ON A FUZZY DETECTION SCHEME FOR WEAK STOCHASTIC SIGNALS *

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Abstract

In this paper an application of fuzzy testing of hypothesis to purely stochastic signal detection problem is considered when the signal-to-noise ratio approaches zero. We first obtain the general relationship between the test statistic of the locally optimum fuzzy detector and that of the locally optimum detector. Based on this result, the test statistics of the locally optimum fuzzy detector for stochastic signals are obtained. Several aspects of the locally optimum fuzzy nonlinearity for stochastic signals are also described. Finally, performance characteristics of the locally optimum fuzzy detector are briefly discussed.

1. Introduction

Signal detection schemes based on fuzzy set theory have recently been investigated in the literature [2, 4, 5]. The detection schemes seem to be practically appealing since the noise distribution is often not known exactly. In [5] we discussed the weak *known-signal* detection problem with fuzzy information based on techniques of fuzzy testing of statistical hypothesis [*e.g.*, 6] and found the detector structure. The performance characteristics of the detector were also compared with those of the combined system of the quantizer and locally optimum (LO) detector (*i.e.*, the LOQ detector). The assumption of known signals in [5] is a realistic one since it is not difficult to find many examples which can be modeled as the known-signal detection problem in modern communication systems.

In this paper an application of fuzzy testing of hypothesis to detection of purely stochastic signals when the signal-to-noise ratio approaches zero is considered as a natural extension of our previous studies considered in [5, 6]. It should be noted that a considerable amount of study [*e.g.*, 3, 7] has been devoted to detection of purely stochastic (or random) signals in various noise circumstances. This is because it is convenient and reasonable to assume less about the signal than is required for known or parametric assumptions when the representation of desired a signal structure is difficult. For example, in acoustical applications, random dispersion due to turbulence and inhomogeneities in propagation media and insufficient understanding of the signal generating mechanism may lead us to adopt the purely stochastic signal model [3].

* *This research was supported by Korea Science and Engineering Foundation (KOSEF) under Grant 921-0800-002-2.*

2. Preliminaries

2.1. Observation Model

Let us consider the well-known signal-detection problem which can be expressed by the following hypotheses:

$$H_0: \ Y_i = W_i \ \ (\textit{signal is absent}), \tag{1}$$

versus

$$H_1: \ Y_i = \theta S_i + W_i \ \ (\textit{signal is present}), \tag{2}$$

for $i = 1, 2, ..., n$. In (1) and (2), Y_i is the observation, S_i is the stochastic signal component, and W_i is the purely-additive noise (PAN) component at the i-th sampling instant. The positive real quantity θ is the amplitude parameter which controls the signal strength. The stochastic signal component S_i is a random variable which has finite mean μ_i and variance σ_i^2, $i = 1, 2, ..., n$. The joint probability density function (pdf) of $\bar{S} = (S_1, S_2, \ldots, S_n)$ is denoted by $f_{\bar{S}}$, and the covariance function of S_i and S_j is denoted by $E\{(S_i - \mu_i)(S_j - \mu_j)\} = K_S(i, j)$. The PAN components W_i, $i = 1, 2, ..., n$, are assumed to be independent and identically distributed (i.i.d.) with common continuous pdf f_W, where the pdf f_W is assumed to be zero-mean and even. It is also assumed that the stochastic signal components are statistically independent of the PAN components. Based on these descriptions, we can express the conditional joint pdf of $\bar{Y} = (Y_1, Y_2, \ldots, Y_n)$ assuming θ as

$$f_{\bar{Y}}(\bar{y} \mid \theta) = \int_{X^n} f_{\bar{S}}(\bar{s}) \prod_{i=1}^{n} f_W(y_i - \theta s_i)\, d\bar{s}, \tag{3}$$

where $\bar{y} = (y_1, y_2, \ldots, y_n)$, $\bar{s} = (s_1, s_2, \ldots, s_n)$, and X^n is a Euclidean n-dimensional space.

