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STANDING WAVES OF NONLINEAR SCHRÖDINGER EQUATIONS WITH OPTIMAL CONDITIONS FOR POTENTIAL AND NONLINEARITY

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Abstract. We consider the singularly perturbed nonlinear elliptic problem

$$\varepsilon^2\Delta v - V(x)v + f(v) = 0, \quad v > 0, \quad \lim_{|x|\to\infty} v(x) = 0.$$

Under almost optimal conditions for the potential V and the nonlinearity f, we establish the existence of single-peak solutions whose peak points converge to local minimum points of V as $\varepsilon \to 0$. Moreover, we exhibit a threshold on the condition of V at infinity between existence and nonexistence of solutions.

1. **Introduction.** In this paper, we study standing wave solutions for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + f(\psi) = 0, \quad (t,x) \in \mathbf{R}\times\mathbf{R}^N, \tag{1.1}$$

where $\hbar$ denotes the Plank constant and i is the imaginary unit. We always assume that V and f are continuous.

A solution of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. We assume that f satisfies $f(\exp(i\theta)v) = \exp(i\theta)f(v)$ for $\theta, v \in \mathbf{R}$; then $f(\psi) = g(|\psi|)\psi$ for some real valued function g. Then, the function $\psi(x,t)$ is a standing wave solution of (1.1) if and only if v satisfies the equation

$$\frac{\hbar^2}{2}\Delta v - (V(x) - E)v + f(v) = 0 \quad \text{in} \quad \mathbf{R}^N.$$

We are interested in positive solutions of (1.1) decaying to 0 at infinity for small $\hbar > 0$. For small $\hbar > 0$, these standing waves are referred as semi-classical states. For convenience sake, we write V for $V - E$ and consider the following equation

$$\varepsilon^2\Delta v - V(x)v + f(v) = 0, \quad v > 0 \quad \text{in} \quad \mathbf{R}^N, \quad \lim_{|x|\to\infty} v(x) = 0 \tag{1.2}$$

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when $\varepsilon > 0$ is sufficiently small. Defining $u(x) = v(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, we see that equation (1.2) is equivalent to

$$\Delta u - V_\varepsilon u + f(u) = 0,\ u > 0 \quad \text{in} \quad \mathbf{R}^N,\ \lim_{|x|\to\infty} u(x) = 0. \tag{1.3}$$

Note that for each $x_0 \in \mathbf{R}^N$ and $R > 0$, V_ε converges uniformly to $V(x_0)$ on $B(x_0/\varepsilon, R)$ as $\varepsilon \to 0$. Thus, for $a = V(x_0) > 0$, we have the following limiting equation

$$\Delta u - au + f(u) = 0,\ u > 0 \quad \text{in} \quad \mathbf{R}^N,\ \lim_{|x|\to\infty} u(x) = 0. \tag{1.4}$$

A natural question is whether there exists a solution of (1.3) close to a solution of (1.4) for small $\varepsilon > 0$. In fact, when $\inf_{x\in\mathbf{R}^N} V(x) > 0$, $N = 1$ and $f(u) = u^3$, Floer and Weinstein in [19] constructed a solution close to $U(\cdot - \frac{x_0}{\varepsilon})$ provided that potential V has a non-degenerate critical point x_0 and U is a unique radially symmetric solution of (1.4) with $a = V(x_0)$. Later, Oh in [33] obtained the same result in higher dimension for $f(u) = |u|^{p-1}u$ with $1 < p < \frac{N+2}{N-2}$. On the other hand, it was shown by Wang [35] that if there exists a solution u_ε of (1.3) close to $U(\cdot - \frac{x_0}{\varepsilon})$, the point x_0 should be a critical point of $V \in C^1(\mathbf{R}^N)$.

The arguments in [19, 33] are based on the Lyapunov-Schmidt reduction which requires a linearized non-degeneracy of a solution of (1.4). The linearized nondegeneracy of a solution u means that if $\Delta\phi - a\phi + f'(u)\phi = 0$ and $\phi \in H^{1,2}(\mathbf{R}^N)$, then $\phi = \sum_{i=1}^N a_i \frac{\partial u}{\partial x_i}$ for some $a_1, \cdots, a_n \in \mathbf{R}$. There have been many further works using the Lyapunov-Schmidt reduction method; refer to [1, 2, 13, 14, 27, 28] and references therein.

In general, it is not easy to check the linearized non-degeneracy of a solution of equation (1.4) for general type of nonlinearity f. Furthermore, the linearizing process is not possible when f is not smooth, but just continuous. To overcome the strong restrictions on the nonlinearity in the approach through the Lyapunov-Schmidt reduction, Rabinowitz initiated a variational approach in [34]. In [34] he employed the mountain pass argument [6] to prove the existence of a positive solution of (1.2) for small $\varepsilon > 0$ provided that

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbf{R}^N} V(x) > 0.$$

These solutions concentrate around the global minimum points of V as $\varepsilon \to 0$. The variational approach has been developed further by del Pino, Felmer, and many others; refer to [11, 12, 15, 16, 17, 18, 22, 24] and references therein.

On the other hand, Berestycki and Lions in [7] showed the existence of least energy solutions to limit problem (1.4) with $a = m$ when the nonlinearity f satisfies the following conditions :

(f1) $\lim_{t\to 0^+} f(t)/t = 0$;
(f2) $\limsup_{t\to\infty} f(t)/t^p < \infty$ for some $p \in (1, \frac{N+2}{N-2})$, $N \geq 3$;
(f3) There exists $T > 0$ such that $\frac{1}{2}mT^2 < F(T)$, where $F(t) = \int_0^t f(s)\,ds$.

Pohozaev's identity (see (2.4) below) says that these conditions are almost necessary and sufficient conditions for the existence of solutions of (1.4). However, in all previous mentioned works, even when they adopt a variational method, they assume stronger conditions on f than (f1), (f2) and (f3). Recently, Byeon and Jeanjean in [9] could prove the existence of a solution of (1.2) concentrating around local minimum points of V for small $\varepsilon > 0$ assuming only the conditions (f1), (f2) and (f3) when $\inf_{x\in\mathbf{R}^N} V(x) > 0$.

On the other hand, it is easy to see that if $\inf_{x\in\mathbf{R}^N} V(x) < 0$, there exist no positive solutions of problem (1.2) for small $\varepsilon > 0$. Thus, a very natural question is whether there still exists a positive solution of problem (1.2) even if $\inf_{x\in\mathbf{R}^N} V(x) = 0$. In fact, Byeon and Wang [12] studied a case $\inf_{x\in\mathbf{R}^N} V(x) = 0$ and

$$\liminf_{|x|\to\infty} V(x) > 0$$

while in [3], Ambrosetti, Felli and Malchiodi studied a case

$$\liminf_{|x|\to\infty} V(x)|x|^\sigma > 0$$

for some $\sigma \geq 2$. From subsequent works [4], [5], [6], [25] by the Lyapunov-Schmidt reduction method, it is known that there exist solutions of (1.2) concentrating around stable critical points of V for small $\varepsilon > 0$ when $V \geq 0$ and $\liminf_{|x|\to\infty} V(x)|x|^2 > 0$. On the other hand, the main result in [26] implies that if $\liminf_{|x|\to\infty} V(x)|x|^\sigma = 0$ for some $\sigma > 2$, there exists no solution of (1.2) for $p \in (1, N/(N-2))$. In a monograph [4], Ambrosetti and Malchiodi raised a question on optimal conditions of V at infinity for the existence of solutions. Recently, Yin-Zhang [36] and Moroz-Van Schaftingen [31] answered independently the question for $f(t) = t^p, p \in (N/(N-2), (N+2)/(N-2))$. Their results in [36] and [31] say that for $f(t) = t^p, p \in (\frac{N}{N-2}, \frac{N+2}{N-2}), N \geq 3$ and small $\varepsilon > 0$, there exists a solution of (1.2) concentrating around positive local minimum points of V when $V \geq 0$. Thus, their result implies that the nonnegativity condition on V is optimal when $f(t) = t^p$ for $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ and $N \geq 3$. It is well known from [8], [21] and [32] that if $p \leq N/(N-2)$ and V has compact support, there exist no positive solutions of (1.2). Thus the exponent $N/(N-2)$ is critical when potential V has compact support.

The main purpose of this paper is to prove the existence of a solution of (1.2) which concentrates around an isolated set of positive local minima of potential V under optimal conditions both on the nonlinearity f and on the potential V, and to find a threshold of the asymptotic behavior of V at infinity between existence and nonexistence of a solution of (1.2). For the nonlinearity $f(t) = t^p$, we will show that if

$$\limsup_{|x|\to\infty} V(x)|x|^2 = 0 \qquad \text{for } \begin{cases} p \in (1,\infty), \ N = 1, 2, \\ p \in (1, \frac{N}{N-2}), \ N \geq 3, \end{cases}$$

$$\limsup_{|x|\to\infty} V(x)|x|^2 \log|x| = 0 \qquad \text{for } p = N/(N-2), N \geq 3,$$

then (1.2) has no solutions for small $\varepsilon > 0$, while if

$$\liminf_{|x|\to\infty} V(x)|x|^2 > 0 \qquad \text{for } \begin{cases} p \in (1,\infty), N = 1, 2, \\ p \in (1, \frac{N}{N-2}), N \geq 3, \end{cases}$$

$$\liminf_{|x|\to\infty} V(x)|x|^2 \log|x| > 0 \qquad \text{for } p = N/(N-2), N \geq 3,$$

then (1.2) has a solution which concentrates around an isolated set of positive local minima of potential V for small $\varepsilon > 0$. Thus we see that the case $p = N/(N-2)$ is critical and in contrast with the case $p \neq N/(N-2)$. The existence result will be established for general nonlinearity f satisfying Berestycki-Lions optimal conditions (f1), (f2), (f3). We prove the existence of a solution by developing further the approaches in [9], [31] and [36], and show the nonexistence of solutions by making use of an averaging argument and the so-called Emden-Fowler transformation.

To state our results precisely, we make the following conditions on V and f.

