# Design of Efficient FIR Filters with Cyclotomic Polynomial Prefilters Using Mixed Integer Linear Programming

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**Abstracts**-The Cyclotomic Polynomial (CP) prefilter design problem is formulated as an optimization problem with linear objective functions by applying the logarithm to the transfer function of the CP prefilter. Then this problem is solved by mixed integer linear programming (MILP). Design examples demonstrate that this method leads to more efficient cascaded FIR prefilter-equalizers than existing methods.

## I. Introduction

Considerable attention has been given in the digital signal processing literature to the design of efficient FIR filters which require fewer arithmetic operations than conventional ones. The design methods in [1]-[7], which lead to FIR filters with cascade structures, can reduce the number of required multiplications and some also reduce additions at the expense of some increase in the number of delays. The computational savings can be significant especially in designing narrow band filters.

One approach to efficient FIR filter design is based on a cascade structure composed of a prefilter, which is often multiplierless, followed by an FIR equalizer [2]-[7]. The prefilter provides reasonable stopband attenuation, and the equalizer makes the overall filter meet the passband and stopband specifications. Most prefilters introduced so far are based on using the recursive running sum (RRS) [3]. The equalizer is designed via a modified Parks-McClellan algorithm [2].

In a recent paper [7], Hartnett and Boudreaux-Bartels proposed the use of cyclotomic polynomials (CP's) [8] to form multiplierless prefilters. Their prefilter consists of cascaded subfilters whose system functions are represented as CP's (Fig.1). This class of CP prefilters includes the RRS as a special case. It has been shown that the CP prefilter-equalizer method performs better and can
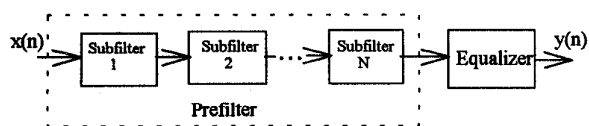


Fig. 1. Prefilter-Equalizer cascade structure

be applied to a wider range of filter types as compared with the other prefilter-equalizer methods.

Although the CP prefilter design leads to efficient FIR filters, the design method is ad hoc and is not able to find "best" prefilter for the specification at hand. The objective of this paper is to develop an optimal procedure for designing CP prefilters with minimal complexity.

In what follows, we first show that the CP prefilter design problem can be formulated as an optimization problem with linear objective functions by applying the logarithm to the transfer function of the CP prefilter. Then the design problem is solved by mixed integer linear programming (MILP) [9]. Through design examples, we demonstrate that the proposed method leads to more efficient cascaded FIR prefilter-equalizers, as compared with the existing method.

## II. The CP Prefilter Design Method

In this section, we will briefly review the CP prefilter-equalizer design method proposed in [7], and then describe a method for designing prefilters with minimal complexity. Both the prefilters and equalizers considered in this section are linear phase FIR filters.

### A. Review on the CP prefilter-equalizer design [7]

The system function of the prefilter $P(z)$ is represented as

$$P(z) = \prod_{q=1}^{Q} F_q(z)^{m_q} \qquad (1)$$

where $F_q(z)$ are CP's in $z^{-1}$, $m_q$ are nonnegative integers and $Q$ is a positive integer (Fig. 1). In order to obtain multiplierless CP prefilters, only the first 104 CP's which contain only the coefficients $\{0, 1, -1\}$ are considered. The CP prefilter-equalizer is designed as follows:

**Step 1.** Choose the maximum length of the prefilter and the equalizer.
**Step 2.** Determine the set of eligible CP's whose zero locations are "consistent" with the desired filter specifications. Among the 104 CP's, keep only those containing zeros within the stopband or within some intrusion into the transient bands, and eliminate all

CP's with passband zeros.

**Step 3.** Determine the order of each eligible CP's selected in Step 2. In (1), $F_q(z)$ are eligible CP's obtained in Step 2. The orders $m_q$ are determined so that $P(z)$ meets the prefilter specifications of maximum length, passband deviation and stopband attenuation. If such $m_q$ are found, conclude the prefilter design and proceed to the equalizer design in Step 4. Otherwise, this step is repeated with an increased value for the maximum length of the prefilter.

