Weighted Median Filters: Generation and Properties

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Abstract

In this paper it is shown - without reference to threshold decomposition or stacking property - that a WM filter of size N, with discrete-time continuous-valued inputs, can be specified by 2^{N-1} mutually consistent linear inequalities relating the weights. Its relation to binary threshold functions is indicated. For WM filters with symmetric weights, it is shown that the specification is the same as for ternary threshold functions. Based on the inequalities specifying a WM filter, some deterministic properties are derived and the generation of WM filter is discussed.

1. Introduction

A natural extension of the median filter is the weighted median(WM) filter. The output $Y(k)$ of the WM filter of span N associated with the integer weights $\mathrm{w} = (w_1, w_2, \cdots, w_N)^T$ is given by

$$Y(k) = \text{median}\{\overbrace{X(k-N_1), \cdots, X(k-N_1)}^{w_1 \text{ times}}, \overbrace{X(k-N_1+1), \cdots, X(k-N_1+1)}^{w_2 \text{ times}}, \cdots \overbrace{X(k+N_2), \cdots, X(k+N_2)}^{w_N \text{ times}}\} \quad (1)$$

where, k is the time index, $\{X_k\}$ is the continuous-valued input sequence, $\sum_{i=1}^{N} w_i = 2M+1$, $N_1 + N_2 + 1 = N$ and M, N_1, N_2 and each w_i are non-negative integers. Obviously the median filter is a special case of WM filters with $w_1 = w_2 = \cdots = w_N = 1$.

The WM filter was suggested by Justusson [1] as an extension of the median filter. Brownrigg [2][3] enumerated distinct 2D WM filters with weights which are symmetric along the two diagonal axes. Recently Wendt et al [4] showed that the WM filter is uniquely represented by threshold functions [5] and is a special case of the stack filter. Based on this result, some properties of WM filters have been analyzed by Yli-Harja et al [6].

In this paper we present some deterministic properties of the WM filter and discuss its generation. An attempt is made to further clarify the relationship between WM filtering and threshold functions. The basic result of Wendt et al [4] is rederived and extended without reference to the stacking property or threshold decomposition. Some observations which explicitly show the equivalence among WM filters are made. In addition, enumeration of WM filters is discussed.

2. Specification of a WM filter

In this section, it will be shown that the WM filter of span N can be specified by exactly 2^{N-1} mutually consistent linear inequalities. The relationship between these inequalities and threshold functions will be pointed out.

The discrete-time continuous-valued data $\mathrm{X} = (X_1, X_2, \cdots, X_N)$ within the window of a WM filter can be ordered in $N!$ different ways. Let z_k, $k = 1, 2, \cdots, N!$, denote each such ordering. For example, when $N = 2$, "$X_1 \le X_2$" may be denoted by z_1 and "$X_2 \le X_1$" by z_2. For an ordering z_k, consider $X_{(i)} = X_j$ for some i, j, $1 \le i, j \le N$, where $X_{(i)}$ is the i^{th} smallest data in X. We decompose the weights into three disjoint sets, $\{w_j\}$, $\Omega_{k(i,j)}$ and $\Omega'_{k(i,j)}$, where $\Omega_{k(i,j)} = \{w_r \mid w_r$ are the weights associated with $X_{(1)}, X_{(2)}, \cdots, X_{(i-1)}\}$, and $\Omega'_{k(i,j)} = \{w_r \mid w_r$ are the weights associated with $X_{(i+1)}, X_{(i+2)}, \cdots, X_{(N)}\}$. With these notations we can describe the following result.