(V1) $V \in C(\mathbf{R}^N, \mathbf{R})$ and $\inf_{\mathbf{R}^N} V(x) \geq 0$;
(V2) There is a bounded domain O such that

$$0 < m \equiv \min_{x \in O} V(x) < \min_{x \in \partial O} V(x);$$

(V3) $\liminf_{|x| \to \infty} V(x)|x|^2 \equiv 4\lambda > 0$;
(V4) $\liminf_{|x| \to \infty} V(x)|x|^2 \log |x| > 0$;
(f1-1) $\lim_{t \to 0^+} f(t)/t^\mu = 0$ for $N \geq 3$ and some $\mu > \frac{N}{N-2}$;
(f1-2) $\lim_{t \to 0^+} f(t)/t^\mu = 0$ for some $\mu > 1$;
(f1-3) $\limsup_{t \to 0^+} f(t)/t^{\frac{N}{N-2}} < \infty$ for $N \geq 3$.

Set $\mathcal{M} \equiv \{x \in O \mid V(x) = m\}$. Without loss of generality, we assume that $0 \in \mathcal{M}$.

Theorem 1.1. *Let $N \geq 3$. We assume that* **(V1)**, **(V2)**, **(f2)** *and* **(f3)** *hold. In addition, we assume that one of the sets of conditions* **(A1) = {(f1-1)}**, **(A2) = {(V3), (f1-2)}**, **(A3) = {(V4), (f1-3)}** *hold. Then for sufficiently small $\varepsilon > 0$, (1.2) has a positive solution u_ε satisfying the following properties:*

- **(i)** *there exists a maximum point x_ε of u_ε such that $\lim_{\varepsilon \to 0} dist(x_\varepsilon, \mathcal{M}) = 0$, and $w_\varepsilon(x) \equiv u_\varepsilon(\varepsilon(x - x_\varepsilon))$ converges (up to a subsequence) uniformly to a positive, least energy solution of (1.4) with $a = m$;*
- **(ii)** *if* **(A1)** *holds, there exist $c, C > 0$ such that*

$$u_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon})/|x - x_\varepsilon|^{N-2};$$

- **(iii)** *if* **(A2)** *holds, there exist $c, C > 0$ such that*

$$u_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon})|x - x_\varepsilon|^{-\omega_\varepsilon} \chi_{\mathbf{R}^N \setminus B(x_\varepsilon, 1)} + C \exp(-c\frac{|x - x_\varepsilon|}{\varepsilon}) \chi_{B(x_\varepsilon, 1)},$$

where $\omega_\varepsilon \equiv \frac{(N-2) + \sqrt{(N-2)^2 + 4\lambda/\varepsilon^2}}{2}$ and $\chi_A(x) = 1$ for $x \in A$, $\chi_A(x) = 0$ for $x \notin A$;
- **(iv)** *if* **(A3)** *holds, for any $\alpha > 0$, there exist $c, C > 0$ such that*

$$u_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon})/|x - x_\varepsilon|^{N-2} \big| \log |x_\varepsilon - x| \big|^\alpha.$$

Remark 1. To obtain a similar existence result for $N = 1, 2$, we assume that **(V1)**, **(V2)**, **(V3)** and **(f1-2)** hold, that there exists $T > 0$ such that if $N = 2$, $\frac{1}{2}mT^2 < F(T)$ and if $N = 1$, $\frac{1}{2}mt^2 > F(t)$ for $t \in (0, T)$, $\frac{1}{2}mT^2 = F(T)$ and $mT < f(T)$. In addition, we assume for $N = 2$ that for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|f(t)| \leq C_\alpha \exp(\alpha t^2)$ for all $t \in \mathbf{R}^+$. Then the existence result of Theorem 1.1 with property **(iii)** holds. We can prove the existence combining the arguments of this paper and [10]. We leave the proof to the readers.

For a nonexistence result, we consider the following general exterior problem

$$\Delta u - W(x)u + u^p \leq 0, \quad u > 0 \quad , \quad \mathbf{R}^N \setminus \Omega \tag{1.5}$$

where Ω is a bounded open set.

Theorem 1.2. *Let $p > 1$ and $N \geq 1$. Assume that when $p(N-2) < N$,*

$$W(x)|x|^2 \leq \frac{2}{p-1}[\frac{2}{p-1} - (N-2)] \text{ for sufficiently large } |x| > 0,$$

and that when $p(N-2) = N$,

$$W(x)|x|^2 \log |x| \leq \frac{(N-2)^2}{2} \text{ for sufficiently large } |x| > 0.$$

Then, (1.5) has no C^2 solutions.

Theorem 1.2 improves the well-known nonexistence results in [8], [21] and [32] for the Lane-Emden equation

$$\Delta u + u^p = 0, \quad u > 0 \text{ in } \mathbf{R}^N. \tag{1.6}$$

Namely, if $p > 1$ and $p(N-2) \leq N$, then (1.6) has no positive supersolutions in any exterior domain.

An immediate consequence of Theorem 1.2 is the following nonexistence for problem (1.2).

Theorem 1.3. *Let $p > 1$ and $N \geq 1$. Assume that for $p(N-2) < N$,*

$$\lim_{|x|\to\infty} V(x)|x|^2 = 0$$

and for $p(N-2) = N$,

$$\lim_{|x|\to\infty} V(x)|x|^2 \log|x| = 0$$

Then, (1.2) has no C^2 solutions for any $\varepsilon > 0$.

We remark that in Theorems 1.2 and 1.3 for $N = 1$, each side condition near $\pm\infty$ in the assumptions is sufficient for the nonexistence.

In Section 2, we will prove Theorem 1.1. For the proof, we truncate the nonlinear function as in [15], [31] and [36]. Here we will take the truncation with relation to the asymptotic behavior of V near infinity Then we define an energy functional with a positive forcing term on energy function; by the forcing term, we get a lower estimate. Then, following the scheme developed in [9], we construct a set $X_\varepsilon \subset H_0^{1,2}(B(0, b/\varepsilon))$ of approximate solutions for large $b > 0$. Then, for large $b > 0$, we show that there exists a critical point $u_{\varepsilon,b} \in H_0^{1,2}(B(0, b/\varepsilon))$ of a modified energy functional on $H_0^{1,2}(B(0, b/\varepsilon))$ near the set X_ε of approximate solutions. Then, we find appropriate comparison functions depending on the truncation so that the solution of the modified equation decays to 0 faster than the comparison function near infinity. This implies that the critical point is a solution of the original problem on $B(0, b/\varepsilon)$ uniformly for large $b > 0$. Then, taking $b \to \infty$, we get a required solution on $\mathbf{R}^N$. We need the compact exhaustion of $\mathbf{R}^N$ by balls since the energy functional with some appropriately truncated nonlinearity can not belong to C^1 when the negative part of nonlinearity f exists.

In Section 3, we will prove Theorem 1.2. For the proof, assuming there exists a C^2 solution, by a averaging process, we reduce the problem to an ordinary differential inequality. Taking the Emden-Fowler transformation, we can get a contradiction via some elementary arguments.

After we had finished this work, we got to know the existence of related papers [26] and [30] which studied the nonexistence of positive supersolutions of (1.5) in exterior domains. In particular, the first result of Theorem 1.2 for $N \geq 2$ was established in Theorem 1.2 of [30]. Our approach is based on simple ODE arguments quite different from the approaches in [26] and [30].

2. **Proof of Theorem 1.1: existence.** We shall work with equation (1.3). Let H_ε be the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon = \left(\int_{\mathbf{R}^N} |\nabla u|^2 + V_\varepsilon u^2 \, dx\right)^{1/2}$$

and $\|u\|$ be the standard norm on $H^{1,2}(\mathbf{R}^N)$. From now on, for any set $B \subset \mathbf{R}^N$ and $\varepsilon, s > 0$, we define $B_\varepsilon \equiv \{x \in \mathbf{R}^N \mid \varepsilon x \in B\}$ and $B^s \equiv \{x \in \mathbf{R}^N \mid \operatorname{dist}(x, B) \leq s\}$. We may assume that $0 \in \mathcal{M}$ and $B(0,R) \subset O$ for some $R > 0$, and that for small $\delta > 0$ and $s \in [0, 5\delta]$, ∂O^s is smooth, $\inf_{x \in O^{5\delta}} V(x) = m$ and $\operatorname{dist}(\mathcal{M}, \mathbf{R}^N \setminus O) \geq 5\delta$. We can take a smaller neighborhood O of $\mathcal{M}$ and a sufficiently small $\delta > 0$ so that $V(x)T^2 < 2\int_0^T f(s)\,ds$ for any $x \in O^{5\delta}$. Since we look for positive solutions, we may assume that $f(t) = 0$ for all $t \leq 0$. If there exists $T_0 > T$ satisfying $f(T_0) = 0$, we may assume from the maximum principle that $f(t) = 0$ for $t \geq T_0$. Then, there exists $C_0 > 0$ such that

$$f(t) \geq -C_0 t \text{ for } \quad t \geq 0. \tag{2.1}$$

We note that for $N \geq 3$, the Hardy inequality says that

$$\int_{\mathbf{R}^N} |\nabla u|^2\,dx \geq \frac{(N-2)^2}{4}\int_{\mathbf{R}^N} \frac{u^2}{|x|^2}\,dx, \quad u \in C_0^\infty(\mathbf{R}^N). \tag{2.2}$$

Let β_ε be a continuous function on $[R/\varepsilon, \infty)$ satisfying $\beta_\varepsilon(|x|) \geq |x|^2$. Now, we define the truncated function g_ε of f

$$g_\varepsilon(x,t) \equiv \chi_{O_\varepsilon} f(t) + (1 - \chi_{O_\varepsilon}) \min\{\varepsilon^2/\beta_\varepsilon(|x|), f(t)/t\}t$$

and $G_\varepsilon(x,t) \equiv \int_0^t g_\varepsilon(x,s)\,ds$. We define

$$f^+(t) \equiv \max\{f(t), 0\} \text{ and } f^-(t) \equiv \max\{-f(t), 0\}.$$

Then, we see that

$$g_\varepsilon(x,t) = \chi_{O_\varepsilon} f^+(t) + (1 - \chi_{O_\varepsilon}) \min\{\varepsilon^2/\beta_\varepsilon(|x|), f^+(t)/t\}t - f^-(t).$$

Setting

$$g_\varepsilon^+(x,t) \equiv \chi_{O_\varepsilon} f^+(t) + (1 - \chi_{O_\varepsilon}) \min\{\varepsilon^2/\beta_\varepsilon(|x|), f^+(t)/t\}t,$$

$g_\varepsilon^-(x,t) \equiv f^-(t)$ and $G_\varepsilon^\pm(x,t) \equiv \int_0^t g_\varepsilon^\pm(x,s)\,ds$, we have $G_\varepsilon(x,t) = G_\varepsilon^+(x,t) - G_\varepsilon^-(x,t)$.