**Step 4.** Design the equalizer using either the modified Parks-McClellan algorithm or the subset selection method.

In [7], an iterative algorithm for finding the order $m_q$ in Step 3 are provided. However, this algorithm is ad hoc and in general cannot provide a CP prefilter with minimum complexity. In implementing the prefilter, some prefilter subsections are combined into one efficient multiplierless structure, and some are implemented recursively. This approach provides savings in number of additions at the expense of delays. Next we present an optimal method for determining the order $m_q$.

### B. The proposed design method

The CP prefilter with minimal complexity may be designed by solving the following optimization problem.

$$\text{Minimize} \quad \sum_{q=1}^{Q} m_q \left( a_q + c \cdot d_q \right) \qquad (complexity\,measure)$$

$$\text{Subject to} \quad \left\| \left| D(e^{j\omega}) \right| - s \cdot \left| P(e^{j\omega}) \right| \right\| \le r_p \quad (in\,passbands) \qquad (2)$$

$$\left| s \cdot P(e^{j\omega}) \right| \le r_s \qquad (in\,stopbands)$$

where $a_q$ and $,d_q$ respectively, are the number of adders and delays required in implementing the $q$-th eligible polynomial, $c$ is a constant which is determined depending on the complexity of an adder and a delay, $0 < c < 1$, $D(e^{j\omega})$ is the desired frequency response of the cascaded prefilter-equalizer, $s$ is a positive scale factor, which keeps $s \cdot \left| P(e^{j\omega}) \right| \le \left| D(e^{j\omega}) \right|$, $r_p$ and $r_s$, respectively, are the ripples of passband and stopband, and $P(e^{j\omega}) = \prod_{q=1}^{Q} F_q(e^{j\omega})^{m_q}$. Our objective is to find $m_q$. It is obvious that this design problem cannot be solved by using conventional filter design methods such as the remez exchange algorithm. Due to the nonlinearity between $F_q(e^{j\omega})$ and $m_q$, linear programming (LP) cannot be applied directly to this prefilter design. However, after taking the logarithm on $\left| P(e^{j\omega}) \right|$, this problem can be formulated as an MILP problem. We define

$$P_{dB}(e^{j\omega}) = \sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega}) \qquad (3)$$

where $P_{dB}(e^{j\omega}) = 20 \log \left| P(e^{j\omega}) \right|$ and $F_{dBq}(e^{j\omega}) = 20 \log \left| F_q(e^{j\omega}) \right|$. Now the optimization in (2) is rewritten as

$$\text{Minimize} \quad \sum_{q=1}^{Q} m_q \left( a_q + cd_q \right) \qquad (complexity\,measure)$$

$$\text{Subject to} \quad \sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega}) + s_{dB} \le r_{dBp} \qquad (in\,passbands) \quad (4)$$

$$\sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega}) + s_{dB} \ge r_{dBs} \qquad (in\,stopbands)$$

$$\sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega_m}) + s_{dB} \le 20 \log \left| D(e^{j\omega}) \right| \quad (scaling\,constraints)$$

where $s_{dB} = 20 \log(s)$, $r_{dBp} = \left| -20 \log \left| D(e^{j\omega}) \right| - r_p \right|$ and $r_{dBs} = -20 \log(r_s)$. In (4), optimal $m_q$ values and proper scale factor $s$ can be determined by using MILP, if $c$, $r_{dBp}$, $r_{dBs}$ and $F_{dBq}(e^{j\omega})$ are given.

As described in Section II. A., the number of adders required for prefiltering can be reduced either by combining some CP's or by recursively implementing them. To exploit this fact, we obtain such efficient structures of the CP's and add them in the set of eligible CP's. Now the number of eligible CP's is denoted as $Q'$, $Q' \ge Q$