Lemma 1: Let the order of the input X to the WM filter with weight w be specified by z_k. Then the output $Y = X_{(i)} = X_j$ iff

$$\xi^{\mathbf{T}}_{\mathbf{k(i\,j)}} \mathrm{w} > 0 \quad (2a)$$

$$\hat{\xi}^{\mathbf{T}}_{\mathbf{k(i\,j)}} \mathrm{w} > 0 \quad (2b)$$

simultaneously. Where, $\xi_{\mathbf{k(i\,j)}} = (\xi^k_1, \cdots, \xi^k_{j-1}, 1, \xi^k_{j+1}, \cdots, \xi^k_N)^T$, $\hat{\xi}_{\mathbf{k(i\,j)}} = (-\xi^k_1, \cdots, -\xi^k_{j-1}, 1, -\xi^k_{j+1}, \cdots, -\xi^k_N)^T$, $\xi^k_r = 1$ if $w_r \in \Omega_{k(i,j)}$; -1 otherwise, for all $r \ne j, 1 \le r = 1, 2, \cdots, N$.

Proof: If $Y = X_{(i)} = X_j$ then the following inequalities hold simultaneously:

$$\sum_{w_r \in \Omega_{k(i,j)}} w_r + w_j \ge M + 1 \quad (3a)$$

$$\sum_{w_r \in \Omega_{k(i,j)}} w_r \le M \quad (3b)$$

It is obvious that if (3) holds for some i, j then $Y = X_{(i)} = X_j$ for a given z_k. (For a given z_k there is only one such i, j satisfying (3), since the output of a WM filter is unique.) It is sufficient to prove that (3a) and (3b) are identical to (2a) and (2b) respectively.

From (3a) we have $2\sum_{w_r \in \Omega_{k(i,j)}} w_r + w_j - \sum_{r=1}^{N} w_r \ge 1 > 0$, or $\xi^k_1 w_1 + \cdots + \xi^k_{j-1} w_{j-1} + w_j + \xi^k_{j+1} + \cdots + \xi^k_N > 0$, or $\xi^{\mathbf{T}}_{\mathbf{k(i,j)}} \mathrm{w} > 0$. Similarly, we can show the equivalence between (2b) and (3b)∎

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This Lemma indicates that the output of a WM filter can be obtained without the knowledge of the actual value of the weights, if the vectors $\xi_{k(i,j)}$ and $\hat{\xi}_{k(i,j)}$ satisfying (2), are known for each z_k. For example, if we know that $(1,1,-1)^T$ and $(1,-1,1)^T$ are the vectors satisfying (2), for a z_k representing the ordering "$X_2 \leq X_1 \leq X_3$", then the output is X_1. This is because when the output is X_j, the j^{th} element in both $\xi_{k(i,j)}$ and $\hat{\xi}_{k(i,j)}$ satisfying (2) is 1. Let $s_k = \{\xi_{k(i,j)}, \hat{\xi}_{k(i,j)}$ satisfying (2)$\}$, $\mathcal{R} = \{s_k \mid k = 1, 2, \ldots, N!\}$ and $\mathcal{S} = \bigcup_{k=1}^{N!} s_k$. Clearly, given an ordering z_k and $\mathcal{R}$, the output can be easily computed by picking s_k, which gives $\xi_{k(i,j)}$ and $\hat{\xi}_{k(i,j)}$ satisfying (2). The set $\mathcal{S}$ is the collection of all distinct vectors in $\mathcal{R}$. It should be noted that the number of distinct vectors in $\mathcal{S}$ is at most 2^{N-1}, because there are only 2^N different binary valued vectors, and all the inequalities associated with the vectors in $\mathcal{R}$ (see (2)) should be consistent. Let us assume that the number of distinct vectors in $\mathcal{S}$ is $q \leq 2^{N-1}$ and rewrite it as,

$$\mathcal{S} = \bigcup_{k=1}^{N!} s_k = \{\xi_n \mid n = 1, 2, \cdots, q\} \tag{4}$$

$\mathcal{S}$ is associated with a WM filter, and includes all vectors ξ_n, for which $\xi_n^T w > 0$