For $u \in H_\varepsilon$, we define

$$\Gamma_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V_\varepsilon u^2\,dx - \int_{\mathbf{R}^N} G_\varepsilon(x,u)\,dx + \Big(\int_{O_\varepsilon^\delta \setminus O_\varepsilon} \frac{1}{\varepsilon} u^2\,dx - 1\Big)_+^2.$$

If V has compact support and $f(t) < 0$ for some $t > 0$, Γ_ε could not belong to $C^1(H_\varepsilon)$. In order to circumvent this situation, for large $b > R$ with $O \subset B(0, b/2)$, we define

$$H_\varepsilon^b \equiv \{u \in H_\varepsilon \mid u(x) = 0 \text{ for } |x| \geq b/\varepsilon\}.$$

Then, it is standard to verify that $\Gamma_\varepsilon \in C^1(H_\varepsilon^b)$ for each $b > 0$. We should note that

$$g_\varepsilon(x,t) \leq f(t), \quad t \in \mathbf{R}. \tag{2.3}$$

We define an energy functional for limiting problem (1.4) by

$$L_a(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + au^2\,dx - \int_{\mathbf{R}^N} F(u)\,dx, \quad u \in H^{1,2}(\mathbf{R}^N),$$

where $F(t) = \int_0^t f(s)\,ds$. Berestycki and Lions proved that for $a = V(x), x \in O^{5\delta}$, there exists a least energy solution of (1.4) if f satisfies (f1), (f2) and (f3) with $a = m$, and that for each solution u of (1.4),

$$\frac{N-2}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx + N\int_{\mathbf{R}^N} a\frac{u^2}{2} - F(u)\,dx = 0. \tag{2.4}$$

Let S_a be the set of least energy solutions U of (1.4) satisfying

$$U(0) = \max_{x\in\mathbf{R}^N} U(x).$$

Then, there exist $C, c > 0$ such that

$$U(x) \le C\exp(-c|x|), \quad U \in S_a, \tag{2.5}$$

and the set S_a is compact in $H^{1,2}(\mathbf{R}^N)$(see [9]). We define $E_m = L_m(U)$ for $U \in S_m$. For each $t > 0$ and $U \in S_m$, we define $U_t(x) = U(\frac{x}{t})$. Then, from (2.4), we see that

$$\begin{aligned} L_m(U_t) &= \int_{\mathbf{R}^N} \frac{t^{N-2}}{2}|\nabla U|^2 + m\frac{t^N}{2}U^2 - t^N F(U)\,dx \\ &= \Big(\frac{t^{N-2}}{2} - \frac{(N-2)t^N}{2N}\Big)\int_{\mathbf{R}^N} |\nabla U|^2\,dx. \end{aligned} \tag{2.6}$$

Thus, there exists $t_0 > 1$ such that $L_m(U_t) < -1$ for $t \ge t_0$, uniformly in $U \in S_m$. Let $\varphi \in C_0^\infty(\mathbf{R}^N)$ be such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| \le \delta$ and $\varphi(x) = 0$ for $|x| \ge 2\delta$. Define $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. For each $x \in \mathcal{M}^\delta$ and $U \in S_m$, let

$$U_\varepsilon^x(y) \equiv \varphi_\varepsilon(y - \frac{x}{\varepsilon})U(y - \frac{x}{\varepsilon}).$$

We define a set X_ε of approximating solutions by

$$X_\varepsilon = \{U_\varepsilon^x(y) \mid x \in \mathcal{M}^\delta, U \in S_m\}.$$

We assume that $b > 0$ is sufficiently large so that $\mathrm{supp}(U_\varepsilon^x) \subset B(0, b/t_0\varepsilon)$ for all $U_\varepsilon^x \in X_\varepsilon$. Then, we define $X_\varepsilon^d(b) \equiv \{u \in H_\varepsilon^b \mid \|u - X_\varepsilon\|_\varepsilon \le d\}$. For some $d > 0$ and large $b > 0$, we will find a solution $u_{\varepsilon,b}$ in $X_\varepsilon^d(b)$ for sufficiently small $\varepsilon > 0$, independent of large $b > 0$, which satisfies

$$\Delta u_{\varepsilon,b} - V_\varepsilon u_{\varepsilon,b} + f(u_{\varepsilon,b}) = 0 \quad \text{in} \quad B(0, b/\varepsilon), \quad u_{\varepsilon,b} = 0 \quad \text{on} \quad \partial B(0, b/\varepsilon).$$

Then, taking a limit of $u_{\varepsilon,b}$ as $b \to \infty$, we will get a solution u_ε of original problem on $\mathbf{R}^N$.

For a fixed $U \in S_m$, we define

$$W_\varepsilon(y) = U(y)\varphi(\varepsilon y) \quad \text{and} \quad W_{\varepsilon,t}(y) = U(\frac{y}{t})\varphi(\varepsilon y),$$

where U is a fixed element in S_m. Then, it follows from (2.5) and (2.6) that for large $b > R$ and $t \in (0, t_0)$, $W_{\varepsilon,t} \in H_\varepsilon^b$ and

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(W_{\varepsilon,t}) = \Big(\frac{Nt^{N-2}}{2} - \frac{(N-2)t^N}{2}\Big)E_m \tag{2.7}$$

uniformly for $t \in (0, t_0]$ and large $b > R$. Thus we obtain that

$$D_\varepsilon \equiv \max_{s\in[0,t_0]} \Gamma_\varepsilon(W_{\varepsilon,s}) \to E_m \quad \text{as} \quad \varepsilon \to 0. \tag{2.8}$$

For small $d > 0$, we can take $\eta \in (0, 1)$ such that

$$W_{\varepsilon,t} \in X_\varepsilon^d(b) \quad \text{for} \quad t \in (1-\eta, 1+\eta).$$

We define

$$\begin{aligned} \Phi_\varepsilon^d(b) = \ & \{\gamma \in C([0,t_0], H_\varepsilon^b) \ |\gamma(t) = W_{\varepsilon,t} \ \text{ for } \ t \in (0, t_0] \setminus (1-\eta, 1+\eta)\} \\ & \cap\{\gamma \in C([0,t_0], H_\varepsilon^b) \ |\gamma(t) \in X_\varepsilon^d(b) \ \text{ for } \ t \in (1-\eta, 1+\eta)\} \end{aligned}$$

and

$$C_\varepsilon^d(b) = \inf_{\gamma\in\Phi_\varepsilon^d(b)} \max_{s\in[0,t_0]} \Gamma_\varepsilon(\gamma(s)).$$

From (2.7), we can see that $\Gamma_\varepsilon(\gamma(t_0)) < -1$ for any sufficiently small $\varepsilon > 0$ and $\gamma \in \Phi_\varepsilon^d(b)$.

Proposition 1. *For any small $d > 0$, it holds that*

$$\lim_{\varepsilon\to 0} C_\varepsilon^d(b) = E_m \quad \textit{uniformly for large } b > 0.$$

Proof. Let $W_{\varepsilon,0} = \lim_{t\to 0} W_{\varepsilon,t}$, i.e., $W_{\varepsilon,0} = 0$. Then, it follows from (2.5), (2.6) and (2.7) that

$$C_\varepsilon^d(b) \le \max_{t\in[0,t_0]} \Gamma_\varepsilon(W_{\varepsilon,t}) \to E_m \text{ as } \varepsilon \to 0.$$

Now, it suffices to show that $\liminf_{\varepsilon\to 0} C_\varepsilon^d(b) \ge E_m$ uniformly for large $b > 0$. We choose $\phi \in C_0^\infty(\mathbf{R}^N, [0,1])$ such that $\phi(x) = 0$ for $x \notin O^\delta$ and $\phi(x) = 1$ for $x \in O$. Set $\phi_\varepsilon(x) = \phi(\varepsilon x)$. Then, $\|\nabla\phi_\varepsilon\|_{L^\infty} \le C\varepsilon$ for some $C > 0$. For $\gamma \in \Phi_\varepsilon^d(b)$, we see that for any $s \in [0, t_0]$,

$$\begin{aligned}
&\Gamma_\varepsilon(\gamma(s)) = \Gamma_\varepsilon(\phi_\varepsilon\gamma(s) + (1-\phi_\varepsilon)\gamma(s))\\
&= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla\phi_\varepsilon\gamma(s)|^2 + V_\varepsilon(\phi_\varepsilon\gamma(s))^2\,dx - \int_{\mathbf{R}^N} G_\varepsilon(x,\phi_\varepsilon\gamma(s))\,dx\\
&\quad + \frac{1}{2}\int_{\mathbf{R}^N} |\nabla(1-\phi_\varepsilon)\gamma|^2 + V_\varepsilon((1-\phi_\varepsilon)\gamma)^2 dx - \int_{\mathbf{R}^N} G_\varepsilon(x,(1-\phi_\varepsilon)\gamma)\,dx\\
&\quad + \int_{\mathbf{R}^N} \nabla(\phi_\varepsilon\gamma(s))\cdot\nabla((1-\phi_\varepsilon)\gamma(s)) + V_\varepsilon\phi_\varepsilon(1-\phi_\varepsilon)(\gamma(s))^2\,dx\\
&\quad + \int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon(x,\phi_\varepsilon\gamma(s)) + G_\varepsilon(x,(1-\phi_\varepsilon)\gamma(s)) - G_\varepsilon(x,\gamma(s))\,dx\\
&\quad + \Big(\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{1}{\varepsilon}(\gamma(s))^2\,dx - 1\Big)_+^2.
\end{aligned}$$