2.2. Assumptions

In order to handle the observation Y_i as fuzzy information, let us introduce some definitions. Let (X^n, B_{X^n}, F) be a probability space where X^n is a Euclidean n-dimensional space, B_{X^n} is the Borel σ-field, and F is a probability measure over X^n. A *fuzzy information system* τ is a fuzzy partition of the real line X by means of fuzzy events. An n tuple of elements in τ, $\bar{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_n)$, $\kappa_i \in \tau$, $i = 1, 2, ..., n$, is called the *sample fuzzy information of size n* based on which a decision will be made. The set consisting of all possible *sample fuzzy information* is called the *fuzzy random sample* *of size n* and is denoted by $\tau^{(n)}$. The *probability distribution* of $\tau^{(n)}$ is given by

$$P(\overline{\kappa}) = \int_{X^n} \lambda_{\overline{\kappa}}(\overline{y})\, dF(\overline{y}), \tag{4}$$

where the integral is the Lebesgue-Stieltjes integral, $\lambda_{\overline{\kappa}}(\overline{y})$ is called the *membership function* of $\overline{\kappa}$, and it is assumed that

$$\lambda_{\overline{\kappa}}(\overline{y}) = \prod_{i=1}^{n} \lambda_{\kappa_i}(y_i). \tag{5}$$

From (4) we see that the conditional probability of $\overline{\kappa}$ assuming θ can be expressed as

$$P(\overline{\kappa} \mid \theta) = \int_{X^n} \lambda_{\overline{\kappa}}(\overline{y}) f_{\overline{Y}}(\overline{y} \mid \theta)\, d\overline{y}. \tag{6}$$

More details on these descriptions can be found in [6].

It is also assumed in this paper that the inputs to both of the LOQ and LOF detection processors are $Q_i = Q(Y_i)$, where $Q(\cdot)$ is the quantizer characteristic. The LOQ detector makes a decision based on Q_i, while the LOF detector makes a decision regarding Q_i as fuzzy information. In Figure 1(a), a typical input-output characteristic of the m-level quantizer for stochastic signal detection is shown in which we assume that the quantizer characteristic is even-symmetric. The parameters $\{\pm b_i\}_{i=1}^{m-1}$ and $\{l_i\}_{i=1}^{m}$ in Figure 1(a) are called the breakpoints and quantization levels of the quantizer, respectively.

Based on the above construction we can now consider a specific fuzzy information. For example, a fuzzy information τ_1 obtained from the quantizer means that the observed value lies approximately in $[b_1, b_2]$ (see Figure 1(b)). The shape of the membership function $\lambda_{\tau_i}(y)$ can arbitrarily be given provided that it satisfies the constraint of orthogonality. In this paper, we assume that there is a self-noise whose variance is considerably small compared to the variance of the PAN. In this case, we see from the discussions in [1] that one of the convenient and reasonable membership functions for the fuzzy information from the quantizer is the trapezoidal membership function which is illustrated in Figure 1(b). The parameter Δ in Figure 1(b) is called the *incredibility*.

3. Detector Test Statistics

Based on the fuzzy set theoretic extension of the generalized Neyman-Pearson lemma, it was shown in [6] that the test statistic of the LOF detector can be expressed as

$$T_{LOF}(\overline{\kappa}) = \frac{\dfrac{d^{\nu} P(\overline{\kappa} \mid \theta)}{d\theta^{\nu}}\Big|_{\theta=0}}{P(\overline{\kappa} \mid \theta=0)}, \tag{7}$$

where ν is the first non-zero derivative of $P(\overline{\kappa} \mid \theta)$ at $\theta = 0$. Using (6) and (7), it can be shown that the following relationship holds between the LOF detector test statistic $T_{LOF}(\overline{\kappa})$ and the LO detector test statistic $T_{LO}(\overline{y})$:

$$T_{LOF}(\overline{\kappa}) = \frac{\int_{X^n} \lambda_{\overline{\kappa}}(\overline{y}) T_{LO}(\overline{y}) \prod_{j=1}^{n} f_W(y_j)\, d\overline{y}}{\prod_{i=1}^{n} P(\kappa_i \mid \theta=0)} \tag{8}$$

In this section, we derive the LOF detector test statistic for stochastic signals using (8).

