), $\mathbf{K} = \mathbf{0}$ may be also used. Since $\dot{\mathbf{x}}(k-1) = \mathbf{f}(\mathbf{x}(k-1)) + \mathbf{g}(\mathbf{x}(k-1))\mathbf{u}(k-1)$, we have

$$\begin{aligned}\mathbf{u}(k) = \overline{\mathbf{g}}^{+}\big(&-\mathbf{f}(\mathbf{x}(k-1)) - \left(\mathbf{g}(\mathbf{x}(k-1)) - \overline{\mathbf{g}}\right)\mathbf{u}(k-1) \\ &+ \mathbf{A}_m\mathbf{x}(k-1) + \mathbf{B}_m \cdot \mathbf{R}(k-1)\big)\end{aligned} \tag{12}$$

B. Approximate Discrete Model of Nonlinear Continuous Plant

An approximate discrete model of nonlinear continuous plant is to be expressed in terms of the relationship between $\mathbf{x}(k-1)$ and $\mathbf{x}(k)$. To this end, let $\mathbf{x}(k)$ denote a state vector at k -th sampling instant. Assuming a zero-order-holder in digital-analog conversion, $\mathbf{u}(k)$, which is constructed by using $\mathbf{x}(k-1)$ and $\mathbf{u}(k-1)$, is constant between $(k-1)$ -th sampling instant and k -th sampling instant. As the result, $\mathbf{x}(k-1)$, $\mathbf{x}(k)$ and $\mathbf{u}(k)$ form a causal sequence like Fig. 3(a).

Although $\mathbf{u}(k)$ remains constant between $(k-1)$ -th and k -th sampling instant, $\mathbf{x}$ changes continuously in this interval. To express this situation, the interval is divided with d steps like Fig. 3(b).



(a) (b) $i = 0, \ldots, d-1$

Fig. 3. Sequence of $\mathbf{x}(k)$, $\mathbf{x}(k-1)$ and $\mathbf{u}(k)$

If d is made infinite, the continuity of plant is achieved. In order to express the change of $\mathbf{x}$ at *intermediate steps* within this interval, a new variable $\boldsymbol{\chi} \in \Re^{n \times 1}$ is introduced instead of $\mathbf{x}$, which is to have the following values at $i = 0,\cdots,d-1$:

$$\boldsymbol{\chi}_{i+1} = \boldsymbol{\chi}_i + \frac{L}{d}\left(\mathbf{f}(\boldsymbol{\chi}_i) + \mathbf{g}(\boldsymbol{\chi}_i)\mathbf{u}(k)\right) \tag{13}$$

where $\boldsymbol{\chi}_0 = \mathbf{x}(k-1)$ and $\boldsymbol{\chi}_d = \mathbf{x}(k)$.

To solve (13) leads to the relationship between $\mathbf{x}(k-1)$ and $\mathbf{x}(k)$, which, however, is difficult since $\mathbf{f}$ and $\mathbf{g}$ are nonlinear functions. Hence we use, instead, a first-order Taylor series of $\mathbf{f}(\boldsymbol{\chi}_i)$ and $\mathbf{g}(\boldsymbol{\chi}_i)$ as follows:

$$\mathbf{f}(\boldsymbol{\chi}_i) \cong \mathbf{f}(\boldsymbol{\chi}_0) + \mathbf{F}(\boldsymbol{\chi}_0)\left(\boldsymbol{\chi}_i - \boldsymbol{\chi}_0\right) \tag{14}$$

where $\mathbf{f}(\boldsymbol{\chi}_i) = [f_1(\boldsymbol{\chi}_i) \ \cdots \ f_n(\boldsymbol{\chi}_i)]^T$, $\boldsymbol{\chi}_i = [\chi_1 \ \cdots \ \chi_n]^T$ and $[\mathbf{F}(\boldsymbol{\chi}_0)]_{a,b} = \left(\partial f_a / \partial \chi_b\right)\left(\boldsymbol{\chi}_0\right)$ for $a = 1,\cdots,n$ and $b = 1,\cdots,n$.

Here, $\mathbf{g}(\boldsymbol{\chi}_i) \in \Re^{n\times p}$, a matrix function, can be expressed by

$$\mathbf{g}(\boldsymbol{\chi}_i) = \begin{bmatrix} g_{11}(\boldsymbol{\chi}_i) & \cdots & g_{1p}(\boldsymbol{\chi}_i) \\ \vdots & \cdots & \vdots \\ g_{n1}(\boldsymbol{\chi}_i) & \cdots & g_{np}(\boldsymbol{\chi}_i) \end{bmatrix} \tag{15}$$

For the sake of convenience, we obtain $\mathbf{g}(\boldsymbol{\chi}_i)\mathbf{u}(k)$, instead of $\mathbf{g}(\boldsymbol{\chi}_i)$, as the following:

$$\mathbf{g}(\boldsymbol{\chi}_i)\mathbf{u}(k) \cong \mathbf{g}(\boldsymbol{\chi}_0)\mathbf{u}(k) + \mathbf{G}(\boldsymbol{\chi}_0, \mathbf{u}(k)) \cdot (\boldsymbol{\chi}_i - \boldsymbol{\chi}_0) \tag{16}$$

where $[\mathbf{G}\left(\boldsymbol{\chi}_0, \mathbf{u}(k)\right)]_{a,b} = \sum_{j=1}^{p} \frac{\partial g_{aj}}{\partial \chi_b}(\boldsymbol{\chi}_0) \cdot u_j(k)$ for $a = 1,\cdots,n$ and $b = 1,\cdots,n$ and $\mathbf{u}(k) = [u_1(k) \ \cdots \ u_p(k)]^T$.