We see from (2.1) that

$$\int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon(x,\phi_\varepsilon\gamma)\,dx \ge -\int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon^-(x,\phi_\varepsilon\gamma)\,dx \ge -\frac{C_0}{2}\int_{O_\varepsilon^\delta\setminus O_\varepsilon} (\phi_\varepsilon\gamma)^2\,dx$$

and

$$\begin{aligned}
\int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon(x,(1-\phi_\varepsilon)\gamma(s))\,dx &\ge -\int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon^-(x,(1-\phi_\varepsilon)\gamma(s))\,dx\\
&\ge -\frac{C_0}{2}\int_{O_\varepsilon^\delta\setminus O_\varepsilon} ((1-\phi_\varepsilon)\gamma(s))^2\,dx.
\end{aligned}$$

By the definition of $g_\varepsilon(x,t)$, we see that

$$-\int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon(x,\gamma)\,dx \ge -\int_{O_\varepsilon^\delta\setminus O_\varepsilon} G_\varepsilon^+(x,\gamma)\,dx \ge -\frac{1}{2}\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{\varepsilon^2}{\beta_\varepsilon(|x|)}(\gamma(s))^2\,dx.$$

Then, it follows that for some C, independent of small $\varepsilon > 0$ and large $b > 0$,

$$\begin{aligned}
&\Gamma_\varepsilon(\gamma(s)) \\
&\geq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla \phi_\varepsilon \gamma(s)|^2 + V_\varepsilon(\phi_\varepsilon\gamma(s))^2\, dx - \int_{\mathbf{R}^N} F(\phi_\varepsilon\gamma(s))\, dx \\
&\quad + \frac{1}{2}\int_{\mathbf{R}^N} |\nabla(1-\phi_\varepsilon)\gamma|^2 + V_\varepsilon((1-\phi_\varepsilon)\gamma)^2 - \frac{\varepsilon^2}{|x|^2}((1-\phi_\varepsilon)\gamma)^2\, dx \\
&\quad - C\varepsilon\int_{O_\varepsilon^\delta\setminus O_\varepsilon} |\nabla\gamma(s)|^2 + (\gamma(s))^2\, dx \\
&\quad - C_0\int_{O_\varepsilon^\delta\setminus O_\varepsilon} (\gamma(s))^2\, dx - \frac{1}{2}\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{\varepsilon^2}{|x|^2}(\gamma(s))^2\, dx \\
&\quad + \Big(\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{1}{\varepsilon}(\gamma(s))^2\, dx - 1\Big)_+^2.
\end{aligned}$$

If $\int_{O_\varepsilon^\delta\setminus O_\varepsilon}(\gamma(s))^2\, dx \geq \sqrt{\varepsilon}$, then

$$\begin{aligned}
&-C_0\int_{O_\varepsilon^\delta\setminus O_\varepsilon} (\gamma(s))^2\, dx + \Big(\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{1}{\varepsilon}(\gamma(s))^2\, dx - 1\Big)_+^2 \\
&\geq -C_0\int_{O_\varepsilon^\delta\setminus O_\varepsilon} (\gamma(s))^2\, dx + (\frac{1}{\sqrt{\varepsilon}} - 1)^2.
\end{aligned}$$

Since there exists M_0, independent of small $\varepsilon > 0$ and large $b > 0$, such that $\int_{O_\varepsilon^\delta\setminus O_\varepsilon}(\gamma(s))^2 dx \leq M_0$ for any $\gamma \in \Phi_\varepsilon^d(b)$, we see that

$$\liminf_{\varepsilon\to 0}\Big\{ - C_0\int_{O_\varepsilon^\delta\setminus O_\varepsilon} (\gamma(s))^2\, dx + \Big(\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{1}{\varepsilon}(\gamma(s))^2\, dx - 1\Big)_+^2\Big\} \geq 0$$

uniformly for large $b > 0$. Since $V(x) \geq m$ for any $x \in O^\delta$, we see that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^N} |\nabla \phi_\varepsilon \gamma(s)|^2 + V_\varepsilon(\phi_\varepsilon\gamma(s))^2\, dx - \int_{\mathbf{R}^N} F(\phi_\varepsilon\gamma(s))\, dx \\
&\geq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla \phi_\varepsilon \gamma(s)|^2 + m(\phi_\varepsilon\gamma(s))^2\, dx - \int_{\mathbf{R}^N} F(\phi_\varepsilon\gamma(s))\, dx.
\end{aligned}$$

Hence, we have

$$\max_{s\in[0,t_0]}\Big(\frac{1}{2}\int_{\mathbf{R}^N} |\nabla \phi_\varepsilon \gamma(s)|^2 + V_\varepsilon(\phi_\varepsilon\gamma(s))^2\, dx - \int_{\mathbf{R}^N} F(\phi_\varepsilon\gamma(s))\, dx\Big) \geq E_m.$$

The Hardy inequality in (2.2) implies that for small $\varepsilon > 0$,

$$\frac{1}{2}\int_{\mathbf{R}^N} |\nabla(1-\phi_\varepsilon)\gamma|^2 + V_\varepsilon((1-\phi_\varepsilon)\gamma)^2 - \frac{\varepsilon^2}{|x|^2}((1-\phi_\varepsilon)\gamma)^2\, dx \geq 0.$$

Lastly, we see that $\lim_{\varepsilon\to 0}\int_{O_\varepsilon^\delta\setminus O_\varepsilon} \frac{\varepsilon^2}{|x|^2}(\gamma(s))^2\, dx = 0$ uniformly for large $b > 0$ and $\gamma \in \Phi_\varepsilon^d(b)$. Then, we have

$$\liminf_{\varepsilon\to 0}\max_{s\in[0,t_0]}\Gamma_\varepsilon(\gamma(s)) \geq E_m \quad \text{uniformly for large } b > 0 \text{ and } \gamma \in \Phi_\varepsilon^d(b),$$

which completes the proof. □

We define

$$\Gamma_\varepsilon^\alpha = \{u \in H_\varepsilon \mid \Gamma_\varepsilon(u) \leq \alpha\}.$$

For a set $A \subset H_\varepsilon$ and $\alpha, b > 0$, we define $A^\alpha \equiv \{u \in H_\varepsilon \mid \|u - A\|_\varepsilon \leq \alpha\}$ and $A^\alpha(b) \equiv \{u \in H_\varepsilon^b \mid \|u - A\|_\varepsilon \leq \alpha\}$.

Proposition 2. *For sufficiently small $d_1 > d_2 > 0$, there exist constants $\omega > 0$ and $\varepsilon_0 > 0$, independent of large $b > 0$, such that $\|\Gamma'_\varepsilon(u)\|^*_{H^b_\varepsilon} \geq \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^{d_1}(b) \setminus X_\varepsilon^{d_2}(b))$ and $\varepsilon \in (0, \varepsilon_0)$.*

Proof. To the contrary, suppose that for small $d_1 > d_2 > 0$, there exist $\{b_i\}_{i=1}^\infty \subset [R, \infty)$, $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} \varepsilon_i = 0$ and $u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_1}(b_i) \setminus X_{\varepsilon_i}^{d_2}(b_i)$ satisfying

$$\limsup_{i\to\infty} \Gamma_{\varepsilon_i}(u_{\varepsilon_i}) \leq E_m$$

and $\lim_{i\to\infty} \|\Gamma'_{\varepsilon_i}(u_{\varepsilon_i})\|^*_{H^{b_i}_\varepsilon} = 0$. We may assume that $d_1 \leq \frac{E_m}{6}$. For our convenience, we write ε for ε_i and b for b_i. We may regard u_ε as an element in H_ε by defining $u_\varepsilon(x) = 0$ for $|x| \geq b/\varepsilon$.

We shall get a contradiction by showing that $u_\varepsilon \in X_\varepsilon^{d_2}(b)$ for sufficiently small $\varepsilon > 0$. Clearly this will be the case if u_ε is, as $\varepsilon \to 0$, arbitrarily close to a function of the form $(\varphi_\varepsilon U_\varepsilon)(\cdot - \frac{x_\varepsilon}{\varepsilon})$ with $x_\varepsilon \in \mathcal{M}^\delta$, $U_\varepsilon \in S_m$.

By the compactness of S_m and $\mathcal{M}^\delta$, there exist $Z \in S_m$ and $x_\varepsilon \in \mathcal{M}^\delta$ such that

$$\|u_\varepsilon - \varphi_\varepsilon(\cdot - x_\varepsilon/\varepsilon)Z(\cdot - x_\varepsilon/\varepsilon)\|_\varepsilon \leq 2d_1 \tag{2.9}$$

for small $\varepsilon > 0$. Taking a subsequence, we may assume that $\lim_{\varepsilon\to 0} x_\varepsilon = x_0 \in \mathcal{M}^\delta$. We denote $u_\varepsilon^1 = \varphi_\varepsilon(\cdot - x_\varepsilon/\varepsilon)u_\varepsilon$ and $u_\varepsilon^2 = u_\varepsilon - u_\varepsilon^1$.

