Modal analysis of rotor system using modulated coordinates for asymmetric rotor system with anisotropic stator

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ABSTRACT
A new modal analysis method for general rotor systems with time-varying parameters is proposed. The essence of method is to introduce the modulated coordinates to derive the equivalent infinite order time-invariant matrix equation from the finite order time-varying matrix equation. Time-varying parameter equation is inevitable for general rotors, of which rotating and stationary parts both possess asymmetric properties. An extensive modal analysis method is rigorously developed for the infinite order time-invariant system and an approximate, yet effective modal analysis technique is introduced with the reduced order system, in order to derive four types of directional frequency response functions. It is shown theoretically that they can be effectively used for detection of the presence of system anisotropy and asymmetry. A numerical example with a simple rotor model and a flexible asymmetric rotor are also provided to demonstrate the theoretical findings and practicality of the proposed method.

INTRODUCTION
The present study proposes a new modal analysis method, employing the modulated coordinates, for general rotor systems, of which rotating and stationary parts are asymmetric. For asymmetrical rotors with isotropic stators, the periodically time-varying linear differential equation expressed in stationary coordinates can be transformed to the time-invariant linear differential equations expressed in the rotating coordinates. Then the modal analysis becomes essentially the same as the ordinary complex modal analysis method developed for anisotropic rotors, which possess asymmetric properties only in the stator part. However, the asymmetric rotor system with anisotropic stator cannot be transformed to a finite order equation of motion with the time-invariant parameters. It is well known that the concept of directional frequency response functions (dFRFs) is very useful for the vibration analysis of rotating machinery with asymmetric and/or anisotropic properties. In particular, the reverse dFRF has been found to be a sensitive indicator to presence of anisotropic or asymmetric properties. However, the usefulness of dFRFs has not been fully proven in the presence of both the asymmetric and anisotropic properties in the rotor system.

Unlike the asymmetric rotor system with isotropic stator, the general rotor cannot be transformed to a finite order equation of motion with the time-invariant parameters, regardless of the coordinate system taken in the formulation. Thus, there requires a method to deal with the equation of motion with time-varying parameters. To solve such an equation of motion with time-varying parameters, the harmonic balance method (HBM) is often adopted, which assumes the solution to be in a form of Fourier series [7]. Although the method is very simple and effective in calculating the natural frequencies, this approach has been rarely used for obtaining complete modal solutions and further responses. A direct
method has been introduced to perform a modal analysis of rotor systems with periodically time-varying parameters by employing time-varying eigenvectors but confined to real coordinate systems. This paper introduces a generalized theory of complex modal analysis method using modulated complex stationary coordinates for asymmetric rotor systems with anisotropic stators. The method introduces modulated complex stationary coordinates to derive an equivalent, infinite-order time-invariant equation of motion. Then, the characteristics of eigenvalues and eigenvectors are theoretically investigated thoroughly by using the equivalent time-invariant equation of motion. Two analytical and numerical examples are treated to demonstrate the effectiveness of the proposed method.

1. ANALYTICAL MODAL ANALYSIS

1.1 Equation of motion in the modulated coordinates

The equation of motion of an asymmetric rotor system with anisotropic stator part can be written, using the complex stationary coordinates, as [1]

\[
M_p \ddot{p}(t) + M_b \ddot{p}(t) + M_r \ddot{p}(t)e^{2j\Omega t} + C_p \dot{p}(t) + C_b \dot{p}(t) + C_r \dot{p}(t)e^{2j\Omega t} + K_p p(t) + K_b p(t) + K_r p(t)e^{2j\Omega t} = g(t),
\]

where \(M_i, C_i, K_i\) denote the complex valued \(N \times N\) generalized mass, damping and stiffness matrices, respectively; the subscripts \(f, b\) and \(r\) refer to the mean value, and, the deviatoric values for anisotropy (stationary asymmetry) and asymmetry (rotating asymmetry), respectively; \(p(t) = y(t) + jz(t)\) and \(g(t) = f_y(t) + jf_z(t)\) are the \(N \times 1\) complex response and input vectors, respectively; \(y(t)\) and \(z(t)\) are the real valued response vectors, and, \(f_y(t)\) and \(f_z(t)\) are the real valued input vectors, in the direction of \(Y\) and \(Z\) in the stationary coordinates, respectively; \(\Omega\) is the rotational speed of the shaft; \(N\) is the dimension of the complex coordinate vector; \(j\) is the imaginary number; the bar indicates the complex conjugate. Note here that the time-varying parameters associated with the harmonic frequency of twice the rotational speed appear due to the asymmetry in the rotor part.