Substituting (14) and (16) into (13) leads to

$$\boldsymbol{\chi}_{i+1} = \left(\mathbf{I}_n + \frac{L}{d}\mathbf{H}_{k-1}\right)\boldsymbol{\chi}_i + \frac{L}{d}\left(-\mathbf{H}_{k-1}\boldsymbol{\chi}_0 + \mathbf{f}(\boldsymbol{\chi}_0) + \mathbf{g}(\boldsymbol{\chi}_0)\cdot\mathbf{u}(k)\right) \tag{17}$$

where $\mathbf{H}_{k-1} = \mathbf{F}(\boldsymbol{\chi}_0) + \mathbf{G}(\boldsymbol{\chi}_0, \mathbf{u}(k))$ and $\mathbf{I}_n$ denotes an $n\times n$ identity matrix. From (17), if the inverse of $\mathbf{H}_{k-1}$ exists, we derive $\boldsymbol{\chi}_d$ as follows:

$$\boldsymbol{\chi}_d = \boldsymbol{\chi}_0 + \left(\left(\mathbf{I}_n + \frac{L}{d}\mathbf{H}_{k-1}\right)^d - \mathbf{I}_n\right)\cdot\mathbf{H}_{k-1}^{-1}\cdot\left(\mathbf{f}(\boldsymbol{\chi}_0) + \mathbf{g}(\boldsymbol{\chi}_0)\cdot\mathbf{u}(k)\right) \tag{18}$$

As $d \to \infty$, (18) can be rewritten as

$$\begin{aligned}\mathbf{x}(k) &= \mathbf{x}(k-1) + \mathbf{C}\left(k-1\right)\cdot\left(\mathbf{f}(\mathbf{x}(k-1)) + \mathbf{g}(\mathbf{x}(k-1))\cdot\mathbf{u}(k)\right) \\ &\triangleq \boldsymbol{\Phi}_L^a\left(\mathbf{x}(k-1), \mathbf{u}(k)\right)\end{aligned} \tag{19}$$

where

$$\mathbf{C}\left(k-1\right) = \lim_{d\to\infty}\left(\left(\mathbf{I}_n + \frac{L}{d}\mathbf{H}_{k-1}\right)^d - \mathbf{I}_n\right)\cdot\mathbf{H}_{k-1}^{-1} \tag{20}$$

Note that $\boldsymbol{\Phi}_L^a$ of (19) is the *approximate discrete plant model* of (1) and (19) is the resulting expression of $\mathbf{x}(k)$ in terms of $\mathbf{x}(k-1)$ and $\mathbf{u}(k)$. Using the properties of matrix exponential function $e^{\mathbf{A}}$ for $n\times n$ matrix $\mathbf{A}$, we have

$$\mathbf{C}(k-1) = (e^{L\cdot\mathbf{H}_{k-1}} - \mathbf{I}_n)\cdot\mathbf{H}_{k-1}^{-1} \tag{21}$$

Note that $\mathbf{C}(k-1)$ is a function of a sampling period L.

C. Approximate Discrete Model of the Closed Loop System under Discrete TDC

Substituting (12) into (19) yields

$$\begin{aligned}\mathbf{x}(k) &= \mathbf{x}(k-1) + \mathbf{C}_{k-1}\cdot\Big(\mathbf{f}_{k-1} - \mathbf{g}_{k-1}\overline{\mathbf{g}}^+\mathbf{f}_{k-1} \\ &- \mathbf{g}_{k-1}\overline{\mathbf{g}}^+\left((\mathbf{g}_{k-1} - \overline{\mathbf{g}})\mathbf{u}(k-1) - (\mathbf{A}_m\mathbf{x}(k-1) + \mathbf{B}_m\cdot\mathbf{R}(k-1))\right)\Big)\end{aligned} \tag{22}$$

where $\mathbf{C}(k-1) = \mathbf{C}_{k-1}$, $\mathbf{f}_{k-1} = \mathbf{f}(\mathbf{x}(k-1))$ and $\mathbf{g}_{k-1} = \mathbf{g}(\mathbf{x}(k-1))$.

Here we define vector $\mathbf{e}(k)$ as

$$\mathbf{e}(k) = \left[\mathbf{x}(k) - \mathbf{x}_m(k) \ \ \left(\mathbf{u}(k) - \mathbf{u}_m(k)\right)/K_s\right]^T \triangleq [\mathbf{e}_1(k) \ \ \mathbf{e}_2(k)]^T \tag{23}$$

where $\mathbf{u}_m(k) = \mathbf{g}\left(\mathbf{x}_m(k)\right)^+\left(\mathbf{A}_m\mathbf{x}_m(k) + \mathbf{B}_m\mathbf{R}(k-1) - \mathbf{f}\left(\mathbf{x}_m(k)\right)\right)$ with $\mathbf{g}(\mathbf{x}_m(k))^+$ denoting a pseudoinverse matrix of $\mathbf{g}(\mathbf{x}_m(k))$; and K_s denotes a scaling factor that reduces the size of $\mathbf{u}(k) - \mathbf{u}_m(k)$. Since in tracking control problem the size of tracking error, $\mathbf{e}_1(k)$, is important, it is necessary to reduce the size of $\mathbf{u}(k) - \mathbf{u}_m(k)$ in (23) so that in the error norm, $\|\mathbf{e}(k)\|$, the $\mathbf{e}_1(k)$ term may be dominant. In addition, it is easy to derive a discrete model of the reference model (5) as the following:

$$\mathbf{x}_m(k) = \mathbf{D}_1\cdot\mathbf{x}_m(k-1) + \left(\mathbf{D}_1 - \mathbf{I}_n\right)\mathbf{A}_m^{-1}\mathbf{B}_m\cdot\mathbf{R}(k-1) \tag{24}$$

where $\mathbf{D}_1 = e^{L\cdot\mathbf{A}_m}$.

Using Taylor Series expansion, we can express $\mathbf{f}_{k-1}$ as

$$\begin{aligned}\mathbf{f}_{k-1} &= \mathbf{f}(\mathbf{x}_m(k-1)) + \mathbf{F}\left(\mathbf{x}_m(k-1)\right)\cdot\left(\mathbf{x}(k-1) - \mathbf{x}_m(k-1)\right) \\ &+ \mathbf{O}_1\left(\mathbf{x}(k-1), \mathbf{x}_m(k-1)\right)\end{aligned} \tag{25}$$

where $\mathbf{F}$ is the same as was introduced in (14) and the a-th element of a $n\times 1$ vector $\mathbf{O}_1\left(\mathbf{x}(k-1), \mathbf{x}_m(k-1)\right)$ is defined as

$$\left[\mathbf{O}_1\left(\mathbf{x}(k-1), \mathbf{x}_m(k-1)\right)\right]_a = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2 f_a}{\partial x_i \partial x_j}(\mathbf{q})e_{1i}(k-1)e_{1j}(k-1)$$

where $e_{1j}(k-1)$ denotes the *j*-th element of $\mathbf{e}_1(k-1)$ defined in (23), and $\mathbf{q}$ lies somewhere on the line segment joining $\mathbf{x}(k-1)$ to $\mathbf{x}_m(k-1)$.