It is a significant fact that, even though the inequalities in $\mathcal{S}$ are not arranged in pairs as in $\mathcal{R}$, it is still possible to obtain the inequalities $\xi_{k(i,j)}^T w > 0$ and $\hat{\xi}_{k(i,j)}^T w > 0$ satisfying (2) – and thus the output – for any ordering z_k. For example, consider Fig. 1 which illustrates the procedure for obtaining the output for some z_k. In Fig. 1a, we show the set $\mathcal{S}$ for the WM filter with weights $(2,1,1,1)^T$. Suppose that the weights are unknown, but $\mathcal{S}$ is known. Then, we can generate the set of 2^{N-1} inequalities $\{\xi_n^T w > 0 \mid \xi_n \in \mathcal{S}\}$, where $w = (w_1, w_2, w_3, w_4)^T$. Such inequalities are listed in Fig 1b. For the input $X_1 \leq X_3 \leq X_4 \leq X_2$, the output, Y, of the WM fiter is $X_{(2)} = X_3$, because only that pair of inequalities which is a necessary and sufficient condition for $Y = X_{(2)} = X_3$ is found in the set of inequalities defined by $\mathcal{S}$ while the others are not (Fig. 1c). Since the output of a WM filter can be found from $\mathcal{S}$ for any z_k, we can say that a WM filter can be completely specified by $\mathcal{S}$.

$$\begin{array}{c}(1,1,1,1)\ (1,1,1,-1)\ (1,1,-1,1)\ (1,-1,1,1)\\ (-1,1,1,1)\ (1,1,-1,-1)\ (1,-1,-1,1)\ (1,-1,1,-1)\end{array}$$

a. Vectors in $\mathcal{S}$

$$\begin{array}{c}w_1 + w_2 + w_3 + w_4 > 0; w_1 + w_2 + w_3 - w_4 > 0;\\ w_1 + w_2 - w_3 + w_4 > 0; w_1 - w_2 + w_3 + w_4 > 0;\\ -w_1 + w_2 + w_3 + w_4 > 0; w_1 + w_2 - w_3 - w_4 > 0;\\ w_1 - w_2 - w_3 + w_4 > 0; w_1 - w_2 + w_3 - w_4 > 0.\end{array}$$

b. Inequalities from $\mathcal{S}$

$$Y = X_{(1)} = X_1 \Rightarrow \begin{cases} w_1 - w_2 - w_3 - w_4 > 0 \\ w_1 + w_2 + w_3 + w_4 > 0 \end{cases}$$
$$Y = X_{(2)} = X_3 \Rightarrow \begin{cases} w_1 - w_2 + w_3 - w_4 > 0 \\ -w_1 + w_2 + w_3 + w_4 > 0 \end{cases}$$
$$Y = X_{(3)} = X_4 \Rightarrow \begin{cases} w_1 - w_2 + w_3 + w_4 > 0 \\ -w_1 + w_2 - w_3 + w_4 > 0 \end{cases}$$
$$Y = X_{(4)} = X_2 \Rightarrow \begin{cases} w_1 + w_2 + w_3 + w_4 > 0 \\ -w_1 + w_2 - w_3 - w_4 > 0 \end{cases}$$

c. Inequalities for different outputs

Figure 1

In the above discussion we showed that $\mathcal{S}$ has at most 2^{N-1} vectors. In other words, at most 2^{N-1} consistent inequalities have to be specified to define a WM filter. In what follows, we show that the number of vectors, q, in $\mathcal{S}$ should be equal to 2^{N-1}. This is shown in Lemma 3 using the result stated in Lemma 2.

Lemma 2: If $\sum_{i=1}^{L} w_{r(i)} - \sum_{i=L+1}^{N} w_{r(i)} > 0$; $1 \leq r(i) \leq N, i = 1, 2, \cdots, N, r(p) \neq r(q)$ if $p \neq q, r(i)$ are the indices of the elements in w; is one of the inequalities which is satisfied by the weights w of a WM filter, then for an input X, with either $X_{r(1)} \leq X_{r(2)} \leq \cdots \leq X_{r(N)}$ or $X_{r(1)} \geq X_{r(2)} \geq \cdots \geq X_{r(N)}$, the output, Y, is one of the samples $X_{r(1)}, X_{r(2)}, \cdots, X_{r(L)}$.