3.1. The Case of Non-Zero Mean Stochastic Signals

Let us first assume that at least one of μ_i, $i = 1, 2, \ldots, n$, is not zero. Then it can be shown that the LOF detector test statistic (8) becomes

$$T_{LOF}(\overline{\kappa}) = \sum_{i=1}^{n} \mu_i\, g_{LOF}(\kappa_i), \tag{9}$$

where

$$g_{LOF}(\kappa_i) = \frac{E\{\lambda_{\kappa_i}'(y)\}}{E\{\lambda_{\kappa_i}(y)\}} \tag{10}$$

is an LOF nonlinearity for stochastic signals. It is interesting to note that this nonlinearity is exactly the same as the known signal LOF nonlinearity [5]. In (10), $E\{\cdot\}$ denotes the statistical expectation with respect to f_W.

From (9), we see that the test statistic is in the form of the generalized correlator detectors. We also see that in this case the LOF detector test statistic depends only on the mean values of the stochastic signals. This implies that if a stochastic signal component has a non-zero mean, no other statistical characteristic of the stochastic signal components than the mean is necessary in constructing the LOF detector, and the test statistic is exactly the same as that for known signal detection with μ_i replaced with known signal components.

3.2. The Case of Zero Mean Stochastic Signals

Now let us assume that μ_i, $i = 1, 2, \ldots, n$, are all zero. If we assume that the stochastic signals are correlated, then it can also be shown that the LOF detector test statistic is

$$T_{LOF}(\overline{\kappa}) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} K_S(i,j)\, g_{LOF}(\kappa_i)\, g_{LOF}(\kappa_j) + \sum_{i=1}^{n} \sigma_i^2\, h_{LOF}(\kappa_i), \tag{11}$$

where

$$h_{LOF}(\kappa_i) = \frac{\int_{-\infty}^{\infty} \lambda_{\kappa_i}(y_i) f_W''(y_i)\, dy_i}{P(\kappa_i \mid \theta=0)} \tag{12}$$

is also an LOF nonlinearity for stochastic signals. From (11) we see that when the stochastic signal components are zero-mean, only the second-order statistics of the stochastic signal components are crucial in making a decision.

4. Locally Optimum Fuzzy Nonlinearity for Stochastic Signals

One of the important factors which characterize the detector structure is the detector nonlinearity. Hence more details on the characteristics of the LOF nonlinearities would be helpful and important in describing and analyzing LOF detectors. In this section we will discuss several characteristics of the nonlinearity h_{LOF}.

Let us first consider an alternative expressions of (12). Applying integration by parts to (12) twice, we have

$$h_{LOF}(\kappa_i) = \frac{E\{\lambda_{\kappa_i}''(y)\}}{E\{\lambda_{\kappa_i}(y)\}}. \tag{13}$$

For trapezoidal membership functions, it can be shown that we have Property 1 for the LOF nonlinearity h_{LOF}.

Property 1. If we consider the trapezoidal membership function, an alternative form of $h_{LOF}(\cdot)$ is, for any $\pm\tau_i$, $i = 1,2,\ldots,m-2$,

$$h_{LOF}(\pm\tau_i)=\frac{D(b_i)-D(b_{i+1})}{G(b_{i+1})-G(b_i)}, \tag{14}$$

where

$$D(\eta)\triangleq f_W(\eta-\frac{\Delta}{2})-f_W(\eta+\frac{\Delta}{2}) \tag{15}$$

and

$$G(\eta)\triangleq\int_{\eta-(\Delta/2)}^{\eta+(\Delta/2)} F_W(\xi)\,d\xi. \tag{16}$$

From (14) we see that h_{LOF} depends only on the values of the pdf at the four points $b_i\pm(\Delta/2)$ and $b_{i+1}\pm(\Delta/2)$ and of the cdf for the two intervals of length Δ $[b_i-(\Delta/2),\ b_i+(\Delta/2)]$ and $[b_{i+1}-(\Delta/2),\ b_{i+1}+(\Delta/2)]$ in which the membership grade varies. Expressions for the LOF nonlinearity h_{LOF} for τ_0 and $\pm\tau_{m-1}$ can also be obtained to be

$$h_{LOF}(\tau_0)=\frac{D(b_1)}{\frac{\Delta}{2}-G(b_1)} \tag{17}$$

and

$$h_{LOF}(\pm\tau_{m-1})=\frac{D(b_{m-1})}{\Delta-G(b_{m-1})}, \tag{18}$$

respectively.