Using (22), (23), (24) and (25), we have

$$\mathbf{e}(k) = \mathbf{M}_{k-1}\mathbf{e}(k-1) + \mathbf{N}_{k-1} \triangleq \mathbf{E}_{CL}^a\left(\mathbf{e}(k-1)\right) \tag{26}$$

where $\mathbf{M}_{k-1}$ and $\mathbf{N}_{k-1}$ are given by the top of next page. Note that $\mathbf{E}_{CL}^a$ of (26) is the *approximate discrete model* of the closed loop system. We can think of this model as perturbation of the nominal model

$$\mathbf{e}(k) = \mathbf{M}_{k-1}\mathbf{e}(k-1) \tag{27}$$

D. Consistent Property of Approximate Discrete Model with Exact Discrete Model

In order to use the approximate discrete model (26) for the stability analysis, the *accuracy* of the approximate model should be established with respect to the exact model. The accuracy is represented by the concept of *consistency* shown in [23] and [24].

Before getting into details, let us make some important definitions. For vector $\mathbf{x}^T = \left[x_1,\ldots,x_n\right] \in \Re^n$, $\|\mathbf{x}\|$ is defined as $\|\mathbf{x}\| \triangleq (x_1^2 + \cdots + x_n^2)^{1/2}$. For matrix $\mathbf{A} \in \Re^{n\times n}$, $\|\mathbf{A}\|$ is defined by $\|\mathbf{A}\| \triangleq \max_i |\lambda_i(\mathbf{A})|$ where λ_i denotes the *i*-th eigenvalue of $\mathbf{A}$.

We consider the difference equations corresponding to the exact plant model and its approximation, respectively,

$$\mathbf{x}(k) = \boldsymbol{\psi}_L^e\left(\mathbf{x}(k-1), \mathbf{u}_L(k)\right) \tag{28}$$

$$\mathbf{x}(k) = \boldsymbol{\psi}_L^a\left(\mathbf{x}(k-1), \mathbf{u}_L(k)\right) \tag{29}$$

where $\boldsymbol{\psi}_L^e$ denotes the exact plant model, $\boldsymbol{\psi}_L^a$ the approximate model, and $\mathbf{u}_L$ discrete control input.

The followings are definition and lemma presented in [24].

$$\mathbf{M}_{k-1}=\begin{bmatrix}\mathbf{I}_n+\mathbf{C}_{k-1}\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}\mathbf{A}_m+\mathbf{C}_{k-1}\left(\mathbf{I}-\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}\right)\mathbf{F}\left(\mathbf{x}_m(k-1)\right) & K_s\cdot\left(\mathbf{C}_{k-1}\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}(\bar{\mathbf{g}}-\mathbf{g}_{k-1})\right)\\ \bar{\mathbf{g}}^{+}\left(\mathbf{A}_m-\mathbf{F}\left(\mathbf{x}_m(k-1)\right)\right)/K_s & \bar{\mathbf{g}}^{+}\left(\bar{\mathbf{g}}-\mathbf{g}_{k-1}\right)\end{bmatrix} \tag{30}$$

$$\begin{aligned}\mathbf{N}_{k-1}=&\begin{bmatrix}\left(\mathbf{C}_{k-1}\left(\mathbf{I}_n-\mathbf{g}_{k-1}\cdot\bar{\mathbf{g}}^{+}\right)+\left(\mathbf{C}_{k-1}\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}\mathbf{A}_m-\mathbf{D}_1+\mathbf{I}_n\right)\mathbf{A}_m^{-1}\right)\\ \mathbf{0}\end{bmatrix}\mathbf{f}\left(\mathbf{x}_m(k-1)\right)+\begin{bmatrix}\mathbf{C}_{k-1}\left(\mathbf{I}_n-\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}\right)\\ -\bar{\mathbf{g}}^{+}/K_s\end{bmatrix}\mathbf{O}_1\left(\mathbf{x}(k-1),\mathbf{x}_m(k-1)\right)\\ &+\begin{bmatrix}\left(\left(\mathbf{C}_{k-1}\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}\mathbf{A}_m-\mathbf{D}_1+\mathbf{I}_n\right)\mathbf{A}_m^{-1}\mathbf{g}\left(\mathbf{x}_m(k-1)\right)+\mathbf{C}_{k-1}\mathbf{g}_{k-1}\bar{\mathbf{g}}^{+}(\bar{\mathbf{g}}-\mathbf{g}_{k-1})\right)\mathbf{u}_m(k-1)\\ \left(-\mathbf{u}_m(k)+\bar{\mathbf{g}}^{+}(\bar{\mathbf{g}}-\mathbf{g}_{k-1}+\mathbf{g}\left(\mathbf{x}_m(k-1)\right))\mathbf{u}_m(k-1)\right)/K_s\end{bmatrix}\end{aligned} \tag{31}$$

Definition 1 [24]. The family $(\mathbf{u}_L,\boldsymbol{\psi}_L^a)$ is said to be *one-step consistent* with $(\mathbf{u}_L,\boldsymbol{\psi}_L^e)$ if, for compact sets $\mathfrak{A}\subset\Re^n$ and $\mathfrak{A}'\subset\Re^p$, there exists a function $\rho\in$ class K_∞ and a constant $L^*>0$ such that, for all $\mathbf{x}\in\mathfrak{A}$, all $\mathbf{u}_L\in\mathfrak{A}'$ and $L\in(0,L^*)$, we have

$$\|\boldsymbol{\psi}_L^e(\mathbf{x},\mathbf{u}_L)-\boldsymbol{\psi}_L^a(\mathbf{x},\mathbf{u}_L)\|\le L\rho(L). \tag{32}$$

A sufficient condition for one-step consistency is given as the following lemma:

Lemma 1 [24]. If

(A1) $(\mathbf{u}_L,\boldsymbol{\psi}_L^a)$ is one step consistent with $(\mathbf{u}_L,\boldsymbol{\psi}_L^{Euler})$, where $\boldsymbol{\psi}_L^{Euler}:=\mathbf{x}+L\dot{\mathbf{x}}=\mathbf{x}+L(\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}_L)$,

(A2) for compact sets $\mathfrak{A}\subset\Re^n$ and $\mathfrak{A}'\subset\Re^p$ there exist $\rho\in\mathrm{K}_\infty$, $M>0$, $L^*>0$ such that, for all $L\in(0,L^*)$, all $\mathbf{x},\mathbf{y}\in\mathfrak{A}$ and all $\mathbf{u}_L\in\mathfrak{A}'$,

(A2a) $\|\mathbf{f}(\mathbf{y})+\mathbf{g}(\mathbf{y})\mathbf{u}_L\|<M$

(A2b) $\|\mathbf{f}(\mathbf{y})+\mathbf{g}(\mathbf{y})\mathbf{u}_L-(\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}_L)\|\le\rho\left(\|\mathbf{x}-\mathbf{y}\|\right)$,

then $(\mathbf{u}_L(k),\boldsymbol{\psi}_L^a)$ is one-step consistent with $(\mathbf{u}_L(k),\boldsymbol{\psi}_L^e)$.