Proof: Let $\sum_{i=1}^{L} w_{r(i)} = u$ and $\sum_{i=L+1}^{N} w_{r(i)} = v$, then $u + v = 2M + 1$ and $u - v > 0$. Then $2u > 2M + 1$ or $u \geq M + 1$. But the output is that sample for which the sum of the weights is $M + 1$. Thus the output is one of the samples $X_{r(1)}, X_{r(2)}, \cdots, X_{r(L)}$ because the sum of their corresponding weights, $u \geq M + 1$.[]

Example: Consider the WM filter with $w = (1,2,3,2,1)$. The inequality $w_1 + w_3 + w_4 - w_2 - w_5 > 0$ is satisfied by the weights. For the input $X = (-2.1, 3, -6, 0, 8.2)$ where $X_3 < X_1 < X_4 < X_2 < X_5$ it is easily verified that the output is one of X_1, X_3 or X_4.

We can now show that the number of elements in $\mathcal{S}$ is 2^{N-1}.

Lemma 3: If $\mathcal{S} = \{\xi_n \mid n = 1, 2, \cdots, q\}$ represents a WM filter with weights w, then $q = 2^{N-1}$.

Proof: Assume that $q = 2^{N-1} - 1$. Then $\exists$ a vector, say $\xi \notin \mathcal{S}$ implies that the sign of $\xi^T w$, which is either positive or negative, need not be specified in defining WM filters. Assume that the inequality $\xi^T w < 0$ is inconsistent with the inequalities, $\{\xi_n^T w > 0 \mid \xi_n \in \mathcal{S}\}$. This indicates that the sign of $\xi^T w$, which is positive, is specified by the WM filter. This is a contradiction. Thus, $\xi \in \mathcal{S}$. Now, assume that both $\xi^T w > 0$ and $-\xi^T w > 0$ are consistent with the inequalities in $\{\xi_n^T w > 0 \mid \xi_n \in \mathcal{S}\}$. Consider two WM filters with different weights w^1 and w^2, which satisfy $\xi^T w^1 > 0$ and $-\xi^T w^2 > 0$ but have the same $\mathcal{S}$ (Since both $\xi^T w^1 > 0$ and $-\xi^T w^2 > 0$ and are assumed to be consistent with the inequalities $\{\xi_n^T w > 0 \mid \xi_n \in \mathcal{S}\}$, there must exist such weights w^1 and w^2). By hypothesis, the filters have identical $\mathcal{S}$, therefore they must be identical. However, by using Lemma 2, we can easily show that their outputs are not identical for a certain input ordering. This implies that $\xi^T w^1$ and $\xi^T w^2$ must have the same sign, i.e. $\xi \in \mathcal{S}$. This proof can be extended easily for $q < 2^{N-1} - 1$.[]

Two WM filters whose weights are different can be compared by comparing their respective sets $\mathcal{S}$. If two filters are the same then their sets $\mathcal{S}$ are identical. Conversely if their sets $\mathcal{S}$ are identical then the corresponding filters are identical.

From the discussion above it is clear that a WM filter may be specified by the inequalities associated with the vectors in $\mathcal{S}$, even if the actual weights are unknown. In practise, the filter may be of little use if the weights are not given. Thus, it may be required to compute the weights of a WM filter, given the set $\mathcal{S}$. The weights can be obtained by solving a linear/integer programming problem with some objective function[7] and constraints specified by $\xi_n^T w > 0$ for all $\xi_n \in \mathcal{S}$, where $w = (w_1, w_2, \cdots, w_N)^T$ are the unknown weights.

For convenience, the results above are summarized in part (A) and (B) of the following theorem, both of which indicate that $\mathcal{S}$ in (4) completely specifies a WM filter.