We see that when the membership function is trapezoidal the expressions (13) and (14) are more convenient to handle than (12) since we can calculate the numerator of (13) and (14) with ease. In addition, it can be shown that $h_{LOF}(\cdot)$ is a decreasing function of Δ, which is a natural and reasonable observation since the detector nonlinearity can physically be considered as a weighting function for the observation containing noise and a large value of Δ implies that the LOF detector puts low confidence in the observed information.

Now let us consider a consequential property from (17) and (18).

Property 2. If we assume that the continuous noise pdf f_W is even, zero-mean, and unimodal with $f_W(0)$ being the only maximum value, then $h_{LOF}(\tau_0)<0$ and $h_{LOF}(\tau_{m-1})>0$.

The LO nonlinearity $h_{LO}(y)=f_W''(y)/f_W(y)$ is an even function of y when f_W is even. It is noteworthy that the same observation can be found for h_{LOF}.

Property 3. If $\lambda_{\tau_i}(y)=\lambda_{-\tau_i}(-y)$ and $f_W(y)$ is even, the nonlinearity $h_{LOF}(\cdot)$ is an even function of τ_i.

Properties 2 and 3 imply that the LO and LOF nonlinearities are of similar characteristic.

5. Performance Characteristic

In this section, we examine some performance characteristics of the LOF detector for stochastic signals obtained in Section 3 and compare them with those of the LOQ detector. Specifically, we performed three computer simulations, letting $n=50$, $m=4$, and the false-alarm probability (P_{fa}) equal to 10^{-2}. Each simulation for obtaining the detection probabilities (P_d) of the LOQ and LOF detectors was accomplished by 10^5 Monte-Carlo runs to make the relative error about 0.1%. For simplicity, we assumed that the stochastic signal components are i.i.d. with the standard normal pdf. We also assumed that the pdf of the PAN components is standard normal. To generate the stochastic signal and PAN components we used the GGNML subroutine of the IMSL.

In the first simulation, we assumed the ideal situation; that is, it is assumed that we have the perfect statistical information on the stochastic signal and PAN components and no self-noise is present. For the LOF detectors, we considered two values of Δ, 0.1 and 0.4. The detector thresholds and P_d were obtained through Monte-Carlo simulations. Note that the Monte-Carlo simulation is one of the conventional and reasonable methods, although it is no doubt that the method is based on a heuristic approach.

Figure 2 shows the plots of the detection probabilities of the LOQ and LOF detectors as functions of the stochastic-signal strength parameter θ. From Figure 2 we can see that there is no difference among the performance characteristics of the detectors. This seems to be due to the fact that the order of the ordered fuzzy information space is preserved. We may conclude that the LOF detector can replace the LOQ detector in the ideal situation although the LOF detector regards the output of the quantizer as fuzzy information.

In the second simulation, we again assumed that we have the perfect statistical information on the stochastic signal and PAN components. We assumed, however, that the self-noise is present in the second simulation. We let the self-noise be normal with mean zero and variance (or power) 0.01. (Since the variance of the PAN is assumed to be 1, we see that the power of the self-noise is 20dB lower than that of the PAN). Now let us denote the LOQ detector for noise of variance γ by $LOQ(\gamma)$. In the second simulation, we used approximate approach to find the thresholds, since finding the exact thresholds through the Monte-Carlo simulations is too time-consuming and thus physically cumbersome to implement. To find the thresholds of the LOF detectors, we used the approximate value,

$$threshold \approx Z_\alpha\left\{n\sum_{\kappa\in\tau}P(\kappa\mid\theta=0)\,h_{LOF}^2(\kappa)\right\}^{\frac{1}{2}}, \tag{19}$$

where Z_α is the $100(1-\alpha)th$ percentile of the standard normal distribution. Equation (19) can be obtained with the central limit theorem. The detector threshold of the $LOQ(1)$ was also obtained based on the asymptotic approximation. It should be noted that some errors can be made by this asymptotic approximation.