Proof: See [24]. ■

We consider the exact discrete plant model of (1), similar to $\boldsymbol{\Phi}_L^a$ defined in (19),

$$\mathbf{x}(k)=\boldsymbol{\Phi}_L^e\left(\mathbf{x}(k-1),\mathbf{u}(k)\right) \tag{33}$$

where $\boldsymbol{\Phi}_L^e$ denotes the exact discrete plant model of (1). Also we consider the exact discrete model of the closed loop system of (33) under (12), similar to $\mathbf{E}_{CL}^a$ defined in (26),

$$\mathbf{e}(k)=\mathbf{E}_{CL}^e\left(\mathbf{e}(k-1)\right) \tag{34}$$

where $\mathbf{E}_{CL}^e$ denotes the exact discrete model of the closed loop system. Using Lemma 1, we can prove the following lemma.

Lemma 2. $(\mathbf{u},\boldsymbol{\Phi}_L^a)$ is one step consistent with $(\mathbf{u},\boldsymbol{\Phi}_L^e)$.

Proof: We are going to use Lemma 1 to prove Lemma 2, by showing that all the conditions of Lemma 1 are met, in the order of (A2a), (A2b), and (A1), consecutively.

If $\mathbf{y}\in\mathfrak{A}$ and $\mathbf{u}\in\mathfrak{A}'$ for compact sets $\mathfrak{A}\subset\Re^n$ and $\mathfrak{A}'\subset\Re^p$, there exists M such that $\|\mathbf{f}(\mathbf{y})+\mathbf{g}(\mathbf{y})\mathbf{u}\|<M$. Therefore (A2a) is satisfied.

In addition, taking Taylor expansion of $\mathbf{f}$ and $\mathbf{g}$ leads to the following relationship: for all $\mathbf{x},\mathbf{y}\in\mathfrak{A}$ and all $\mathbf{u}\in\mathfrak{A}'$,

$$\begin{aligned}\|\mathbf{f}(\mathbf{y})+\mathbf{g}(\mathbf{y})\mathbf{u}-(\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u})\|&=\|\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})+(\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{x}))\mathbf{u}\|\\&=\|(\partial\mathbf{f}+\partial\mathbf{g})\cdot(\mathbf{y}-\mathbf{x})+\mathbf{g}(\mathbf{x})(\mathbf{y}-\mathbf{x})\|\le A\cdot\|\mathbf{y}-\mathbf{x}\|\end{aligned} \tag{35}$$

where $A=\max(\|\partial\mathbf{f}+\partial\mathbf{g}\|,\|\mathbf{g}(\mathbf{x})\|)$, with $[\partial\mathbf{f}]_{i,j}=(\partial f_i/\partial x_j)(\mathbf{q}_i)$, $[\partial\mathbf{g}]_{i,j}=\sum_{r=1}^{p}(\partial g_{ir}/\partial x_j)(\mathbf{q}'_{ir})\cdot u_r$ where u_r denotes r-th element of $\mathbf{u}$; $\mathbf{q}_i$ and $\mathbf{q}'_{ij}$ (for $i=1,\ldots,n$, and $j=1,\ldots,p$) lie somewhere on the line segment joining $\mathbf{x}$ to $\mathbf{y}$. Thus, (A2b) is met.

Finally, Condition (A1) is investigated as follows: From (19) and the definition of $\boldsymbol{\psi}_L^{Euler}$ in Lemma 1, we have

$$\|\boldsymbol{\Phi}_L^a(\mathbf{x},\mathbf{u})-\boldsymbol{\psi}_L^{Euler}(\mathbf{x},\mathbf{u})\|\le K_c\cdot\|\mathbf{C}(\mathbf{x},\mathbf{u})-L\cdot\mathbf{I}_n\| \tag{36}$$

where $\mathbf{C}(\mathbf{x},\mathbf{u})=(e^{L\cdot\mathbf{H}(\mathbf{x},\mathbf{u})}-\mathbf{I}_n)\cdot\mathbf{H}(\mathbf{x},\mathbf{u})^{-1}$ where $\mathbf{H}(\mathbf{x},\mathbf{u})$ is the same as was introduced in (17) and $K_c=\|\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}\|$. If $\mathbf{x}\in\mathfrak{A}$ and $\mathbf{u}\in\mathfrak{A}'$ for compact sets $\mathfrak{A}\subset\Re^n$ and $\mathfrak{A}'\subset\Re^p$, K_c in (36) is bounded. As to $\|\mathbf{C}(\mathbf{x},\mathbf{u})-L\cdot\mathbf{I}_n\|$ in (36), by (21) and the definition of matrix exponential function, we have

$$\begin{aligned}\left\|\mathbf{C}(\mathbf{x},\mathbf{u})-L\cdot\mathbf{I}_n\right\|&=\left\|\sum_{i=2}^{\infty}\frac{(L)^i\,\mathbf{H}(\mathbf{x},\mathbf{u})^{i-1}}{i!}\right\|\le\sum_{i=2}^{\infty}\frac{(L)^i\,\lambda_{\max}^{\;i-1}}{i!}\\&=\frac{1}{\lambda_{\max}}\left(\sum_{i=0}^{\infty}\frac{(L)^i\,\lambda_{\max}^{\;i}}{i!}\right)-L-\frac{1}{\lambda_{\max}}=L\left(\frac{e^{L\lambda_{\max}}-1}{\lambda_{\max}L}-1\right)=L\rho_1(L)\end{aligned} \tag{37}$$

where $\lambda_{\max}(\neq0)$ denotes absolute maximum eigenvalue of $\mathbf{H}(\mathbf{x},\mathbf{u})$ and $\rho_1(L)=((e^{L\lambda_{\max}}-1)/\lambda_{\max}L-1)$.