Theorem 1: (A) Suppose that the weight vector, denoted by w, of WM filter with span N is unknown, but it is known that the relationships among weights are represented by 2^{N-1} consistent inequalities $\xi_n^T w > 0, n = 1, 2, \cdots, 2^{N-1}$, where $\{\xi_n \mid n = 1, 2, \cdots, 2^{N-1}\}$ are distinct binary vectors consisting of -1 and -1. Then the output of the WM filter can be obtained for any input, and the weights of a WM filter, which is equal to the given filter, can be generated by solving the 2^{N-1} inequalities simultaneously.
(B) Two WM filters, of span N having different weights, w^1 and w^2, are equal iff they have the same binary vector set $\mathcal{S} = \{\xi_n \mid n = 1, 2, \cdots, 2^{N-1}\}$ for which $\xi_n^T w^1 > 0$ and $\xi_n^T w^2 > 0$ for every n. Here ξ_n is a $N \times 1$ vector consisting $+1$ and -1 (see (2)). ▌

In [6] it was shown, by using the result in [4], that a WM filter can be specified by a self-dual threshold function when the input sequence consists of quantized values. Here a threshold function is, $f(u_n)$, is defined as

$$f(u_n) = \begin{cases} 1, & \text{if } u_n^T w \geq t; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

where, $u_n, n = 1, 2, \cdots, 2^N$ are $N \times 1$ binary vectors consisting of 1 and 0's, $w = (w_1, w_2, \cdots, w_N)^T$ is the weight vector associated with the threshold function, and t is a threshold value. When the function is self-dual, $\overline{f(u_n)} = f(\overline{u_n})$, where $\overline{(.)}$ is the boolean inversion, and $t = (\sum_{r=1}^{N} w_r + 1)/2$. The threshold function is uniquely specified if the output for each binary input vector $u_n, n = 1, 2, \cdots, 2^{N-1}$ is known, or equivalently the inequalities associated with each vector u_n, $u_n^T w \geq$ or $< t$ is known. Thus 2^N consistent linear inequalities completely specify a threshold function. For a self-dual threshold function, the number of inequalities required to specify it is reduced to 2^{N-1}, because $\overline{f(u_n)} = f(\overline{u_n})$. By choosing only those vectors u_n for which $f(u_n) = 1$ it can be easily verfied that the inequalities in (6) above reduce to the same form as those associated with the vectors in $\mathcal{S}$. Therefore the result in [6], saying that a WM filter can be specified by a self-dual threshold function, is equivalent to our result summarized in Theorem 1. The difference is merely in the way these inequalities are interpreted. For binary inputs, each inequality corresponds to the binary output for the associated binary input vector. In the analysis presented here, every ordering of the inputs is associated with two inequalities which give the actual output. An advantage of the approach presented in this paper is the fact that we have not limited ourselves to quantized inputs. Binary input is merely a special case in the analysis presented in this paper.

A somewhat more interesting relation comes into play for WM filters which have symmetric weights. The inequalities relating the weights can be obtained in the same manner as before. For $N = 2L + 1$, the form of the inequalities is

$$\sum_{i=1}^{L} \xi_i^k \, 2w_i + w_{L+1} \geq 1 \text{ or } \leq -1 \tag{6a}$$

For $N = 2L$, the form of the inequality is

$$\sum_{i=1}^{L} \xi_i^k \, 2w_i \geq 1 \text{ or } \leq -1 \tag{6b}$$

where, $\xi_i^k = \pm 1$, or 0; $k = 1, 2, \ldots, 3^L$ and L is a positive integer. By comparision, a function $f(x_1, x_2, \cdots, x_L)$ is a ternary threshold function[8] of the variables $x_i (= 1, 0, -1), i = 1, 2, \cdots, L$, if there exists a set of weights $w = (w_1, w_2, \cdots, w_N)$, and thresholds $\{T_1, T_{-1}\}$, such that

$$f(x_1, x_2, \cdots, x_L) = \begin{cases} 1, & \text{if } \sum_{j=1}^{L} x_j w_j \geq 1; \\ -1, & \text{if } \sum_{j=1}^{L} x_j w_j \leq -1; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Notice that if the output is restricted to be binary values – i.e. take values ± 1 only – then the inequalities which describe the ternary threshold function have the same form as those describing the symmetric WM filter. In fact, the inequalities describing the WM filter of span N with symmetric weights can be interpreted as a ternary threshold function with weights $2w_1, 2w_2, \cdots, 2w_L$ and thresholds $\{1, -1\}$ for $N = 2L$ and $\{1 - w_{L+1}, -1 - w_{L+1}\}$ for $N = 2L + 1$ respectively. It is expected that a study of ternary threshold functions will give us a better insight into the properties of the symmetrically weighted WM filter.