Then, we claim that

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1). \tag{2.10}$$

Suppose there exist $y_\varepsilon \in B(x_\varepsilon/\varepsilon, \delta/\varepsilon)\backslash B(x_\varepsilon/\varepsilon, \delta/2\varepsilon)$ and $r > 0$ satisfying

$$\liminf_{\varepsilon\to 0} \int_{B(y_\varepsilon, r)} (u_\varepsilon)^2\, dy > 0.$$

Taking a subsequence, we can assume that $\lim_{\varepsilon\to 0} \varepsilon y_\varepsilon = y_0$ for some $y_0 \in \mathcal{M}^{2\delta}$ and that $u_\varepsilon(\cdot + y_\varepsilon) \to \tilde{W}$ weakly in $H^1(\mathbf{R}^N)$ for some $\tilde{W} \in H^1(\mathbf{R}^N) \setminus \{0\}$. Moreover $\tilde{W}$ satisfies

$$\Delta\tilde{W}(y) - V(y_0)\tilde{W}(y) + f(\tilde{W}(y)) = 0 \quad \text{for} \quad y \in \mathbf{R}^N.$$

Since $V(y_0) \geq m$, we deduce from [23] that

$$\frac{1}{2}\int_{\mathbf{R}^N} |\nabla\tilde{W}|^2 + V(y_0)(\tilde{W})^2\, dy - \int_{\mathbf{R}^N} F(\tilde{W})\, dy \geq E_{V(y_0)} \geq E_m.$$

From the weak convergence, we see that for large $R > 0$,

$$\liminf_{\varepsilon\to 0} \int_{B(y_\varepsilon, R)} |\nabla u_\varepsilon|^2\, dy \geq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla\tilde{W}|^2\, dy. \tag{2.11}$$

Thus, combining (2.11) and (2.4) with $a = V(y_0)$, we see that

$$\liminf_{\varepsilon\to 0} \int_{B(y_\varepsilon, R)} |\nabla u_\varepsilon|^2\, dy \geq \frac{N}{2} L_{V(x_0)}(\tilde{W}) \geq \frac{N}{2} E_m > 0. \tag{2.12}$$

Then, taking $d_1 \leq \sqrt{NE_m}/4$, we reach a contradiction with (2.9). Due to the nonexistence of such a sequence $\{y_\varepsilon\}_\varepsilon$, we deduce from a result of Lions (see [29, Lemma I.1]) that

$$\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon/\varepsilon, \delta/\varepsilon)\backslash B(x_\varepsilon/\varepsilon, \delta/2\varepsilon)} |u_\varepsilon|^{p+1}\, dx = 0.$$

Consequently, we can derive by using (f1), (f2) and boundedness of $\{\|u_\varepsilon\|_{L^2(O_\varepsilon)}\}_\varepsilon$ that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^N} G_\varepsilon(x,u_\varepsilon)-G_\varepsilon(x,u_\varepsilon^1)-G_\varepsilon(x,u_\varepsilon^2)\,dx=0.$$

Then, since $\{u_\varepsilon\}_\varepsilon$ is bounded uniformly for large $b>0$, we deduce that

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon) &= \Gamma_\varepsilon(u_\varepsilon^1)+\Gamma_\varepsilon(u_\varepsilon^2)\\ &\quad+\int_{\mathbf{R}^N}\psi_\varepsilon(1-\psi_\varepsilon)|\nabla u_\varepsilon|^2+V_\varepsilon\psi_\varepsilon(1-\psi_\varepsilon)u_\varepsilon^2\,dx\\ &\quad-\int_{\mathbf{R}^N} G_\varepsilon(x,u_\varepsilon)-G_\varepsilon(x,u_\varepsilon^1)-G_\varepsilon(x,u_\varepsilon^2)\,dx+o(1)\end{aligned}$$

and thus, inequality (2.10) follows.

We now estimate $\Gamma_\varepsilon(u_\varepsilon^2)$. It follows from (2.9) that $\|u_\varepsilon^2\|_\varepsilon\le 4d_1$ for small $\varepsilon>0$. From conditions (f1) and (f2), there exists $C>0$ that

$$F(t)\le mt^2/4+Ct^{2N/(N-2)},\ t\ge 0.$$

Then we see that

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon^2) &\ge \int_{O_\varepsilon}\frac{1}{2}|\nabla u_\varepsilon^2|^2+\frac{1}{2}V_\varepsilon(u_\varepsilon^2)^2-F(u_\varepsilon^2)\,dx\\ &\quad+\int_{\mathbf{R}^N\setminus O_\varepsilon}\frac{1}{2}|\nabla u_\varepsilon^2|^2+\frac{1}{2}V_\varepsilon(u_\varepsilon^2)^2-\frac{\varepsilon^2}{2\beta_\varepsilon(|x|)}(u_\varepsilon^2)^2\,dx\\ &\ge \int_{O_\varepsilon}\frac{1}{2}|\nabla u_\varepsilon^2|^2+\frac{1}{2}V_\varepsilon(u_\varepsilon^2)^2-\frac{m}{4}(u_\varepsilon^2)^2-C(u_\varepsilon^2)^{2N/(N-2)}\,dx\\ &\quad+\int_{\mathbf{R}^N\setminus O_\varepsilon}\frac{1}{2}|\nabla u_\varepsilon^2|^2+\frac{1}{2}V_\varepsilon(u_\varepsilon^2)^2-\frac{\varepsilon^2}{2|x|^2}(u_\varepsilon^2)^2\,dx\\ &\ge \int_{O_\varepsilon}\frac{1}{2}|\nabla u_\varepsilon^2|^2-C(u_\varepsilon^2)^{2N/(N-2)}\,dx+\frac{1}{4}\int_{\mathbf{R}^N}V_\varepsilon(u_\varepsilon^2)^2\,dx\\ &\quad+\int_{\mathbf{R}^N\setminus O_\varepsilon}\frac{1}{2}|\nabla u_\varepsilon^2|^2-\frac{\varepsilon^2}{2|x|^2}(u_\varepsilon^2)^2\,dx\\ &= \int_{\mathbf{R}^N}\frac{1}{4}|\nabla u_\varepsilon^2|^2\,dx-C\int_{O_\varepsilon}(u_\varepsilon^2)^{2N/(N-2)}\,dx+\frac{1}{4}\int_{\mathbf{R}^N}V_\varepsilon(u_\varepsilon^2)^2\,dx\\ &\quad+\int_{\mathbf{R}^N}\frac{1}{4}|\nabla u_\varepsilon^2|^2\,dx-\int_{\mathbf{R}^N\setminus O_\varepsilon}\frac{\varepsilon^2}{|x|^2}(u_\varepsilon^2)^2\,dx\\ &\ge \int_{\mathbf{R}^N}\frac{1}{4}|\nabla u_\varepsilon^2|^2\,dx-C\int_{\mathbf{R}^N}(u_\varepsilon^2)^{2N/(N-2)}\,dx+\frac{1}{4}\int_{\mathbf{R}^N}V_\varepsilon(u_\varepsilon^2)^2\,dx\\ &\quad+\int_{\mathbf{R}^N}\frac{1}{4}|\nabla u_\varepsilon^2|^2\,dx-\int_{\mathbf{R}^N}\frac{\varepsilon^2}{|x|^2}(u_\varepsilon^2)^2\,dx.\end{aligned}$$

Since the Hardy inequality in (2.2) implies that for small $\varepsilon>0$,

$$\int_{\mathbf{R}^N}\frac{1}{4}|\nabla u_\varepsilon^2|^2\,dx-\int_{\mathbf{R}^N}\frac{\varepsilon^2}{|x|^2}(u_\varepsilon^2)^2\,dx\ge 0,$$

we obtain by Sobolev's inequality that for some $C, c > 0$,

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon^2) \quad &\geq \quad \frac{1}{4}\|\nabla u_\varepsilon^2\|_{L^2}^2(1 - cC\|\nabla u_\varepsilon^2\|_{L^2}^{\frac{4}{N-2}}) + \frac{1}{4}\int_{\mathbf{R}^N} V_\varepsilon (u_\varepsilon^2)^2\,dx \\ &\quad + \Big(\frac{1}{4} - \frac{4\varepsilon^2}{(N-2)^2}\Big)\|\nabla u_\varepsilon^2\|_{L^2}^2.\end{aligned}$$

Thus, taking $d_1 \in (0, (\frac{1}{4cC})^{(N-2)/4})$, we see that for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(u_\varepsilon^2) \geq \frac{1}{8}\|u_\varepsilon^2\|_\varepsilon^2.$$

Now let $W_\varepsilon(x) = u_\varepsilon^1(x + x_\varepsilon/\varepsilon)$. Taking a subsequence we can assume that $W_\varepsilon \to W$ weakly in $H^1(\mathbf{R}^N)$ for some $W \in H^1(\mathbf{R}^N)$. Moreover, W satisfies

$$\Delta W(x) - V(x_0)W(x) + f(W(x)) = 0 \ \text{ for } \ x \in \mathbf{R}^N.$$

From the maximum principle, we see that W is positive. Let us prove that $W_\varepsilon \to W$ strongly in $H^1(\mathbf{R}^N)$. Suppose there exist $R > 0$ and a sequence $\{z_\varepsilon\}_\varepsilon$ with $z_\varepsilon \in B(x_\varepsilon/\varepsilon, \delta/\varepsilon)$ satisfying

$$\liminf_{\varepsilon\to 0} |z_\varepsilon - x_\varepsilon/\varepsilon| = \infty \quad \text{and} \quad \liminf_{\varepsilon\to 0}\int_{B(z_\varepsilon,R)} (u_\varepsilon^1)^2\,dx > 0.$$

We may assume that $\varepsilon z_\varepsilon \to z_0 \in O$ as $\varepsilon \to 0$. Then, $\hat{W}_\varepsilon(x) = u_\varepsilon^1(x + z_\varepsilon)$ converges weakly to $\hat{W}$ in $H^1(\mathbf{R}^N)$ satisfying

$$\Delta\hat{W} - V(z_0)\hat{W} + f(\hat{W}) = 0 \ \text{ in } \ \mathbf{R}^N.$$

At this point we get a contradiction as before. Then using (f1), (f2) and [29, Lemma I.1], we get that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^N} F(W_\varepsilon)\,dx = \int_{\mathbf{R}^N} F(W)\,dx. \tag{2.13}$$

Then, the weak convergence of W_ε to W in $H^1(\mathbf{R}^N)$ implies that

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon^1) \\ &= \liminf_{\varepsilon\to 0}\frac{1}{2}\int_{\mathbf{R}^N} |\nabla W_\varepsilon(x)|^2 + V(\varepsilon x + x_\varepsilon)W_\varepsilon^2(x)\,dx - \int_{\mathbf{R}^N} F(W_\varepsilon(x))\,dx \\ &\geq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla W|^2 + V(x_0)W^2\,dx - \int_{\mathbf{R}^N} F(W)\,dx \\ &\geq E_m.\end{aligned} \tag{2.14}$$

Since $\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon) \leq E_m$ and $\Gamma_\varepsilon(u_\varepsilon^2) \geq \frac{1}{8}\|u_\varepsilon^2\|_\varepsilon^2$ for small $\varepsilon > 0$, we see from (2.10) that

$$\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon^1) \leq E_m \quad \text{and} \quad \lim_{\varepsilon\to 0}\|u_\varepsilon^2\|_\varepsilon = 0. \tag{2.15}$$