In designing prefilters, it is important to determine proper value of prefilter passband deviation $r_{dBp}$ and prefilter stopband attenuation $r_{dBs}$. In [7], they are determined through trial and error. It was observed that values of $r_{dBp}$ and $r_{dBs}$ which are 6~8dB and 15~20dB, respectively, greater than the desired filter specifications provide good results. In our algorithm, we systematically determine $r_{dBp}$ and $r_{dBs}$ by incorporating the problem for deciding $r_{dBp}$ and $r_{dBs}$ with the optimization in (4). Specifically, we assume that

$$r_{dBs} = \delta_{dBs} + \alpha r_{dBp} \qquad (5)$$

where $\delta_{dBs}$ is the stopband attenuation of the overall filter and $\alpha$ is a constant. The problem in (4) is rewritten with $Q'$, instead of $Q$, and the equality constraint in (5).

$$\text{Minimize} \quad \sum_{q=1}^{Q} m_q \left( a_q + cd_q \right) \qquad (complexity\,measure)$$

$$\text{Subject to} \quad \sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega}) + s_{dB} \le r_{dBp} \qquad (in\,passbands)$$

$$\sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega}) + s_{dB} \ge r_{dBs} \qquad (in\,stopbands) \quad (6)$$

$$\sum_{q=1}^{Q} m_q F_{dBq}(e^{j\omega_m}) + s_{dB} \le 20 \log \left| D(e^{j\omega}) \right| \quad (scaling\,constraints)$$

$$r_{dBs} = \delta_{dBs} + \alpha r_{dBp} \qquad (ripple\,relation)$$

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This problem can be solved by MILP, treating $m_q$, $s_{dB}$, $r_{dBp}$ and $r_{dBs}$ as variables. We solve this problem for several values of $\alpha$. Note that increasing $\alpha$ tends to increase $r_{dBs}$ and the complexity of the prefilter. On the other hand, as $\alpha$ decreases $r_{dBs}$ tends to decrease and this will increase the complexity of the equalizer. For a given $\alpha$, the prefilter designed through (6) will have the minimal $r_{dBp}$ because minimizing the complexity measure in (6) minimizes the attenuation $r_{dBs}$ which is proportional to $r_{dBp}$. This fact indicates that the prefilter design method in (6) leads to an equalizer with reduced complexity.

The prefilter-equalizer design algorithm is summarized as follows:

**Step 1.** Determine the set of eligible CP's. This step is identical to Step 2 in Section II. A.

**Step 2.** Obtain all possible efficient multiplierless building blocks either by combining some eligible CP's or by finding recursive structures of the CP's. We add these blocks in the set of eligible CP's.

**Step 3.** Determine the order $m_q$ by solving the optimization problem in (6). We repeat this for different values of $\alpha$. Consequently, several prefilters are obtained. (For the examples used in this paper, we found that considering $\alpha$, $1 \le \alpha \le 7$ was appropriate.)

**Step 4.** Design the equalizer for each prefilter obtained in Step 3. Among the prefilter-equalizers we select the one with minimal complexity.

Since this algorithm minimizes the number of adders and delays required by the prefilter, estimating the length of the prefilter (Step 1 in Section II. A) is unnecessary.

### III. Filter Design Examples

To compare the proposed CP prefilter design method with the previous one, our method is applied to the filter design problems considered in [7]. In the following, prefilters are designed for $\alpha$=1, 1.5, ..., 6.5, 7, and equalizers are designed by linear programming (LP). Using commercial package in [10], the MILP and LP problems were solved within a few minutes in Sparc 2.

*Example 1* (Lowpass Filter): The specifications in normalized frequency are:

passband: $F \in [0, 0.021]$, stopband: $F \in [0.07, 0.5]$,

ripple: $\delta_{dBp} \le 0.2dB$, $\delta_{dBs} \ge 60dB$

Conventional linear phase equiripple filter requires 51 tap to meet the specifications. The set of eligible CP's obtained in [7] is

$$F_q(z) \in \left\{ C_\tau(z^{-1}) \mid \tau = 2, 3, ..., 13, 14 \right\}$$

where $C_\tau(z^{-1})$ is the $\tau$-th CP. To obtain efficient building blocks, we examined all possible combinations of the eligible CP's and found 13 combinations exhibiting reduced complexity. For example, one of them is

$$C_5(z^{-1})C_{10}(z^{-1}) = (1+z^{-1}+z^{-2}+z^{-3}+z^{-4})(1-z^{-1}+z^{-2}-z^{-3}+z^{-4})$$
$$= \frac{1-z^{-5}}{1-z^{-1}} \frac{1+z^{-5}}{1+z^{-1}} = \frac{1-z^{-10}}{1-z^{-2}}$$