In the following section, some properties of WM filters and its generation is discussed.

3. Properties and Generation

Based on the inequalities corresponding to the vector set $\mathcal{S}$ some properties have been derived. These provide simple criteria for examining the equivalence between WM filters, and are useful in generating WM filters equivalent to a given WM filter. In stating the properties it is assumed that the WM filter with weights $w^1 = (w_1, w_2, \cdots, w_N)^T$ is associated with the set $\mathcal{S} = \{\xi_n \mid n = 1, 2, \cdots, 2^{N-1}\}$

Property 1: The WM filters with weights $w^1 = (w_1, w_2, \cdots, w_N)^T$, $w^2 = (mw_1, mw_2, \cdots, mw_N)^T$ and $w^3 = (w_1/q, w_2/q, \cdots, w_N/q)^T$, are equivalent if m is a integer and q is the greatest common divisor of $w_1, w_2, \cdots, w_N$.

Proof: Follows easily from the inequalities associated with $\mathcal{S}$▌

Property 2: Two equivalent WM filters with weights w^1 and w^2, respectively, are equivalent to another with weights $w^1 + w^2$.

Proof: We have $\xi_n^T w^1 > 0$ and $\xi_n^T w^2 > 0$, for all $\xi_n \in \mathcal{S}$. Thus $\xi_n^T (w^1 + w^2) > 0$ for all $\xi_n \in \mathcal{S}$. ▌

Property 3: Consider a WM filter with weights $w^1 = (w_1, w_2, \cdots, w_N)^T$ and vector set $\mathcal{S}$. Suppose that the value of $d_{min} = \min_n \xi_n^T w^1$ is available. Then the WM filter with positive weights $w^2 = (mw_1 + q_1, mw_2 + q_2, \cdots, mw_N + q_N)^T$ are equivalent if $|q_{max}| \leq md_{min}/N$, where $m > 0$, $q_{max} = \max\{q_1, q_2, \cdots, q_N\}$.

Proof: The two WM filters are equivalent iff $\min_n \xi_n^T w^2 > 0$ for all $\xi_n \in \mathcal{S}$; i.e. $\min_n \{m\xi_n^T w + \xi_n^T q\} > 0$, where, $q = (q_1, q_2, \cdots, q_N)^T$. For this inequality to be true it is sufficient that $md_{min} > \max_n(-\xi_n^T q)$. This bound can be replaced by a more conservative one, viz. $md_{min} > \max_n |\xi_n^T q|$. But $\max_n |\xi_n^T q| \leq \max_n \sum_{i=1}^{N} |\xi_i^n q_i| \leq N|q_{max}|$, where ξ_i^n is the i^{th} element of ξ_n. Thus, if $|q_{max}| < md_{min}/N$ then the WM filters with weights w^1 and w^2 are equivalent.▌

Property 3 is closely related to the input tolerance of threshold gates as discussed by Muroga [pp 46–53, 5].

Property 4: Consider a WM filter with weights $w^1 = (w_1, w_2, \cdots, w_N)^T$ and vector set $\mathcal{S}$. Let a second WM filter have the positive weights $w^2 = (-mw_1 + q, -mw_2 + q, \cdots, -mw_N + q)^T$. Then, (i) for N even, the filters with weights w^1 and w^2 are different for all $q \geq 0$; (ii) for N odd, the two filters are different for all $q \geq 0$, unless w^1 is a median filter; where $m > 0$.