Clearly $\rho_1(L)$ is class K_∞ class function. By (36) and (37),

$$\|\boldsymbol{\Phi}_L^a(\mathbf{x},\mathbf{u})-\boldsymbol{\psi}_L^{Euler}(\mathbf{x},\mathbf{u})\|\le L\rho_2(L) \tag{38}$$

where $\rho_2(L)=K_c\cdot\rho_1(L)\in\mathrm{K}_\infty$. Thus Condition (A1) is met. Now that all the conditions of Lemma 1 are met, $(\mathbf{u},\boldsymbol{\Phi}_L^a)$ is one step consistent with $(\mathbf{u},\boldsymbol{\Phi}_L^e)$. ■

Lemma 3. Let $\mathbf{e}\in\mathfrak{A}''$ for each compact set $\mathfrak{A}''\subset\Re^{n+p}$ and suppose that there exists some constant ϕ_1. Then, there exists a sampling time interval L^* such that for all $L\in(0,L^*)$

$$\|\mathbf{E}_{CL}^e(\mathbf{e})-\mathbf{E}_{CL}^a(\mathbf{e})\|\le L\rho(L)\le\phi_1 \tag{39}$$

where $\rho(L)$ belongs to class K_∞.

Proof: By (23) and definitions of $\mathbf{E}_{CL}^a$ of (26) and $\mathbf{E}_{CL}^e$ of (34), we can obtain the following relationships;

$$\begin{aligned}&\left\|\mathbf{E}_{CL}^e(\mathbf{e})-\mathbf{E}_{CL}^a(\mathbf{e})\right\|\\&=\left\|\left[\boldsymbol{\Phi}_L^e-\mathbf{x}_m\quad(\mathbf{u}-\mathbf{u}_m)/K_s\right]^T-\left[\boldsymbol{\Phi}_L^a-\mathbf{x}_m\quad(\mathbf{u}-\mathbf{u}_m)/K_s\right]^T\right\|\\&=\left\|\boldsymbol{\Phi}_L^e-\boldsymbol{\Phi}_L^a\right\|\end{aligned} \tag{40}$$

By Lemma 2, $(\mathbf{u},\boldsymbol{\Phi}_L^a)$ is one step consistent with $(\mathbf{u},\boldsymbol{\Phi}_L^e)$. Therefore from Definition 1 there exists $\rho(L)\in\mathrm{K}_\infty$ that meets the 1st inequality sign in (39). And $L\rho(L)$ belongs to class K_∞ since $L\in\mathrm{K}_\infty$ and $\rho(L)\in\mathrm{K}_\infty$. Hence for some constant ϕ_1, there exists L^* that satisfies the 2nd inequality sign in (39) for all $L\in(0,L^*)$ ■

IV. Stability Analysis

In this section, by using the approximate discrete model derived in Section III together with the consistency property, we analyze the stability of the exact discrete model of closed loop system. In IV.A, we summarize necessary lemma, theorem and corollary and in IV.B we analyze stability.

A. Preliminaries

Consider a nonlinear control system described by discrete-time equation of the form

$$\mathbf{x}(k+1)=\mathbf{h}(\mathbf{x}(k)) \qquad k\in\mathbb{Z}^+ \tag{41}$$

where $\mathbb{Z}^+:=\{0,1,2,\ldots\}$ and $\mathbf{h}(\mathbf{x}(k))\in\Re^n$ denotes a vector function satisfying $\mathbf{h}(\mathbf{0})=\mathbf{0}$.

Definition 2 [24]. Let $\beta\in KL$ and let $N\subset\mathbb{R}^n$ be an open (not necessarily bounded) set containing the origin.

1. The system (41) is said to be (β,N) *-stable* if the solutions of the system (41) satisfy

$$\|\mathbf{x}(k)\|\le\beta(\|\mathbf{x}(k_0)\|,k-k_0),\ \ \forall x(k_0)\in N,\quad k\ge k_0\ge 0 \tag{42}$$

where k_0 denotes the initial step.

2. The system (41) is said to be (β,N) *-practically stable* if for each $R>0$ the solutions of the system (41) satisfy

$$\|\mathbf{x}(k)\|\le\beta(\|\mathbf{x}(k_0)\|,k-k_0)+R,\ \ \forall x(k_0)\in N,\quad k\ge k_0\ge 0 \tag{43}$$

The following Theorem 1 and Corollary 1 represent respectively the *discrete time versions* of Theorem 5.1 and Corollary 5.3 of [19], the proofs of which have also been presented in [19].

Theorem 1. Let $D\subset\Re^n$ be a domain containing the zero solution ($\mathbf{x}(k)=0$) of system (41) and $\mathbf{h}:D\to\Re^n$ be locally Lipschitz in $\mathbf{x}$. Let $V:D\to\Re$ be a function such that $\forall\|\mathbf{x}\|\in D$,

$$W_1(\mathbf{x}(k))\le V(\mathbf{x}(k))\le W_2(\mathbf{x}(k)) \tag{44}$$

$$\Delta V(\mathbf{x}(k))=V(\mathbf{x}(k+1))-V(\mathbf{x}(k))\le -W_3(\mathbf{x}(k))\quad \forall\|\mathbf{x}(k)\|\ge\mu>0, \tag{45}$$

where $W_1(\mathbf{x}(k))$, $W_2(\mathbf{x}(k))$ and $W_3(\mathbf{x}(k))$ are continuous positive definite functions on D. Take $\delta>0$ such that $B_\delta=\{\mathbf{x}\mid\|\mathbf{x}\|\le\delta\}\subset D$ and suppose that μ is small enough that

$$\max_{\|\mathbf{x}\|\le\mu}W_2(\mathbf{x})<\min_{\|\mathbf{x}\|=\delta}W_1(\mathbf{x}) \tag{46}$$

Let $\eta=\max_{\|\mathbf{x}\|\le\mu}W_2(\mathbf{x})$ and take ρ such that $\eta<\rho<\min_{\|\mathbf{x}\|=\delta}W_1(\mathbf{x})$. Then, there exist a finite step k_1 (dependent on $\mathbf{x}(k_0)$ and μ) and class KL function $\beta(\cdot,\cdot)$ such that $\forall\mathbf{x}(k_0)\in\{\mathbf{x}\in B_\delta\mid W_2(\mathbf{x})\le\rho\}$, the solution of system (41) satisfy

$$\|\mathbf{x}(k)\|\le\beta(\|\mathbf{x}(k_0)\|,k-k_0),\ \ \forall k_0\le k<k_1 \tag{47}$$

$$\mathbf{x}(k)\in\{\mathbf{x}\in B_\delta\mid W_1(\mathbf{x})\le\eta\},\ \ \forall k\ge k_1 \tag{48}$$

That is, the system (41) is (β,N) -practically stable where $N=\{\mathbf{x}\in B_\delta\mid W_2(\mathbf{x})\le\rho\}$.