Then (2.14) implies that $L_{V(x_0)}(W) = E_m$. Also, from [23], we see that $x_0 \in \mathcal{M}$. Clearly, $W(x) = U(x - z)$ with $U \in S_m$ and $z \in \mathbf{R}^N$. Combining (2.13), (2.15) and the fact that $V \geq V(x_0)$ on O, we get from (2.14) that

$$\begin{aligned}
&\int_{\mathbf{R}^N} |\nabla W|^2 + V(x_0)W^2\,dx \\
&\geq \limsup_{\varepsilon\to 0} \int_{\mathbf{R}^N} |\nabla u_\varepsilon^1(x)|^2 + V(\varepsilon x)(u_\varepsilon^1(x))^2\,dx \\
&\geq \limsup_{\varepsilon\to 0} \int_{\mathbf{R}^N} |\nabla u_\varepsilon^1(x)|^2 + V(x_0)(u_\varepsilon^1(x))^2\,dx \\
&\geq \limsup_{\varepsilon\to 0} \int_{\mathbf{R}^N} |\nabla W_\varepsilon(x)|^2 + V(x_0)(W_\varepsilon(x))^2\,dx.
\end{aligned}$$

This proves the strong convergence of u_ε^1 to W in $H^1(\mathbf{R}^N)$. In particular, setting $y_\varepsilon = x/\varepsilon + z$ we have $u_\varepsilon^1 \to \varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)$ strongly in $H^1(\mathbf{R}^N)$, which means that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon^1 - \varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)\|_\varepsilon = 0.$$

Since $\lim_{\varepsilon\to 0}\|u_\varepsilon^2\|_\varepsilon = 0$, we see that $\lim_{\varepsilon\to 0}\|u_\varepsilon - \varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)\|_\varepsilon = 0$, which contradicts that $u_\varepsilon \notin X_\varepsilon^{d_2}(b)$. Then the proof is complete. □

Following Proposition 2, we fix $d > 0$ and corresponding $\omega > 0$ and $\varepsilon_0 > 0$ such that $|\Gamma_\varepsilon'(u)| \geq \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d(b) \setminus X_\varepsilon^{d/2}(b))$, large $b > 0$ and $\varepsilon \in (0, \varepsilon_0)$. Then, we obtain the following proposition.

Proposition 3. *There exists $\alpha > 0$ such that for sufficiently small $\varepsilon > 0$ and large $b > 0$,*

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq C_\varepsilon^d(b) - \alpha \text{ implies that } \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}(b)$$

where $\gamma_\varepsilon(s) = W_{\varepsilon,s}$ and $s \in [0, t_0]$.

Proof. Since $\operatorname{supp}(\gamma_\varepsilon(s)) \subset \mathcal{M}_\varepsilon^{2\delta}$ and the function U of $W_{\varepsilon,s}(x) = U(\frac{x}{s})\varphi(\varepsilon x)$ decays to 0 in exponential order as $|x| \to \infty$, the assertion follows from (2.7). See the arguments of the proof of [9, Proposition 6]. □

Proposition 4. *There exist $\varepsilon_0 > 0$ and $b_0 > R$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $b > b_0$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset X_\varepsilon^d(b) \cap \Gamma_\varepsilon^{D_\varepsilon}$ such that $\Gamma_\varepsilon'(u_n) \to 0$ as $n \to \infty$.*

Proof. By Proposition 3, there exists $\alpha > 0$, independent of small $\varepsilon > 0$ and $b > R$, such that for sufficiently small $\varepsilon > 0$ and $b > R$,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq C_\varepsilon^d(b) - \alpha \quad \text{implies that} \quad \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}(b).$$

If Proposition 4 does not hold for some small $\varepsilon > 0$ and large $b > b_0$, there exists $a(\varepsilon, b) > 0$ such that $|\Gamma_\varepsilon'(u)| \geq a(\varepsilon, b)$ on $X_\varepsilon^d(b) \cap \Gamma_\varepsilon^{D_\varepsilon}$. Moreover, by Proposition 2, there exist $\omega > 0$, independent of $\varepsilon > 0$, and large $b > R$ such that $|\Gamma_\varepsilon'(u)| \geq \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d(b) \setminus X_\varepsilon^{d/2}(b))$. From (2.8) and Proposition 1, we recall that $\lim_{\varepsilon\to 0}(C_\varepsilon^d(b) - D_\varepsilon) = 0$ uniformly for large $b > R$. Then, via a pseudo-gradient flow on $X_\varepsilon^d(b)$, we can deform γ_ε to a path $\tilde{\gamma}_\varepsilon \in \Phi_\varepsilon^d(b)$ satisfying $\Gamma_\varepsilon(\tilde{\gamma}(s)) < C_\varepsilon^d(b), s \in [0, 1]$ (refer to [9, Proposition 7] and [10, Proposition 8]). This contradiction proves the claim. □

Proposition 5. *For $\varepsilon \in (0, \varepsilon_0)$, $b > b_0$, Γ_ε has a critical point $u_{\varepsilon,b} \in X_\varepsilon^d(b) \cap \Gamma_\varepsilon^{D_\varepsilon}$.*

Proof. Let $\{u_n\}_{n=1}^\infty$ be a Palais-Smale sequence as given by Proposition 4 corresponding to a fixed small $\varepsilon > 0$ and large $b > 0$. Since $\{u_n\}_{n=1}^\infty$ is bounded in H_ε, u_n converges weakly to some $u_{\varepsilon,b} \in X_\varepsilon^d(b)$. Then, it follows in a standard way that $u_{\varepsilon,b} \in X_\varepsilon^d(b)$ is a nontrivial critical point of Γ_ε on H_ε^b. From the strong convergence of $\{u_n\}_n$ to $u_{\varepsilon,b}$ in $L^q(B(0,b/\varepsilon))$, $q\in[2,2N/(N-2))$, and the weak convergence of $\{u_n\}_n$ to $u_{\varepsilon,b}$ in H^b_ε, we conclude that $\Gamma_\varepsilon(u)\le D_\varepsilon$. □

Completion of the Proof for Theorem 1.1. We see from Proposition 5 that for small $d>0$, there exist $\varepsilon_0>0$ and $B_0>R$ such that for $\varepsilon\in(0,\varepsilon_0)$ and $B>B_0$, Γ_ε has a critical point $u_{\varepsilon,b}\in X^d_\varepsilon(b)\cap\Gamma_\varepsilon^{D_\varepsilon}$ satisfying

$$\Delta u_{\varepsilon,b}-V_\varepsilon u_{\varepsilon,b}+g_\varepsilon(x,u_{\varepsilon,b})=\frac{4}{\varepsilon}\chi_{O^\delta_\varepsilon\setminus O_\varepsilon}u_{\varepsilon,b}\Big(\int_{O^\delta_\varepsilon\setminus O_\varepsilon}\frac{1}{\varepsilon}u^2\,dx-1\Big)_+\quad\text{in}\quad B(0,b/\varepsilon).\tag{2.16}$$

From the maximum principle, we see that $u_{\varepsilon,b}>0$ on $B(0,b/\varepsilon)$, and that

$$\Delta u_{\varepsilon,b}-V_\varepsilon u_{\varepsilon,b}+g_\varepsilon(x,u_{\varepsilon,b})\ge0\quad\text{in}\quad B(0,b/\varepsilon).\tag{2.17}$$

We also see from (2.3) that

$$\Delta u_{\varepsilon,b}-V_\varepsilon u_{\varepsilon,b}+f(u_{\varepsilon,b})\ge0\quad\text{in}\quad B(0,b/\varepsilon),\tag{2.18}$$

and from the definition of g_ε that

$$\Delta u_{\varepsilon,b}-V_\varepsilon u_{\varepsilon,b}+\frac{\varepsilon^2}{\beta_\varepsilon(|x|)}u_{\varepsilon,b}\ge0\quad\text{in}\quad B(0,b/\varepsilon)\setminus O_\varepsilon.$$

We take $x_{\varepsilon,b}\in\mathcal{M}^\delta_\varepsilon$ such that $\|u_{\varepsilon,b}-(\varphi_\varepsilon U)(\cdot-x_{\varepsilon,b})\|_\varepsilon\le d$. Suppose that there exist $\varepsilon_m>0$ with $\lim_{m\to\infty}\varepsilon_m=0$, large $b_m\ge R$ and $x_m\in O_\varepsilon$ such that $\lim_{m\to\infty}|x_{\varepsilon_m,b_m}-x_m|=\infty$, $\lim_{m\to\infty}\ \mathrm{dist}(x_m,\partial O_\varepsilon)=\infty$ and

$$\liminf_{m\to\infty}u_{\varepsilon_m,b_m}(x_m)>0.$$

Then, by the same argument with (2.12) in the proof of Proposition 2, we get a contradiction for small $d\le\sqrt{NE_m}/4$. Thus, we see that $u_{\varepsilon,b}(x)$ converges to 0 uniformly for large $b>R$ as $x\in O_\varepsilon$ and $\mathrm{dist}(x,\partial O_\varepsilon\cup\{x_{\varepsilon,b}\})\to\infty$.