Equation (1) can be transformed to an infinite order matrix equation, introducing the modulated complex coordinate and force vectors, \(\tilde{p}_n\) and \(\tilde{g}_n\), where the modulation index, \(n\), is an arbitrary integer, defined as

\[
\tilde{p}_n(t) = p(t)e^{2j\Omega t}, \quad \tilde{g}_n(t) = g(t)e^{2j\Omega t}
\]

We can rewrite equation (1) as

\[
M \ddot{p}(t) + C \dot{p}(t) + K p(t) = g(t)
\]

where

\[
M = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \tilde{M}_f & \tilde{M}_b & 0 & 0 & 0 & 0 \\
\cdots & M_b & M_f & M_r & 0 & 0 & 0 \\
\cdots & 0 & \tilde{M}_r & \tilde{M}_f & \tilde{M}_b & 0 & 0 \\
\cdots & 0 & 0 & M_b & M_f & M_r & 0 \\
\cdots & 0 & 0 & 0 & \tilde{M}_b & \tilde{M}_f & \tilde{M}_r \\
\cdots & 0 & 0 & 0 & 0 & M_b & M_f \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \tilde{C}_{f,1} & \tilde{C}_{b,1} & 0 & 0 & 0 & 0 \\
\cdots & C_{b,1} & C_{f,1} & C_{r,0} & 0 & 0 & 0 \\
\cdots & 0 & \tilde{C}_{r,0} & \tilde{C}_{f,0} & \tilde{C}_{b,0} & 0 & 0 \\
\cdots & 0 & 0 & C_{b,0} & C_{f,0} & C_{r,1} & 0 \\
\cdots & 0 & 0 & 0 & \tilde{C}_{r,1} & \tilde{C}_{b,1} & \tilde{C}_{f,1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix},
\]
Note that the differential equation (2) with periodically time-varying parameters is transformed into equation (4) with time-invariant parameters, at the expense of introducing the coordinate vector of infinite dimension.

1.2 Modal analysis

The equation of motion can be written, in the state space, as

\[ \mathbf{A} \mathbf{w}(t) = \mathbf{B} \mathbf{w}(t) + \mathbf{F}(t) \]  

where,

\[ \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & -\mathbf{K} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \end{bmatrix}, \quad \mathbf{w}(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{g}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{F}(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \]

Here, \( \mathbf{0} \) represents the zero vector and matrix of infinite dimension. Assuming the solution form of \( \mathbf{w} = \mathbf{r} e^{i \omega t} \), we can obtain the eigenvalue and its adjoint problems associated with equation (5) as

\[ \lambda^{(i)}_{(r,m)} \mathbf{r}^{(i)}_{(r,m)} = \mathbf{B}^{(i)}_{(r,m)} \mathbf{r}^{(i)}_{(r,m)} \quad \text{and} \quad \mathbf{X}^{T} \mathbf{T}^{T} = \mathbf{B}^{T} \mathbf{T}^{T}, \]  

or the equivalent latent value problem as

\[ \mathbf{D} \lambda^{(i)}_{(r,m)} \mathbf{u}^{(r,m)}_{(r,m)} = \mathbf{0} \quad \text{and} \quad \mathbf{v}^{T} \mathbf{D} \lambda^{(i)}_{(r,m)} = \mathbf{0}, \]  

\[ r = \pm 1, \pm 2, \ldots, m = 0, \pm 1, \pm 2, \ldots, \ i = B, F, \]

where the lambda matrix of degree two is given by

\[ \mathbf{D} \lambda = \lambda^{2} \mathbf{M} + \lambda \mathbf{C} + \mathbf{K} \]

and the right and left eigenvectors, and the latent vectors take the form of

\[ \mathbf{r} = \{\lambda \mathbf{u}, \mathbf{u}\}^{T}, \quad \mathbf{f} = \{\lambda \mathbf{v}, \mathbf{v}\}^{T}, \quad \mathbf{u}_{s} = \{\cdots, \mathbf{u}_{s+1}, \mathbf{u}_{s}, \mathbf{u}_{s-1}, \mathbf{u}_{s}, \mathbf{u}_{s+1}, \cdots\}^{T}, \]

\[ \mathbf{v}_{s} = \{\cdots, \mathbf{v}_{s+1}, \mathbf{v}_{s}, \mathbf{v}_{s-1}, \mathbf{v}_{s}, \mathbf{v}_{s+1}, \cdots\}^{T}. \]