Note that implementing $C_5(z^{-1})C_{10}(z^{-1})$ as $(1+z^{-1}+z^{-2}+z^{-3}+z^{-4})(1-z^{-1}+z^{-2}-z^{-3}+z^{-4})$ requires 8 additions, while $C_5(z^{-1})C_{10}(z^{-1}) = (1-z^{-10})/(1-z^{-2})$ requires only two additions. These efficient building blocks are added to the set of eligible CP's. Now $Q'$=28. We designed 13 prefilters for $1 \le \alpha \le 7$ and the corresponding equalizers by LP. When $\alpha$=6.5, the equalizer with minimal length, which was 4, was obtained. For this $\alpha$ value, the prefilter is

$$P(z) = \left( \frac{1-z^{-9}}{1-z^{-1}} \right) \left( \frac{1-z^{-11}}{1-z^{-1}} \right) \left( \frac{1-z^{-12}}{1-z^{-1}} \right) \left( \frac{1-z^{-13}}{1-z^{-1}} \right) \left( \frac{1-z^{-14}}{1-z^{-1}} \right) (1+z^{-4})$$

The frequency response of the prefilter-equalizer is shown in Fig. 2(a). Table I.A compares the complexity of this prefilter-equalizer with the previous results. Our design reduced 2 multiplications, 3 additions at the expense of 6 delays.

*Example 2* (Bandpass Filter with Center Frequency 0.2): The desired specifications are:

passband: $F \in [0.189, 0.211]$,
stopband: $F \in [0., 0.168] \cup [0.232, 0.5]$,
ripple: $\delta_{dBp} \le 0.5dB$, $\delta_{dBs} \ge 60dB$

For these specifications, a length 111 linear phase FIR filter is required in conventional design. Examining the 15 eligible CP's obtained in [7], we generated 62 efficient blocks. Thus $Q'$=77. The prefilter corresponding to the equalizer with minimal complexity, which was obtained when $\alpha$=6.0, is

$$P(z) = (1+z^{-1})^2 (1-z^{-2})(1-z^{-3})^3 (1-z^{-4})(1+z^{-5})^2 (1-z^{-7})(1-z^{-8})$$
$$\cdot (1+z^{-10})^2 (1-z^{-12})(1+z^{-20})(1-z^{-2}+z^{-4})^2 (1-z^{-3}+z^{-6}) \left( \frac{1-z^{-33}}{1-z^{-5}} \right)$$

Here the length of the equalizer was 6. The frequency response of this prefilter is shown in Fig. 2(b). Table I.B summarizes the complexity of this prefilter-equalizer and the previous results. The proposed method reduced 5 multiplications and 7 additions at the expense of 9 delays.

*Example 3* (Multiband Example): The desired specifications are as follows:

passband: $F \in [0.205, 0.245] \cup [0.36, 0.39]$,
stopband: $F \in [0, 0.17] \cup [0.28, 0.33] \cup [0.42, 0.5]$,
ripple: $\delta_{dBp} \le 0.15 dB$, $\delta_{dBs} \ge 70 dB$

Conventional filters require at least 101 taps to meet the specifications. In this multiband design, it turned out that $Q=7$ and $Q'=18$. The equalizer with minimal complexity was obtained when $\alpha=1.5$, and the length of the equalizer was 34. The corresponding prefilter is

$$P(z) = (1-z^{-1})^4 (1-z^{-2})^5 (1+z^{-3})(1+z^{-5})^6 (1-z^{-6})^3 (1-z^{-7})^6$$

The frequency response of the prefilter-equalizer is shown in Fig. 2(c). The complexity of the prefilter-equalizer is summarized in Table I.C. Our technique reduced 6 multiplications, 14 addtions and 6 delays. Note that our design provided savings in delays as well as in multiplications and additions.

## References

[1] Y. Neuvo, D. Cheng-Yu, and S. K. Mitra, "Interpolated finite impulse response filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 32, pp. 563-570, June 1984.