Proof: The two WM filters are the same iff $\xi_n^T w^2 > 0$, for all $n = 1, 2, \cdots, 2^{N-1}$, $\xi_n \in \mathcal{S}$. It is sufficient to show that $\exists\ \xi_n$, $1 \leq n \leq 2^{N-1}$, for which $\xi_n^T w^2 < 0$. We have $\xi_n^T w^2 = -m \sum_{i=1}^N \xi_i^n w_i + q \sum_{i=1}^N \xi_i^n$. The first term of this expression is always negative. It is required to show that there always exists some ξ_n, $1 \leq n \leq 2^{N-1}$, for which $\sum_{i=1}^N \xi_i^n \leq 0$. (i) For N=2L; $\sum_{i=1}^N \xi_i^n > 0$ for all $n = 1, 2, \cdots, 2^{N-1}$, implies that the number of 1's in each ξ_n is greater than or equal to $L+1$. From Lemma 2, we conclude that the output of the WM filter is $X_{(i)}$, $i = L+1, L+2, \cdots, N$, (i.e. the $(L+1)^{th}$ smallest or a larger sample). From Lemma 2, it would also imply that the output is always $X_{(i)}, i = 1, 2, \cdots, L-1$ (i.e. the $(L+1)^{th}$ largest or a smaller sample). None of the ranked samples satisfies both these requirements simultaneously. Thus $\sum_{i=1}^N \xi_i^n > 0$ for all $n = 1, 2, \cdots, 2^{N-1}$, cannot be true. (ii) For $N = 2L+1$; we follow the same reasoning as above. $\sum_{i=1}^N \xi_i^n > 0$ for all $n = 1, 2, \cdots, 2^{N-1}$, will only be true if the output is always one of $X_{(i)}, i = L+1, \cdots, N$, and one of $X_{(i)}, i = 1, \cdots, L+1$. This is only true when the output is always $X_{(L+1)}$ i.e. the filter with weights w^1 is a median filter. ▮

In the foregoing discussion it was shown that a WM filter of size N can be completely specified by a set of 2^{N-1} consistent linear inequalities, and that this specification is identical to that of a self-dual threshold function of N variables. For a given set of inequalities, we can generate a set of positive integers weights, $w = (w_1, w_2, \cdots, w_N)$ satisfying these inequalities. This can be done either by simultaneous solution of the inequalities or by using Integer Programming (IP). IP is a more effective technique and has been used successfully in optimisation. Here the sum of the weights, $\sum_{i=1}^N w_i$, can be minimised to obtain a representative set of weights.

We may also be interested in enumerating all deterministically distinct WM filters of a given size N. It is obvious that this is the same as enumerating all self-dual threshold functions of N variables. This has been discussed in detail in literature [9][10][11]. All self-dual threshold functions of upto eight variables have been obtained[12]. In 2-D applications the minimum window size is 9, and it may appear that the results obtained in [12] may not be useful. However, in all practical applications, the weights are distributed symmetrically. This reduces the number of inequalities required to define a WM filter. Since the computations required to find all the distinct classes of WM filters of a given size N roughly varies in an exponential manner with the number of inequalities, a reduction in the number of inequalities permits the computation of symmetric WM filters of span 9 or greater.

This problem of enumeration was first discussed by Brownrigg [2][3]. It is worth pointing out that the method presented in [2] and [3] is a bit cumbersome and rather restrictive. It is suggested that the IP approach be used whenever possible.

4. Conclusions

In this paper we have shown that regardless of the input – continuous or discrete valued – to a WM filter of span N, it can be specified by a set of 2^{N-1} mutually consistent linear inequalities involving its weights. This specification is the same as that for binary threshold functions. It has been observed that WM filters with symmetric weights can be interpreted as ternary threshold functions, a relation which needs to be examined in greater detail. A number of properties, which show equivalence between two WM filters, have been derived. Finally, generation and enumeration of WM filters has been discussed. It was pointed out that for $N \leq 8$, the available tables of self-dual threshold functions can be used to find all WM filters of span N.

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