In Figure 3 we show the plots of the detection probabilities of the LOQ and LOF detectors as functions of θ. From Figure 3 we first see that the power function of the $LOQ(1)$ are large than that of the $LOQ(1.01)$ for all values of $\theta\geq 0$, since the $LOQ(1)$ does not take the effect of the self-noise into account. We also see that the LOF detectors have intermediate performance characteristics between the $LOQ(1)$ and $LOQ(1.01)$, and that as Δ becomes large the performance of the LOF detector approaches that of the $LOQ(1.01)$. These results are primarily due to the fact that we calculated the thresholds based on the Zadeh's definition of probability (4) and that the probability mass function (pmf) of the quantizer output level for the LOF detector is more similar to the pmf of the quantizer output level for the $LOQ(1.01)$ than to that for the *LOQ* (1).

6. Concluding Remark

In this paper we obtained the locally optimum fuzzy detector test statistics for stochastic signals. Several aspects of the locally optimum fuzzy nonlinearity for stochastic signals were discussed. We also examined the performance characteristics of the locally optimum fuzzy detector and showed that the locally optimum fuzzy detector has a robustness property.

The assumption of the self-noise can be considered in a different point of view. That is, the same procedure as that in the second computer simulation in Section 5 can be applied to the situation when the actual noise variance is slightly larger than the estimated noise variance.

References

[1] M.K. Brown, "On quantization of noisy signals", *IEEE Trans. Signal Proc.*, vol. SP-39, pp. 836-841, April 1991.

[2] A. Dziech and M.B. Gorzalczany. "Decision making in signal transmission problems with interval-valued fuzzy sets", *Fuzzy Sets and Systems*, vol. 23, pp. 191-203, August 1987.

[3] H.V. Poor and J.B. Thomas, "Locally optimum detection of discrete-time stochastic signals in non-Gaussian noise", *Jour. Acous. Soc. Amer.*, vol. 63, pp. 75-80, January 1978.

[4] J.J. Saade and H. Schwarzlander, "Fuzzy hypothesis testing with hybrid data", *Fuzzy Sets and Systems*, vol. 35, pp. 197-212, April 1990.

[5] J.C. Son, S.Y. Kim, I. Song, and H.M. Kim, "A fuzzy set theoretic approach to weak-known signal detection", *Signal Processing* vol. 28, pp. - , August 1992 (to be published).

[6] J.C. Son, I. Song, and H.Y. Kim, "A fuzzy decision problem based on the generalized Neyman-Pearson criterion", *Fuzzy Sets and Systems*, vol. 47, pp. 65-75, April 1992.

[7] I. Song and S.A. Kassam, "Locally optimum detection of signals in a generalized observation model: The random signal case", *IEEE Trans. Inform. Theory*, vol. IT-36, pp. 515-530, May 1990.