Here, the pair of eigenvalues, equal in subscript value but different in sign of subscript, are dependent upon each other; they can be complex conjugate pairs with or without a shift parameter \( \pm 2n \Omega \), as will be shown later. The superscripts \( B \) and \( F \) implicitly refer to the backward and forward modes, respectively, and the subscript \( r(m) \) refers to the \( r \)-th eigen (latent) solution in the \( m \)-th cluster, as will be shown later (refer to equation (14a)). The eigenvalues and eigenvectors, obtained from equation (6), are normalized so as to satisfy the bi-orthonormality conditions given by

\[ \mathbf{L}^{T} \mathbf{A} \mathbf{r}^{(i)}_{(r,m)} = \delta^{(i)}_{(r,m)} \mathbf{r}^{(i)}_{(r,m)} \quad \text{and} \quad \mathbf{L}^{T} \mathbf{B}^{(i)}_{(r,m)} = \lambda^{k} \mathbf{r}^{(i)}_{(r,m)} \lambda_{(r,m)}^{k} \lambda_{(r,m)}^{k}, \]

\[ r, s = \pm 1, \pm 2, \ldots, N, \quad m, \ell = 0, \pm 1, \pm 2, \ldots, \ i = B, F. \]

From equations (4) and (7), we obtain the relation

\[ \mathbf{p}_{s}(t) = \mathbf{u}_{s}^{(i)}(t) e^{\lambda_{(r,m)}^{(i)} t}, \quad \bar{\mathbf{p}}_{s}(t) = \bar{\mathbf{u}}_{s}^{(i)}(t) e^{\lambda_{(r,m)}^{(i)} t}, \]

implying that
Recall that the subscript \( n \) denotes the order of the modulated coordinates. Since it holds \( i = k, s = -r, \ell = m \), from the above relations, we obtain, for every pair of latent vectors \((\tilde{u}_s^T, u_n^T)^T\) associated with the eigenvalue \( \lambda_{(n)}^r \),
\[
\tilde{u}_{(r,n)}^i = \tilde{u}_{(r,m)\ast}^i, \quad \lambda_{(n)}^r = \overline{\lambda_{(-r)}^r}.
\] (8b)

On the other hand, since it also holds
\[
p_{(s)}(t) = p_{(s)}(t)e^{j2\Omega t} = u_{(s)\ast}^i e^{j2\Omega t} = u_{(s)\ast}^i e^{j2\Omega t},
\] (10a)
we obtain the circulation formula between the eigenvalues associated with the latent vectors \( u_n \) and \( u_m \) as
\[
\lambda_{(n)}^r = \lambda_{(-r)}^r - j2n\Omega, \quad \lambda_{(m)}^r = \overline{\lambda_{(-m)}^m} = \lambda_{(-r)}^r + j2n\Omega = \overline{\lambda_{(-n)}^n}
\] (10b)
which means that, unless \( A \) is a singular matrix (\( M_r \) and thus \( A \) seldom become singular), the shifted eigenvalue by \(-j2n\Omega\) and its complex conjugate also become eigenvalues for any integer value of \( n \). Note that such relation holds only between the same directional modes, forward or backward. For example, if \( \lambda_{(0)}^{F} \) is an eigenvalue, \( \lambda_{(-0)}^{F} - j2n\Omega, \lambda_{(0)}^{F} - j2n\Omega \) also become eigenvalues.

In this case, since \( \lambda_{(0)}^{F} = \lambda_{(0)}^{R} \) is an eigenvalue, we can derive the relations such as \( \overline{\lambda_{(0)}^{F}} = \lambda_{(0)}^{R} \) \( \lambda_{(-0)}^{F} - j2n\Omega, \lambda_{(0)}^{F} - j2n\Omega \) and \( \lambda_{(0)}^{F} = \lambda_{(0)}^{R} \) \( \lambda_{(-0)}^{F} - j2n\Omega + j2n\Omega = \lambda_{(0)}^{R} \) \( \lambda_{(-0)}^{F} + j2n\Omega \) also become eigenvalues.