[2] J. W. Adams and A. N. Willson, Jr., "A new approach to fir filters with fewer multipliers and reduced sensitivity," *IEEE Trans. Circuits Syst.*, vol. 30, pp. 277-283, May 1983.

[3] —,"A novel approach to the design of efficient FIR digital band pass filters," in *Proc. 1984 Int. Symp. Circuits Syst.*, Montreal, Canada, pp. 28-32. May 1984.

[4] J. E. Cabezas and P. S. R. Diniz," FIR filters using interpolated prefilters and equalizers," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 17-32, Jan. 1990.

[5] T. G. Gampbell and Y. Neuvo, "Predictive FIR filters with low computational complexity," *IEEE Trans. Circuits Syst.*, vol 38, pp. 1067-1071, Sept. 1991.

[6] P. P. Vaidyanathan and G. Beitman, "On prefilters for digital filter design," *IEEE Trans. Circuits Syst.*, vol. 32, pp. 494-499, May 1985.

[7] R. J. Hartnett and G. F. Boudreaux-Bartels, "On the use of cyclotomic polynomial prefilters for efficient FIR filter design," *IEEE Trans. Signal Processing*, vol. 41, pp. 1766-1779, May 1993.

[8] D. G. Myers, *Digital Signal Processing: Efficient Convolution and Fourier Transform Techniques*, Sydney, Austalia, Prentice Hall, 1990.

[9] Y. C. Lim and S. R. Parker, "FIR filter design over a discrete power-of-two coefficient space," IEEE Trans. Acoust. Speech, Signal Processing, vol. 31, pp. 583-591, June 1983.

[10] *Using the CPLEX callable library and CPLEX mixed integer library*, CPLEX optimization, Inc. Incline Village, NV, 1994.

Table I. Comparison and summary of Examples.

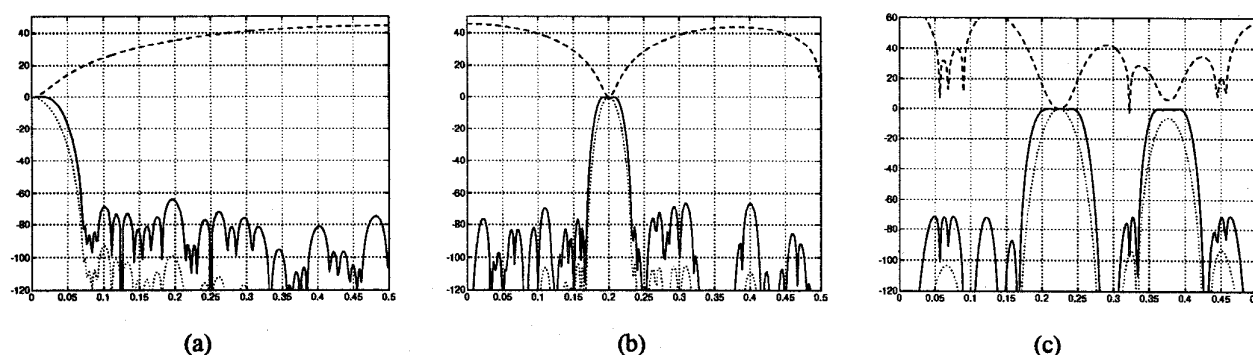| | A. Example 1 (LPF) | | B. Example 2 (BPF) | | C. Example 3 (MBF) | |
|---|---|---|---|---|---|---|
| | CP prefilter by MILP and Equalizer by LP | CP prefilter in [8] and Equalizer by LP | CP prefilter by MILP and Equalizer by LP | CP prefilter in [8] and Equalizer by LP | CP prefilter by MILP and Equalizer by LP | CP prefilter in [8] and Equalizer by LP |
| Multiplier | 2 | 4 | 3 | 8 | 17 | 23 |
| Adder | 13 | 16 | 32 | 39 | 58 | 72 |
| Delay | 65 | 59 | 151 | 142 | 140 | 146 |



(a)  (b)  (c)

Fig. 2. Frequency responses of the prefilter (dotted line), the equalizer (dashed line), and the overall cascaded filter (solid line) in (a) Example 1, (b) Example 2, and (c) Example 3.

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