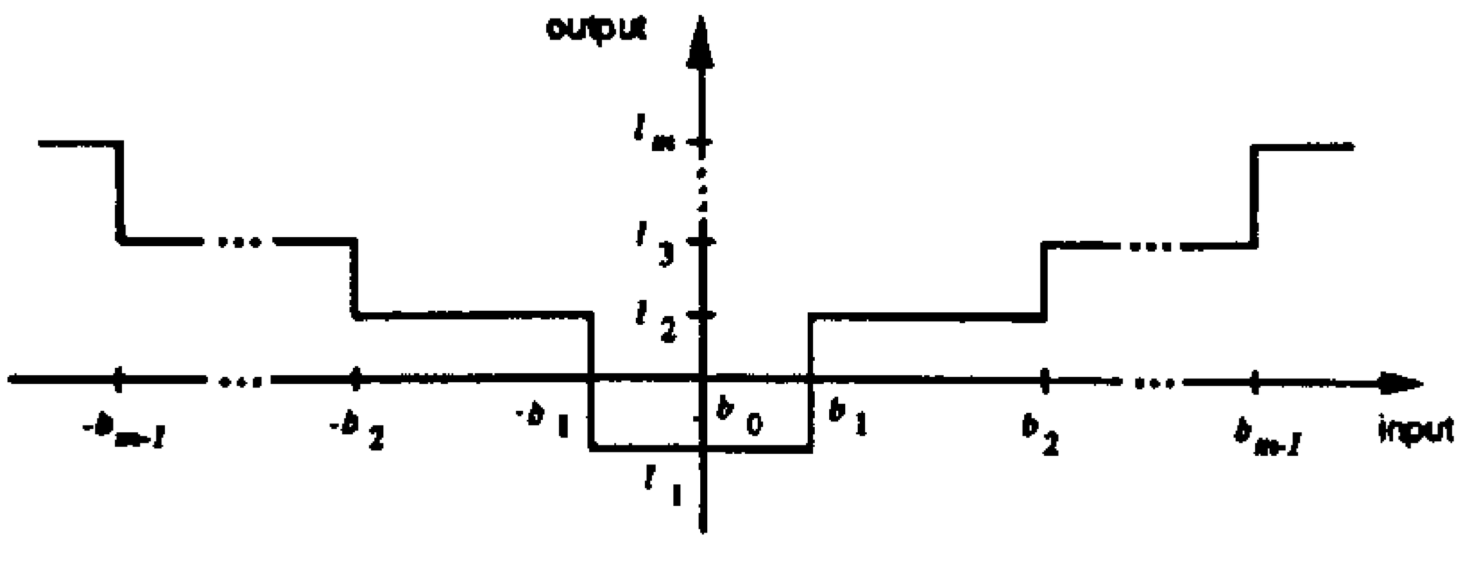

Figure 1. (a) An input-output characteristic of the quantizer for detection of stochastic signals