Proof: The proof of Theorem 1 is similar to the proof of [19, Theorem 5.1] by using [22, Theorem 8], and is omitted here due to space limitation. ■

Corollary 1. Suppose the assumptions of Theorem 1 satisfy

$$W_2(\mathbf{x}(k))\le m_2\|\mathbf{x}(k)\|^q,\ W_i(\mathbf{x}(k))\ge m_i\|\mathbf{x}(k)\|^q \ \textit{for } i=1,3 \tag{49}$$

for some positive constants m_1, m_2, m_3 and q. Suppose $\mu<\delta\cdot(m_1/m_2)^{1/q}$ and $\|\mathbf{x}(k_0)\|<\delta\cdot(m_1/m_2)^{1/q}$. Then solution of (41) satisfies

$$\|\mathbf{x}(k)\|\le C_1\cdot\|\mathbf{x}(k_0)\|e^{-\varphi(k-k_0)},\ \ \forall k_0\le k<k_1 \tag{50}$$

$$\|\mathbf{x}(k)\|\le\mu\cdot(m_2/m_1)^{1/q}\ \ \forall k\ge k_1 \tag{51}$$

where $C_1=(m_2/m_1)^{1/q}$ and $\varphi=m_3/(m_2q)$.

Proof: The proof of Corollary 1 is similar to the proof of [19, Corollary 5.3] and is omitted here due to space limitation. ■

B. Stability Analysis of the Exact Discrete Model of the Closed Loop System

Lemma 4 [21]. $\Delta V(\mathbf{x}(k))$ in (45) has the following relationship:

$$\Delta V(\mathbf{x}(k))=\frac{\partial V}{\partial\mathbf{x}}(\boldsymbol{\kappa}(k))[\mathbf{x}(k+1)-\mathbf{x}(k)] \tag{52}$$

where $\boldsymbol{\kappa}(k)$ denotes a point on the line segment joining $\mathbf{x}(k)$ to $\mathbf{x}(k+1)$.

Proof: See [21]. ■

Since $\mathbf{M}_k$ of (27) is essential to represent the approximate model, we define ξ as

$$\xi\triangleq\max_k\|\mathbf{M}_k\| \tag{53}$$

Now we obtain the main result as follows:

Theorem 2. Suppose $\mathbf{E}^e_{CL}(\mathbf{e}(k))$ is the exact discrete model of closed loop system, $\mathbf{E}^a_{CL}(\mathbf{e}(k))$ the approximate discrete model derived in III.C and $D=\{\mathbf{e}(k)\in\Re^{n+p}\,\|\,\|\mathbf{e}(k)\|<\delta\}$. Let a positive scalar function $V(\mathbf{e}(k))$ of (27) as

$$V(\mathbf{e}(k))=\mathbf{e}^T(k)\mathbf{P}\mathbf{e}(k) \tag{54}$$

where $\mathbf{P}$ denotes a positive definite real symmetric matrix. If for all $\mathbf{e}\in D$,

(B1) there exists ξ satisfying

$$\xi<1 \tag{55}$$

then, (C1) $V(\mathbf{e}(k))$ satisfies the followings;

$$b_1\|\mathbf{e}(k)\|^2\le V(\mathbf{e}(k))\le b_2\|\mathbf{e}(k)\|^2 \tag{56}$$

$$\Delta V(\mathbf{e}(k))=V(\mathbf{e}(k+1))-V(\mathbf{e}(k))\le -b_3\|\mathbf{e}(k)\|^2 \tag{57}$$

$$\left\|\frac{\partial V(\boldsymbol{\kappa}(k))}{\partial\mathbf{e}}\right\|\le b_4\|\mathbf{e}(k)\| \tag{58}$$

where $\boldsymbol{\kappa}(k)$ denotes a point on the line segment joining $\mathbf{e}(k)$ to $\mathbf{M}_k\mathbf{e}(k)$ and b_1, b_2, b_3 and b_4 denote positive constants for $k\in\mathbb{Z}^+$. And for b_1, b_2, b_3 and b_4, if

(B2) the perturbation term $\mathbf{N}_k$ in (26) and ϕ_1 in Lemma 3 satisfy

$$\|\mathbf{N}_k\|+\phi_1\le\vartheta<(b_3/b_4)\sqrt{(b_1/b_2)}\upsilon\delta \tag{59}$$

for some positive constant $\upsilon<1$, then

(C2) $\mathbf{E}^e_{CL}(\mathbf{e}(k))$ is (β,N) -practically stable where $\beta(r,s)=\exp(-\varphi\cdot s)\cdot r$ with $\varphi=(1-\upsilon)b_3/(2b_2)$ and $N=\{\mathbf{e}\,\|\,\|\mathbf{e}\|<\sqrt{b_1/b_2}\delta\}$.

Proof of (C1): Since $\mathbf{P}$ is a positive definite real symmetric matrix, (56) and (58) are satisfied as follows:

$$\lambda_{\min}(\mathbf{P})\|\mathbf{e}(k)\|^2\le V(\mathbf{e}(k))\le\lambda_{\max}(\mathbf{P})\|\mathbf{e}(k)\|^2$$

$$\begin{aligned}\left\|\frac{\partial V}{\partial\mathbf{e}}(\boldsymbol{\kappa}(k))\right\| &=\|2\mathbf{P}\boldsymbol{\kappa}(k)\|=\|\|2\mathbf{P}\mathbf{e}(k)+2\mathbf{P}(\boldsymbol{\kappa}(k)-\mathbf{e}(k))\|\\ &\le 2\lambda_{\max}(\mathbf{P})(\|\mathbf{e}(k)\|+\|\mathbf{M}_k\mathbf{e}(k)-\mathbf{e}(k)\|)\\ &=2\lambda_{\max}(\mathbf{P})(1+\|\mathbf{M}_k-\mathbf{I}\|)\|\mathbf{e}(k)\|\le b_4\|\mathbf{e}(k)\|\end{aligned}$$

where $\lambda_{\min}(\mathbf{P})$ and $\lambda_{\max}(\mathbf{P})$ denote respectively absolute minimum and maximum eigenvalue of matrix $\mathbf{P}$ and $b_4=2\lambda_{\max}(\mathbf{P})(1+\|\mathbf{M}_k-\mathbf{I}\|)$. For later use, let us define