Since $\{\|u_{\varepsilon,b}\|_\varepsilon\}_\varepsilon$ is bounded uniformly for large $b>0$ and $V(x)\ge m$ for $x\in O^{5\delta}$, we conclude from (2.18) and elliptic estimates through the Moser iteration argument that $\{\|u_{\varepsilon,b}\|_{L^\infty(O^{4\delta}_\varepsilon)}\}_\varepsilon$ is bounded uniformly for large $b>R$. Then, we see from the elliptic estimate [20, Theorem 8.17] that there exists $C>0$, independent of small $\varepsilon>0$, large $b>0$ and $y\in O^{3\delta}_\varepsilon$ satisfying

$$\sup_{x\in B(y,1)}u_{\varepsilon,b}(x)\le C\|u_{\varepsilon,b}\|_{L^2(B(y,2))}.$$

Since $u_{\varepsilon,b}\in X^d_\varepsilon(b)$ and $\mathrm{supp}(v)\subset\mathcal{M}^{3\delta}_\varepsilon$ for $v\in X^d_\varepsilon(b)$, we see that

$$\sup_{x\in O^{3\delta}_\varepsilon\setminus O_\varepsilon}u_{\varepsilon,b}(x)\le Cd/m.$$

Then, there exists a large $a>0$ such that

$$\Delta u_{\varepsilon,b}-\frac{m}{2}u_{\varepsilon,b}\ge\Delta u_{\varepsilon,b}-V_\varepsilon u_{\varepsilon,b}+f(u_{\varepsilon,b})\ge0\quad\text{in}\quad O^{3\delta}_\varepsilon\setminus B(x_{\varepsilon,b},a)$$

if $d>0$ is small. Applying a comparison principle, we obtain that for some $C,c>0$, independent of small ε and large $b>R$,

$$u_{\varepsilon,b}(x)\le C\exp(-c\mathrm{dist}(x,\partial O^{3\delta}_\varepsilon\cup\{x_{\varepsilon,b}\})),\quad x\in O^{3\delta}_\varepsilon.\tag{2.19}$$

Then, from (2.16), we see that for small $\varepsilon>0$, independent of large $b>R$, $\Big(\int_{O^\delta_\varepsilon\setminus O_\varepsilon}\frac{1}{\varepsilon}u^2\,dx-1\Big)_+=0$; thus

$$\Delta u_{\varepsilon,b}-V_\varepsilon u_{\varepsilon,b}+g_\varepsilon(x,u_{\varepsilon,b})=0\quad\text{in}\quad B(0,b/\varepsilon),\quad x\in O^{3\delta}_\varepsilon.$$

Moreover, it follows from (2.19) that, for some $C, c > 0$,

$$u_\varepsilon(x) \le C\exp(-c/\varepsilon) \quad \text{for} \quad x \in \partial O_\varepsilon.$$

Case 1. Assume that $(A1)$ holds. Then, we define a comparison function

$$\Upsilon_\varepsilon(|x|) = \frac{1}{|\varepsilon x|^{N-2}}\Big(2 - \frac{\log 2}{\log(\frac{2|\varepsilon x|}{R})}\Big).$$

Then, we have

$$D_0 \equiv \min_{x\in\partial O_\varepsilon} \Upsilon_\varepsilon(x) > 0$$

and for $x \in \mathbf{R}^N \setminus O_\varepsilon$,

$$\begin{aligned}&-\Delta\Upsilon_\varepsilon + V_\varepsilon\Upsilon_\varepsilon - \frac{g_\varepsilon(x, u_{\varepsilon,b})}{u_{\varepsilon,b}}\Upsilon_\varepsilon \\ &\ge \frac{\varepsilon^2\log 2}{(\varepsilon|x|)^N}\Big(\frac{(N-2)}{(\log\frac{2\varepsilon|x|}{R})^2} + \frac{2}{(\log\frac{2\varepsilon|x|}{R})^3}\Big) - \frac{\varepsilon^2}{\beta_\varepsilon(|x|)}\Upsilon_\varepsilon.\end{aligned}$$

Thus, taking $\beta_\varepsilon(|x|) = |x|^2\frac{(\log\frac{2\varepsilon|x|}{R})^2}{(\log 2)^2} \ge |x|^2$ for $|x| \ge R/\varepsilon$, we see that for small $\varepsilon > 0$,

$$-\Delta\Upsilon_\varepsilon + V_\varepsilon\Upsilon_\varepsilon - \frac{g_\varepsilon(x, u_{\varepsilon,b})}{u_{\varepsilon,b}}\Upsilon_\varepsilon \ge 0 \quad \text{in} \quad B(0, b/\varepsilon) \setminus O_\varepsilon.$$

Now, we see that $\frac{C}{D_0}\exp(-\varepsilon/c)\Upsilon_\varepsilon - u_{\varepsilon,b} \ge 0$ on $\partial(B(0, b/\varepsilon) \setminus O_\varepsilon)$. Then from the comparison principle, we obtain that for large $b > R$,

$$u_{\varepsilon,b} \le \frac{C}{D_0}\exp(-\varepsilon/c)\Upsilon_\varepsilon \quad \text{on} \quad B(0, b/\varepsilon) \setminus O_\varepsilon. \tag{2.20}$$

Since $\lim_{t\to 0} f(t)/t^\mu = 0$ for some $\mu > N/(N-2)$, it follows that for sufficiently small $\varepsilon > 0$ and large $b > R$,

$$f(u_{\varepsilon,b})/u_{\varepsilon,b} \le \frac{\varepsilon^2}{\beta_\varepsilon(|x|)} \quad \text{on} \quad B(0, b/\varepsilon) \setminus O_\varepsilon.$$

Then, $u_{\varepsilon,b}$ is a solution of

$$\Delta u_{\varepsilon,b} - V_\varepsilon u_{\varepsilon,b} + f(u_{\varepsilon,b}) = 0 \quad \text{in} \quad B(0, b/\varepsilon), \quad u_{\varepsilon,b} = 0 \quad \text{on} \quad \partial B(0, b/\varepsilon).$$

Then, we see that as $b \to \infty$, $u_{\varepsilon,b}$ converges, along a subsequence, to some $u_\varepsilon \in H_\varepsilon$ uniformly in $C(\mathbf{R}^N)$ and weakly in H_ε. Then, u_ε is a solution of the original problem. By the uniform estimates (2.20), we get the required decay estimate for u_ε.

Case 2. Assume that $(A2)$ holds. We take $\beta_\varepsilon(|x|) = |x|^2(1 + |x|)$. Since $\liminf_{|x|\to\infty} V(x)|x|^2 > 0$, we see that $\mathcal{Z} \equiv \{x \in \mathbf{R}^N \mid V(x) = 0\} \subset \mathbf{R}^N \setminus O$ is compact. Then, we see as in **Case 1** that for any large $R_0 > 1$ and small $l > 0$, there exist $C', c' > 0$, independent of small $\varepsilon > 0$ and large $b > 0$ such that $u_{\varepsilon,b}(x) \le C'\exp(-\frac{c'}{\varepsilon})$ for $\delta/\varepsilon \le |x - x_\varepsilon| \le 2R_0/\varepsilon$ and $\operatorname{dist}(\varepsilon x, \mathcal{Z}) \ge l$. We take $R_0 > 0$ so that $V(x) \ge 2\lambda/|x|^2$ for $|x| \ge R_0$. Let ψ be the positive first eigenfunction of $-\Delta$ on $\mathcal{Z}^{2l}$ with Dirichlet boundary condition. Let λ_1 be the corresponding eigenvalue. We normalize ψ so that $\max_{x\in\mathcal{Z}^{2l}} = 1$. Define $\psi_\varepsilon(x) = \psi(\varepsilon x)$. Then we see that for small $\varepsilon > 0$ and large $b > R$,

$$-\Delta\psi_\varepsilon + V_\varepsilon\psi_\varepsilon - \frac{g_\varepsilon(x, u_{\varepsilon,b})}{u_{\varepsilon,b}}\psi_\varepsilon \ge (\varepsilon^2\lambda_1 - \frac{\varepsilon^2}{(1 + R/\varepsilon)^3})\psi_\varepsilon \ge 0 \quad \text{in} \quad (\mathcal{Z}^{2l})_\varepsilon.$$

Thus, by a comparison principle, we deduce that for some $D > 0$,

$$u_{\varepsilon,b}(x) \le D\psi_\varepsilon \exp(-\frac{c}{\varepsilon}) \quad \text{for} \quad x \in (\mathcal{Z}^l)_\varepsilon,$$

which implies that for some $D', d' > 0$, independent of small $\varepsilon > 0$ and large $b > R$, $u_{\varepsilon,b}(x) \le D' \exp(-\frac{d'}{\varepsilon})$ if $\delta/\varepsilon \le |x - x_\varepsilon| \le 2R_0/\varepsilon$. Now we take a comparison function $\Upsilon_\varepsilon(|x|) = |x|^{-\omega_\varepsilon}$ with $\omega_\varepsilon \equiv \frac{(n-2)+\sqrt{(n-2)^2+4\lambda/\varepsilon^2}}{2}$. Then, we deduce from condition (V3) that for small $\varepsilon > 0$,

$$-\Delta\Upsilon_\varepsilon + V_\varepsilon\Upsilon_\varepsilon \ge \Big(\frac{2\lambda}{\varepsilon^2} - \omega_\varepsilon^2 + (n-2)\omega_\varepsilon\Big) r^{-\omega_\varepsilon - 2} = \frac{\lambda}{\varepsilon^2}|x|^{-\omega_\varepsilon-2}, \quad r \ge R_0/\varepsilon.$$

Thus, we see that for small $\varepsilon > 0$ and large $b > 0$,

$$\Big(-\Delta + V_\varepsilon - \frac{g_\varepsilon(x, u_{\varepsilon,b})}{u_{\varepsilon,b}}\Big)\Upsilon_\varepsilon \ge \frac{\lambda|x|^{-\omega_\varepsilon-2}}{\varepsilon^2} - \frac{\varepsilon^2|x|^{-\omega_\varepsilon}}{|x|^2} \ge 0 \ \text{ in } B(0, \frac{b}{\varepsilon}) \setminus B(0, \frac{R_0}{\varepsilon}).$$

Then, as before, we obtain that for some $C, c > 0$, independent of small $\varepsilon > 0$ and large $b > R$,

$$u_{\varepsilon,b} \le C\exp(-\frac{c}{\varepsilon})|\varepsilon x|^{-\omega_\varepsilon} \quad \text{on} \quad B(0, b/\varepsilon) \setminus B(0, R_0/\varepsilon).$$

Since $\lim_{t\to 0} f(t)/t^\mu = 0$ for some $\mu > 1$, we see that for small $\varepsilon > 0$ and large $b > R$, $f(u_{\varepsilon,b})/u_{\varepsilon,b} \le \varepsilon^2/\beta_\varepsilon(|x|)$ for $x \in B(0, b/\varepsilon) \setminus O_\varepsilon$. Then, $u_{\varepsilon,b}$ is a solution of

$$\Delta u_{\varepsilon,b} - V_\varepsilon u_{\varepsilon,b} + f(u_{\varepsilon,b}) = 0 \quad \text{in} \quad B(0, b/\varepsilon), \quad u_{\varepsilon,b} = 0 \quad \text{on} \quad \partial B(0, b/\varepsilon).$$

Then, as in **Case 1**, we get a solution u_ε of original problem (2.16) satisfying the required decay estimate for u_ε.