1.3 Directional frequency response functions

Directional frequency response matrices (dFRMs) can be represented, in the form of modal summation using the eigensolutions, as:
\[
H(j\omega) = \sum_{n=-\infty}^{\infty} \sum_{j\in\mathbb{R}} \sum_{\ell=-\infty}^{\infty} \left[ u_{(r,n)}^j \overline{\tilde{u}_{(r,m)}^j} \right]^T \frac{1}{j\omega - \lambda_{(n)}^r},
\]
(11a)
where the prime notation in the summation implies the exclusion of \( r = 0 \). Using equation (12), we introduce, among others, four dFRMs, that are important in characterizing the system asymmetry and anisotropy, as

\[
H_{m}(j\omega) = H_{m,p_0}(j\omega) = \sum_{i=B,F} \sum_{n=-\infty}^{\infty} \sum_{r=1}^{N} \frac{1}{j\omega - \lambda^r_{m}(n)} \begin{bmatrix}
\cdots \\
\widetilde{u}_0 \widetilde{V}_0 \\
\widetilde{u}_1 \widetilde{V}_1 \\
\widetilde{u}_2 \widetilde{V}_2 \\
\cdots
\end{bmatrix}
\]

where

\[
G_{m}(j\omega) = \mathcal{G}\{j(\omega - 2n\Omega)\}, \quad \hat{G}_{m}(j\omega) = \hat{\mathcal{G}}\{j(\omega + 2n\Omega)\}.
\]

Here \( \mathcal{G}(j\omega) \) and \( \hat{\mathcal{G}}(j\omega) \) are the Fourier transforms of \( g(t) \) and \( \overline{g}(t) \), respectively. In equation (13), \( H_{m,p_0}(j\omega) \) and \( H_{m,p_0}(j\omega) \) are referred to as the normal and reverse directional frequency response matrix of modulation index \( n \), respectively, which are associated with the shifted input by \( 2n\Omega \) in the frequency domain.

3. MODAL ANALYSIS FOR SIMPLE ASYMMETRIC ROTOR SYSTEM

3.1 Equation of motion

This section illustrates the modal analysis procedure using modulated coordinates with a simple general rotor system shown in Fig. 1. The equation of motion for the system can be written as

\[
\dot{p}(t) + (2\zeta - j\omega \Delta) \dot{p}(t) + p(t) + \delta e^{2n\Omega} \overline{p}(t) + \Delta \overline{p}(t) = g(t)
\]

where \( \Delta \) and \( \delta \) represent the degree of anisotropy and asymmetry, respectively. The reduced equation of motion, including only the coordinates of modulation indices 0, -1 and +1, may be written as

\[
M(p^{(+1)}(t) + C^{(+1)} \dot{p}^{(+1)}(t) + K^{(+1)} p^{(+1)}(t) = g^{(+1)}(t)
\]

where

\[
p^{(+1)} = \begin{bmatrix}
\overline{p}_{0}\eta(t) \\
p_{0}\eta(t) \\
\overline{p}_{1}\eta(t) \\
p_{1}\eta(t) \\
\overline{p}_{2}\eta(t) \\
p_{2}\eta(t)
\end{bmatrix}^T
\]

\[
g^{(+1)} = \begin{bmatrix}
g_{0}\eta(t) \\
g_{0}\eta(t) \\
g_{1}\eta(t) \\
g_{1}\eta(t) \\
g_{2}\eta(t) \\
g_{2}\eta(t)
\end{bmatrix}^T
\]

The twelve eigenvalues and the corresponding latent vectors are related to each other, as given in equation (11), i.e., for \( i = B, F \),
The directional frequency responses can be expressed as

\[
H_\theta(\omega) = \sum_{n=0}^{\infty} \left[ a_{n\theta}^0 \nabla_{n\theta,0} + a_{n\theta}^1 \nabla_{n\theta,1} + b_{n\theta}^0 \nabla_{n\theta,0} + b_{n\theta}^1 \nabla_{n\theta,1} \right]
\]

3.2 Directional frequency response functions

The directional frequency responses can be expressed as

\[
\begin{align*}
L^{(i)}_{(i)} & = L^{(i)}_{(i)} - j2\Omega \\
L^{(i)}_{(i-1)} & = L^{(i)}_{(i)} + j2\Omega \\
L^{(i)}_{(i-1)} & = L^{(i)}_{(i)} - j2\Omega \\
\end{align*}
\]

(16)

\[
\begin{align*}
\frac{\hat{H}}{\hat{H}}_{(i)}(\omega) & = \sum_{n=0}^{\infty} \left[ a_{n\theta}^0 \nabla_{n\theta,0} + a_{n\theta}^1 \nabla_{n\theta,1} + b_{n\theta}^0 \nabla_{n\theta,0} + b_{n\theta}^1 \nabla_{n\theta,1} \right]
\end{align*}
\]

(17a)

\[
\begin{align*}
\frac{\hat{H}}{\hat{H}}_{(i)}(\omega) & = \sum_{n=0}^{\infty} \left[ a_{n\theta}^0 \nabla_{n\theta,0} + a_{n\theta}^1 \nabla_{n\theta,1} + b_{n\theta}^0 \nabla_{n\theta,0} + b_{n\theta}^1 \nabla_{n\theta,1} \right]
\end{align*}
\]