$b_1 = \lambda_{\min}(\mathbf{P})$ and $b_2 = \lambda_{\max}(\mathbf{P})$. Now let us prove that $V(\mathbf{e}(k))$ satisfies (57) under the assumption (B1). Using (27), we obtain $\Delta V(\mathbf{e}(k))$ as follows:

$$\Delta V(\mathbf{e}(k)) = V(\mathbf{e}(k+1)) - V(\mathbf{e}(k)) = \mathbf{e}^T(k)\left(\mathbf{M}_k^T\mathbf{P}\mathbf{M}_k - \mathbf{P}\right)\mathbf{e}(k)$$

If there exists a positive definite matrix $\mathbf{Q}$ such that

$$\mathbf{M}_k^T\mathbf{P}\mathbf{M}_k - \mathbf{P} = -\mathbf{Q} \tag{60}$$

then, (57) is satisfied. From [25], in order to exist matrix $\mathbf{Q}$ for (60), following condition is needed

$$|\psi_{k,i}|^2 < 1 \tag{61}$$

where $\psi_{k,i}$ denotes the i-th eigenvalue of $\mathbf{M}_k$. Since $\xi = \max_{k,i}|\psi_{k,i}|$ by (53), if $\xi < 1$ holds, then $V(\mathbf{e}(k))$ satisfies (57) and b_3 is given by

$$b_3 = (1-\xi^2)\lambda_{\min}(\mathbf{P}) \qquad \blacksquare$$

Proof of (C2): We use $V(\mathbf{e}(k))$ like (54) as a Lyapunov function candidate for $\mathbf{E}_{CL}^e(\mathbf{e}(k))$. From Lemma 3, Lemma 4 and (B2), the difference of $V(\mathbf{e}(k))$ along the trajectory of $\mathbf{E}_{CL}^e(\mathbf{e}(k))$ satisfies

$$\begin{aligned}
\Delta V(\mathbf{e}(k)) &= \frac{\partial V}{\partial \mathbf{e}}(\boldsymbol{\kappa}(k))\left[\mathbf{E}_{CL}^e(\mathbf{e}(k)) - \mathbf{e}(k)\right] \\
&= \frac{\partial V}{\partial \mathbf{e}}(\boldsymbol{\kappa}(k))\left[\mathbf{E}_{CL}^a(\mathbf{e}(k)) - \mathbf{e}(k) + \left(\mathbf{E}_{CL}^e(\mathbf{e}(k)) - \mathbf{E}_{CL}^a(\mathbf{e}(k))\right)\right] \\
&= \frac{\partial V}{\partial \mathbf{e}}(\boldsymbol{\kappa}(k))\left[\mathbf{M}_k\mathbf{e}(k) - \mathbf{e}(k)\right] + \frac{\partial V}{\partial \mathbf{e}}(\boldsymbol{\kappa}(k))\mathbf{N}_k \\
&\qquad + \frac{\partial V}{\partial \mathbf{e}}(\boldsymbol{\kappa}(k))\left[\mathbf{E}_{CL}^e(\mathbf{e}(k)) - \mathbf{E}_{CL}^a(\mathbf{e}(k))\right] \\
&\le -b_3\|\mathbf{e}(k)\|^2 + b_4\|\mathbf{e}(k)\|(\|\mathbf{N}_k\| + \phi_1) \\
&= -(1-\upsilon)b_3\|\mathbf{e}(k)\|^2 - \upsilon b_3\|\mathbf{e}(k)\|^2 + b_4(\|\mathbf{N}_k\| + \phi_1)\|\mathbf{e}(k)\|, 0 < \upsilon < 1 \\
&\le -(1-\upsilon)b_3\|\mathbf{e}(k)\|^2, \quad \forall \|\mathbf{e}(k)\| \ge b_4(\|\mathbf{N}_k\| + \phi_1)/\upsilon b_3
\end{aligned}$$

By using Theorem 1 and Corollary 1, one can show that for all $\|\mathbf{e}(k_0)\| < \sqrt{b_1/b_2}\,\delta$ the solution of the $\mathbf{E}_{CL}^e(\mathbf{e}(k))$ satisfies

$$\|\mathbf{e}(k)\| \le C_2 \cdot \exp[-\varphi(k-k_0)]\,\|\mathbf{e}(k_0)\|, \quad \forall k_0 \le k < k_1$$

and

$$\|\mathbf{e}(k)\| \le B, \quad \forall k \ge k_1$$

for some finite step k_1, where $C_2 = \sqrt{b_2/b_1}$, $\varphi = (1-\upsilon)b_3/(2b_2)$, $B = (b_4/b_3)\sqrt{b_2/b_1}\,(\vartheta/\upsilon)$. $\blacksquare$

Remark 1. The new stability criteria we have derived are expressed in (55) and (59).

Remark 2. From (30) and (31), we know that matrices $\mathbf{M}_k$ and $\mathbf{N}_k$ include $\bar{\mathbf{g}}$, L and $\mathbf{A}_m$ and that the sufficient criteria for stability are for the maximum eigenvalue of $\mathbf{M}_k$, ξ, to be less than unity and for $\mathbf{N}_k$ to satisfy (59). Hence, the parameters that affect the closed-loop stability are $\bar{\mathbf{g}}$, L and $\mathbf{A}_m$. Note that in addition to $\bar{\mathbf{g}}$, the only factor in (10), L and $\mathbf{A}_m$ also play important roles in determining stability. The impact of L has already been mentioned above and observed widely, but that of $\mathbf{A}_m$ has been rarely observed. How they determine stability may be clarified by the close inspection of how $\bar{\mathbf{g}}$, L and $\mathbf{A}_m$ affect the values of ξ and $\|\mathbf{N}_k\|$.