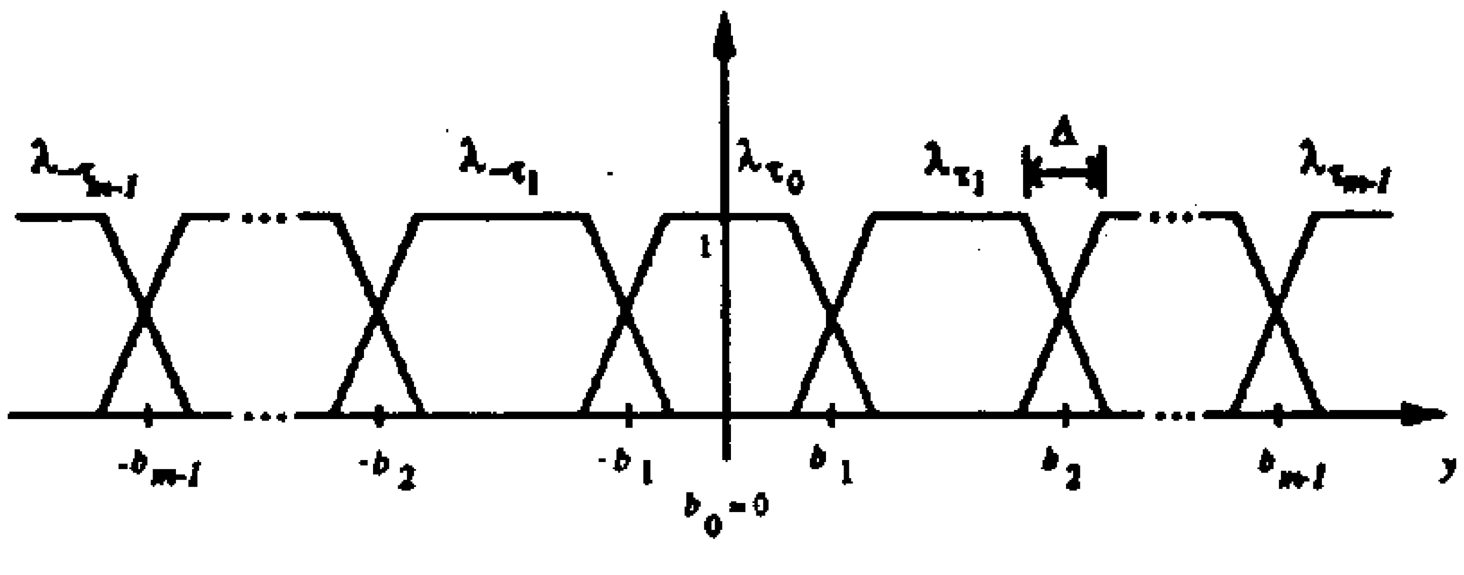

Figure 1. (b) The corresponding membership functions of fuzzy informations

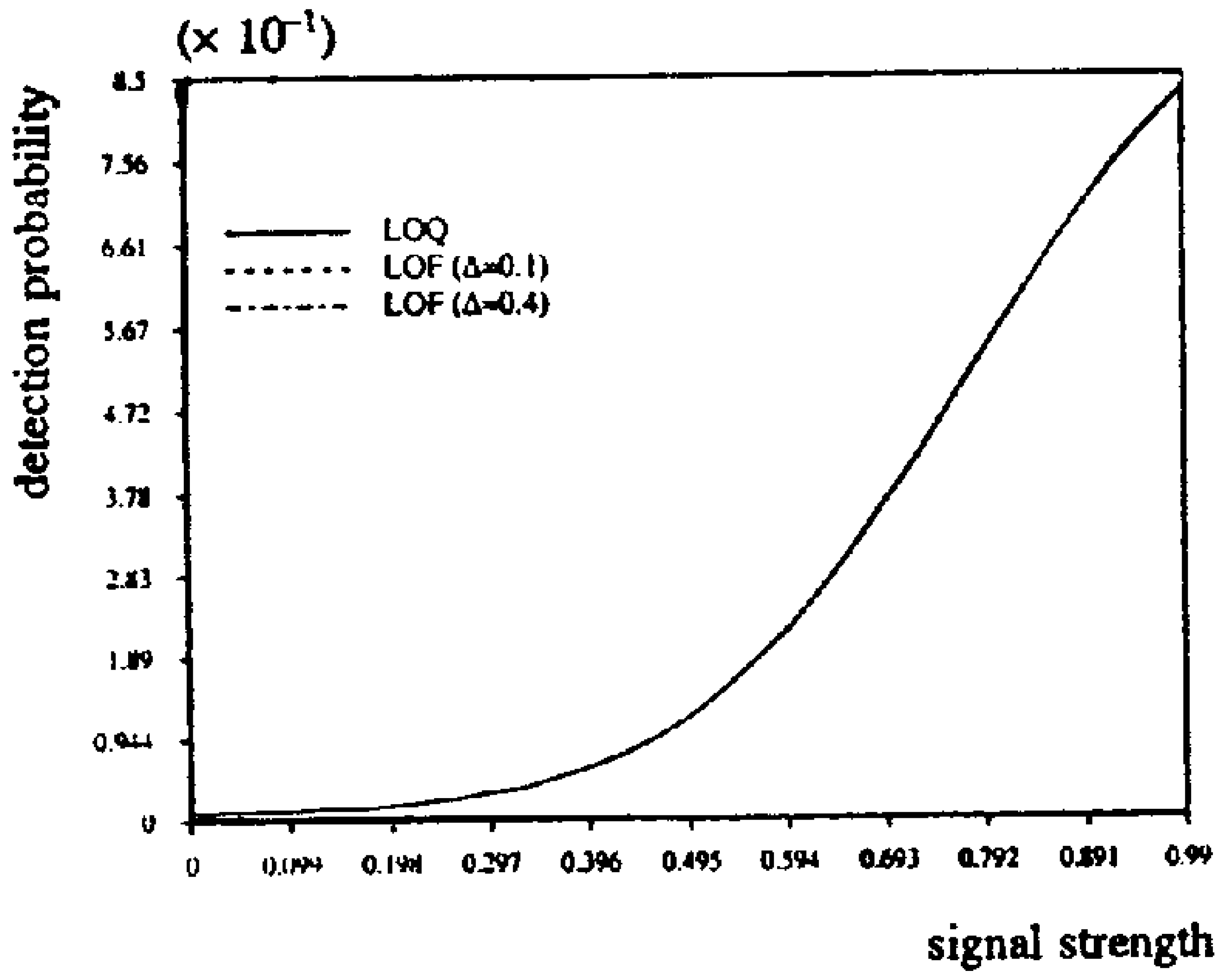

Figure 2. Detection probabilities of the LOQ and LOF detectors when there is no self-noise