Case 3. Assume that $(A3)$ holds. We take $\beta_\varepsilon(|x|) = |x|^2 \log|x|$. Then, since $\liminf_{|x|\to\infty} V(x)|x|^2 \log|x| > 0$, by a similar procedure with the proof of the case that (A2) holds, we see that for any large $R_0 > 0$, there exist $C', c' > 0$ such that $u_\varepsilon(x) \le C' \exp(-\frac{c'}{\varepsilon})$ for $\delta/\varepsilon \le |x - x_\varepsilon| \le 2R_0/\varepsilon$. We take $R_0 > 0$ so that for some $h > 0$, $V(x) \ge h/|x|^2 \log|x|$ for $|x| \ge R_0$. Then, for $\alpha > 0$, we define a comparison function

$$\Upsilon_\varepsilon(|x|) = \frac{1}{|x|^{N-2}(\log|x|)^\alpha}.$$

Then, we see that for some $C > 0$,

$$D_\varepsilon \equiv \min_{x\in\partial O_\varepsilon} \Upsilon_\varepsilon(x) > C\varepsilon^{N-1},$$

and for $x \in B(0, b/\varepsilon) \setminus B(0, \frac{R_0}{\varepsilon})$,

$$\begin{aligned}
&\Big(-\Delta\Upsilon_\varepsilon + V_\varepsilon\Upsilon_\varepsilon - \frac{g_\varepsilon(x, u_{\varepsilon,b})}{u_{\varepsilon,b}}\Upsilon_\varepsilon\Big)/\Upsilon_\varepsilon \\
&\ge -\frac{1}{|x|^2\log(|x|)}\Big((N-2)\alpha + \frac{\alpha(\alpha+1)}{\log|x|}\Big) + \frac{h}{|\varepsilon x|^2 \log|\varepsilon x|} - \frac{\varepsilon^2}{\beta_\varepsilon(|x|)}.
\end{aligned}$$

Note that for small $\varepsilon > 0$ and $|x| \ge R_0/\varepsilon$,

$$\frac{1}{|\varepsilon x|^2 \log|\varepsilon x|} = \frac{1}{\varepsilon}\frac{1}{|x|^2\log|x|}\frac{\log|x|}{(\varepsilon|\log\varepsilon| + \varepsilon|\log x|)} \ge \frac{1}{\varepsilon(1+\varepsilon)}\frac{1}{|x|^2\log|x|}.$$

Thus we see that for small $\varepsilon > 0$, independent of large $b > R$,

$$-\Delta\Upsilon_\varepsilon + V_\varepsilon\Upsilon_\varepsilon - \frac{g_\varepsilon(x, u_{\varepsilon,b})}{u_{\varepsilon,b}}\Upsilon_\varepsilon \ge 0 \quad \text{in} \quad B(0, b/\varepsilon) \setminus B(0, R_0/\varepsilon).$$

Thus, we get that for some $C, c > 0$, independent of small $\varepsilon > 0$ and large $b > R$,

$$u_{\varepsilon,b} \leq C \exp(-\frac{c}{\varepsilon}) \frac{1}{|x|^{N-2}(\log |x|)^\alpha} \quad \text{on} \quad B(0, b/\varepsilon) \setminus B(0, R_0/\varepsilon).$$

Then, for some $c, C > 0$, independent of small $\varepsilon > 0$ and large $b > 0$, it follows from (f1-3) that

$$f(u_{\varepsilon,b})/u_{\varepsilon,b} \leq C \exp(-\frac{c}{\varepsilon}) \frac{1}{|x|^2(\log |x|)^{2\alpha/(N-2)}} \quad \text{on} \quad B(0, b/\varepsilon) \setminus B(0, R_0/\varepsilon).$$

Thus, taking $\alpha > (N-2)/2$, we see that for small $\varepsilon > 0$, independent of large $b > R$, $f(u_{\varepsilon,b})/u_{\varepsilon,b} \leq \varepsilon^2/\beta_\varepsilon(|x|)$ if $x \in \mathbf{R}^N \setminus O_\varepsilon$; then $u_{\varepsilon,b}$ is a solution of

$$\Delta u_{\varepsilon,b} - V_\varepsilon u_{\varepsilon,b} + f(u_{\varepsilon,b}) = 0 \quad \text{in} \quad B(0, b/\varepsilon), \quad u_{\varepsilon,b} = 0 \quad \text{on} \quad \partial B(0, b/\varepsilon).$$

Then, as in **Case 1** and **Case 2**, we get a solution u_ε of original problem (2.16) satisfying the required decay estimate for u_ε. □

3. **Proof of Theorem 1.2: nonexistence.** To the contrary, suppose that there exists a positive supersolution u of (1.5). For $N \geq 2$, let $\bar{u}$ be the spherical average of u,

$$\bar{u}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u(x)\, d\sigma,$$

where $B_r = \{x \in \mathbf{R}^N |\ |x| < r\}$ and $d\sigma$ is the standard volume element on ∂B_r. Then, we see from Jensen's inequality that for large $r > 0$,

$$\frac{d^2\bar{u}}{dr^2} + \frac{N-1}{r}\frac{d\bar{u}}{dr} - \bar{W}(r)\bar{u} + \bar{u}^p \leq 0,$$

where $\bar{W}(r) \equiv \max_{|x|=r} W(x)$. For $N = 1$, we use $\bar{u} = u$, and $r = \pm x \in \mathbf{R}$ for $x \in \mathbf{R}^\pm$ respectively. Setting $w(t) \equiv r^m \bar{u}(r), m = 2/(p-1)$ and $t \equiv \log r$, we see that for t large, w satisfies

$$w'' + aw' + (b - \bar{W}(r)r^2)w + w^p \leq 0, \tag{3.1}$$

where $a = N-2-2m$ and $b = m(m-N+2)$. Note that $a < 0$ for $(N-2)p < N+2$, and $b \geq 0$ for $(N-2)p \leq N$.

Case 1. $b \geq 0$ and $\bar{W}r^2 \leq b$ in a neighborhood of ∞.

We consider two exclusive cases. The first is that $w'(T) < 0$ for some T large. The other case is that w is non-decreasing near ∞.

Now, we assume the first case.

Let $B(r) \equiv b - \bar{W}r^2$. Then, we have $B + w^{p-1} \geq 0$ near ∞. Then, integrating (3.1) over $[T, t]$ for T large, we have

$$w'(t) \leq e^{-a(t-T)}w'(T) - e^{-at}\int_T^t (B + w^{p-1})we^{as}\, ds \leq e^{-a(t-T)}w'(T),$$

which implies that w cannot remain positive as $t \to \infty$ because $a < 0$. On the other hand, if w is non-decreasing and bounded near ∞, then there exists $w_\infty > 0$ such that $w(t) \to w_\infty$ as $t \to \infty$. Then, there exists a sequence $\{t_j\}$ entailing $\lim_{j\to\infty} t_j = \infty$ such that $w'(t_j), w''(t_j) \to 0$ as $j \to \infty$, which implies

$$0 < w_\infty^p \leq \limsup_{j\to\infty} \Big(B(\exp(t_j)) + w(t_j)^{p-1}\Big)w(t_j) \leq 0,$$

a contradiction. The remaining possibility is that w is non-decreasing and unbounded near ∞. Setting $X(t) \equiv e^{\frac{a}{2}t}w(t)$, we have

$$X'' + D(t)X \leq 0, \tag{3.2}$$

where

$$D(t) \equiv B(\exp(t)) - \frac{a^2}{4} + w^{p-1} \to \infty$$

as $t \to \infty$. Multiplying both sides of (3.2) by $\sin t$ and integrating by parts over $[2k\pi, (2k+1)\pi]$ with integer $k > 0$, we obtain

$$\int_{2k\pi}^{(2k+1)\pi} (D-1)X \sin t \, dt \leq -X(2k\pi) - X((2k+1)\pi) \leq 0$$

which leads to a contradiction since $D > 1$ on $[2k\pi, (2k+1)\pi]$ for $k > 0$ is sufficiently large.

Case 2. $b = 0$, i.e., $N = (N-2)p$, and for some $\delta > 0$,

$$W(x)|x|^2 \log|x| \leq \frac{(N-2)^2}{2} + \frac{N(N-2)}{4\log|x|} - \frac{\delta}{\log(\log|x|)}$$

in a neighborhood of ∞.

We introduce a comparison function $\varphi(r) := r^{2-N}(\log r)^{-\frac{N-2}{2}}(\log(\log r))^{\beta}$ for $0 < \beta < \frac{\delta}{N-2}$. Then, we see that φ satisfies

$$\varphi'' + \frac{N-1}{r}\varphi' - W\varphi = [\delta - \beta(N-2) + o(1)]\frac{(\log(\log r))^{\beta-1}}{r^N(\log r)^{N/2}}$$

as r tends to ∞. Hence, $\Delta\varphi(r) - W(r)\varphi(r) \geq 0$ for r sufficiently large. Since $\Delta u - Wu \leq 0$, it follows from comparison principle that for some $C, c > 0$, $u(x) \geq C\varphi(r)$ for $r = |x| \geq c$. Therefore, we see that for large $r > 0$,

$$\begin{aligned} B + w^{p-1} &= -\bar{W}r^2 + r^2\bar{u}^{\frac{2}{N-2}} \\ &\geq \left(C^{\frac{2}{N-2}}(\log(\log r))^{\frac{2\beta}{N-2}} - \bar{W}r^2\log r\right)\frac{1}{\log r} \geq 0. \end{aligned}$$

Then, by the preceding argument to the first case, we see that w is nondecreasing near ∞. We also note that for large $r > 0$,

$$\begin{aligned} B + w^{p-1} &= -\bar{W}r^2 + \frac{r^2}{2}\bar{u}^{\frac{2}{N-2}} + \frac{1}{2}w^{p-1} \\ &\geq \left(\frac{1}{2}C^{\frac{2}{N-2}}(\log(\log r))^{\frac{2\beta}{N-2}} - \bar{W}r^2\log r\right)\frac{1}{\log r} + \frac{1}{2}w^{p-1} \\ &\geq \frac{1}{2}w^{p-1}. \end{aligned}$$

Considering the case of either $\lim_{t\to\infty} w(t) < \infty$ or $\lim_{t\to\infty} w(t) = \infty$, we arrive at a contradiction by the same arguments in Case 1. This completes the proof. □

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