Remark 3. Since $\mathbf{C}_k$ of (21) approaches zero as $L \to 0$, if $L \to 0$, eigenvalues $\mathbf{M}_k$ of (30) are obtained as

$$Eigen(\mathbf{M}) = \left|\bar{\mathbf{g}}^+(\bar{\mathbf{g}} - \mathbf{g}) - \lambda\mathbf{I}_p\right| \cdot \left|\mathbf{I}_n - \lambda\mathbf{I}_n\right| = 0 \tag{62}$$

$\lambda = 1$ and $\lambda\mathbf{I}_p = \bar{\mathbf{g}}^+(\bar{\mathbf{g}} - \mathbf{g})$

Therefore, for $L \to 0$, (55) becomes

$$\|\bar{\mathbf{g}}^+(\bar{\mathbf{g}} - \mathbf{g})\| = \|\mathbf{I}_p - \bar{\mathbf{g}}^+\mathbf{g}\| < 1 \tag{63}$$

Clearly, (63) is equivalent to (10) for $L \to 0$, showing that (55) is the more general criterion from which (10) can be derived.

V. SIMULATION

In order to verify the stability criteria, (55) and (59) of Theorem 2, we have made numerical simulations. Specifically, from (55) and (59) we have derived $\bar{\mathbf{g}}$ and the discrete TDC, with which we have performed simulations to ascertain the correctness of the conditions and the stability analysis. In addition we have compared the result with that obtained from the stability condition by Youcef-Toumi. For the comparison purpose, we have incorporated the same plant dynamics of II.B used to provide the counter-example.

From (59), w is defined as

$$w \triangleq \|\mathbf{N}_k\| + \phi_1 - (b_3/b_4)\sqrt{(b_1/b_2)}\,\upsilon\delta \tag{64}$$

Then we find the set of $\bar{\mathbf{g}}$ that meets $\xi < 1$ and $w < 0$ simultaneously. It is noteworthy that evaluating (64) requires the knowledge of ϕ_1, and eventually the knowledge of the exact discrete plant model in (33), which is impossible to acquire. Instead, we have substituted it with numerical data generated by the fourth-order Runge-Kutta method, widely-accepted for its accuracy.

In the simulations, all the parameter values have been set the same as those in II.B. In addition, we set $\delta = 0.1$, $K_s = 100$, and $\mathbf{P} = \mathbf{I}_2$. Besides, the TDC used in II.B is based on the second form [4] of II.A, whereas the stability analysis has been made on the first form [1]. In [26], it was shown that these two forms are equivalent if some conditions are satisfied. By results of [26], $A_m = -40$ and $B_m \cdot R = \dot{x}_d + 40x_d$ where x_d is same as was introduced in II.B. Fig. 4 shows the evaluations of (55) and (64) with respect to the change of $\bar{g}$. Close inspection reveals that the range of $\bar{g}$ that meets (55) and (64) is $2.56 \le \bar{g} \le 1192$. The respective error response at $\bar{g} = 2.56$ and $\bar{g} = 1192$ is plotted in Fig. 5(a), and the error norms associated with these responses are shown in Fig. 5(b), which obviously are less than 0.1, the value of δ.

The closed-loop system under the discrete TDC demonstrates stable behaviors, thereby confirming the correctness of the proposed criteria, (55) and (59), and the stability analysis. Whereas the condition by Youcef-Toumi yields $\bar{g} > 2.5$, (55) and (59) determine $2.56 \le \bar{g} \le 1192$, not only the lower limit but also the upper limit.

In addition, we have obtained by trial-error the exact range of $\bar{g}$, the boundary between the stability region and the instability region, which has turned out to be $2.554 \le \bar{g} \le 1410$. In light of this range, our criteria yield a sufficient criterion.

VI. CONCLUSION

In this paper, we have presented a new stability analysis of the closed loop system under discrete TDC with finite L. To this end, an approximate discrete model of the closed loop system was derived and then using the derived approximate discrete model and the concepts of consistency and Lyapunov stability, we have analyzed the stability of the exact discrete model of closed loop system. As a result of stability analysis, two sufficient stability criteria were obtained, one of which serves as the general one from which the existing condition by Youcef-Toumi can be derived. The criteria include not only $\bar{\mathbf{g}}$ but also L and $\mathbf{A}_m$, the affect of which was newly found.

Through simulations, the stability criteria have predicted the stability range of $\bar{\mathbf{g}}$, which is more accurate than the one predicted with the criterion by Youcef-Toumi, thereby demonstrating the effectiveness of the criteria and our analysis.

In fairness, since the criteria require the accurate knowledge of plant model (continuous), they are far less practical than the criterion by Youcef-Toumi, which requires the accurate knowledge of $\mathbf{B}(\mathbf{x})$ only. As was mentioned, however, this research is primarily a theoretical investigation on the stability under the assumption of real conditions, endeavoring to gain insights on what determine the stability and how they do.

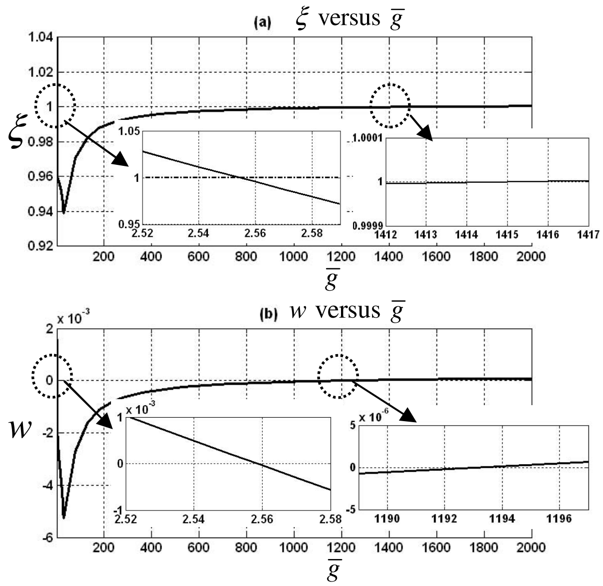


Fig. 4. (a) ξ versus $\bar{g}$ and (b) w versus $\bar{g}$

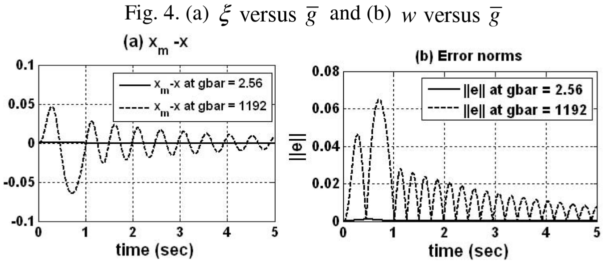


Fig. 5. (a) Time error responses and (b) error norms at $\bar{g} = 2.56$ and $\bar{g} = 1192$

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