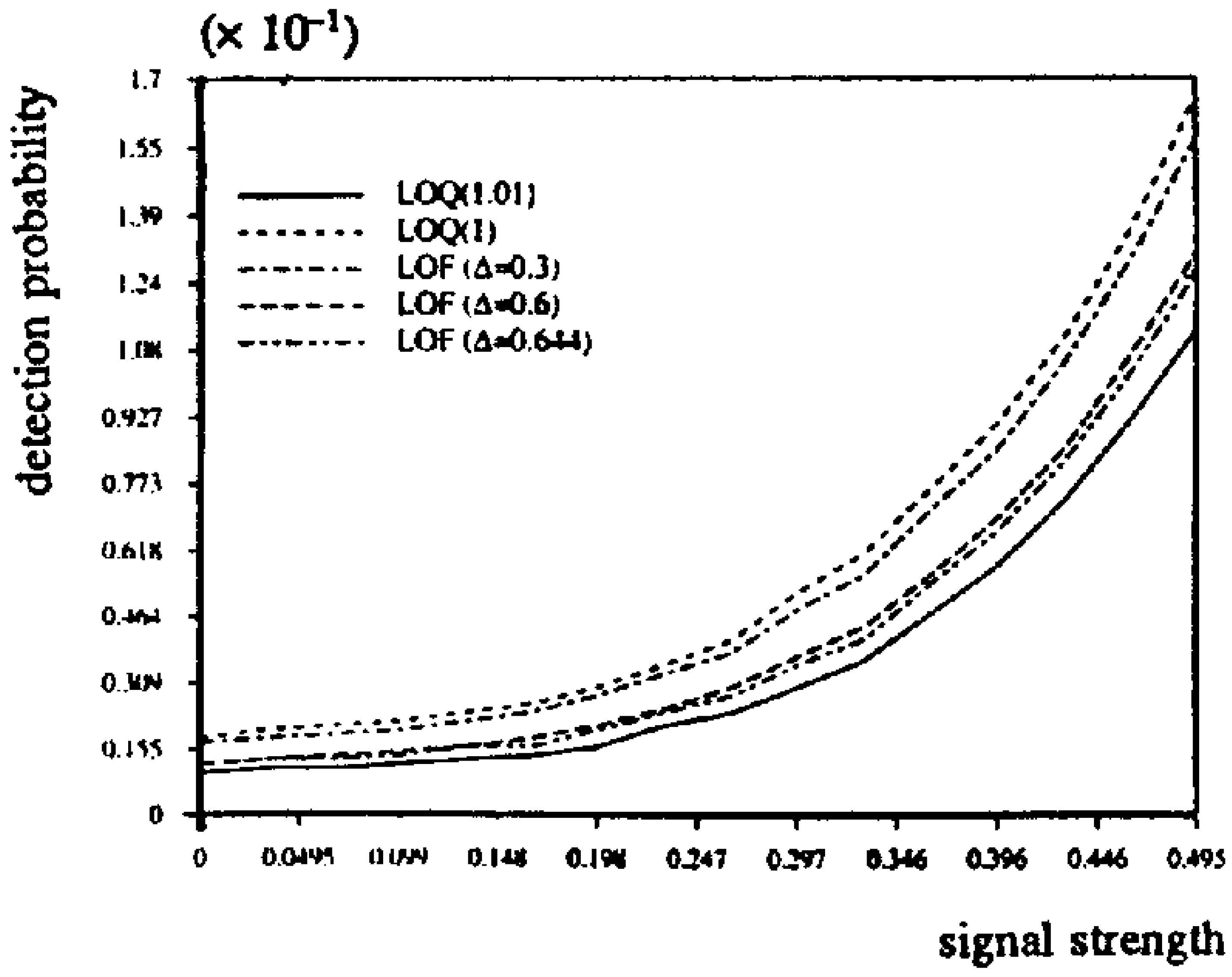

Figure 3. Detection probabilities of the LOQ and LOF detectors when there is self-noise of variance 0.01.