(17b)

\[
\begin{align*}
\frac{\hat{H}}{\hat{H}}_{(i)}(\omega) & = \sum_{n=0}^{\infty} \left[ a_{n\theta}^0 \nabla_{n\theta,0} + a_{n\theta}^1 \nabla_{n\theta,1} + b_{n\theta}^0 \nabla_{n\theta,0} + b_{n\theta}^1 \nabla_{n\theta,1} \right]
\end{align*}
\]

(17c)
Equation (17) represents four kinds of dFRFs: n- and r-dFRFs of modulation indices 0 and -1, respectively [3]. Note that the modes, whose residues are order of one (less than 1) in the n-dFRF of modulation index 0, are referred to as the ‘strong (weak) modes.’ The weak modes tend to vanish as the degree of anisotropy and asymmetry diminishes. The residues of the strong modes are order of 1, $\Delta$, $\delta$, and $\Delta \delta$ for $H_{\mu}(j\omega)$, $H_{\mu}(j\omega)$, $H_{\mu}(j\omega)$, and $H_{\mu}(j\omega)$, respectively. Thus it can be concluded that: $H_{\mu}(j\omega)$, that is the normal dFRF of modulation index 0, is useful to identify the strong and weak modes; $H_{\mu}(j\omega)$ that is the reverse dFRF of modulation index 0, is a good indicator of degree of anisotropy, irrespective of presence of system asymmetry; $H_{\mu}(j\omega)$, that is the reverse dFRF of modulation index -1, is a good indicator of degree of asymmetry, irrespective of presence of system anisotropy; $H_{\mu}(j\omega)$, that is the normal dFRF of modulation index -1, is very sensitive to the coupled effect of system anisotropy and asymmetry. To investigate the nature of dFRFs in symmetric rotor system with anisotropic stator, a numerical simulation is carried out for the parameters given in Tables 1 and 2 for a simple rotor as illustrated in Fig. 1. Figure 2 shows the convergence of natural frequencies due to increase of reduced order of system matrices. Figures 3 to 6 show four types of dFRFs of the rotor with the degree of anisotropy and asymmetry varied. Note from the n-dFRF of modulation index 0 shown in Fig.3, the strong modes (1F, 1B), which are related with the associated isotropic system, are clearly observed, but the weak modes, which exist due to the deviatoric nature from system symmetry (isotropy), are hardly observed. The n-dFRF of modulation index 0 is not sensitive to the change in degree of anisotropy and asymmetry. On the other hand, the r-dFRF of modulation index 0 shown in Fig.4, is very sensitive to degree of anisotropy but insensitive to the asymmetry. The split of neighbouring peaks in the r-dFRF of modulation index 0 is mainly due to the gyroscopic effect. Figure 5 shows that the r-dFRF of modulation index -1 is very sensitive to degree of asymmetry $\delta$, but robust to the change of anisotropy. Note here that the r-dFRFs of modulation indices 0 and -1 are almost decoupled in the sense that the r-dFRF of modulation index 0 (-1) remains almost unchanged due to the degree of asymmetry (anisotropy). On the other hand, as shown in Fig.6, the n-dFRF of modulation index -1 reflects the coupled effect between anisotropy and asymmetry, although the magnitude is only order of the anisotropy times the asymmetry, i.e. $\sim O(\Delta \delta)$.

4. NUMERICAL MODAL ANALYSIS FOR FLEXIBLE ASYMMETRICAL ROTOR

To demonstrate the effectiveness of the proposed modal analysis method, a numerical example is treated, in which a flexible asymmetric rotor supported by anisotropic bearings with an open crack [15] is investigated. The finite element model of the rotor shown in Fig. 7 is analysed. The material and geometrical properties are listed in Table 3. The model consists of twenty-six Rayleigh beam elements, two rigid disks, and two anisotropic bearings. We assume there exists a crack, which causes bending stiffness asymmetry in the shaft. Throughout the simulations, an open crack is assumed to develop at node #12. Figure 8 is the whirl speed chart of the system, calculated from the reduced order model including the modulation indices 0, -1 and +1, for the crack depth to shaft diameter ratio $a/D = 0.48$. The whirl speed chart clearly indicates presence of the strong and weak modes. In this example, the bearing anisotropy does not significantly affect the dynamic behaviour of the system because the bearings are far more rigid than that of the shaft in the low frequency region. Unstable regions exist in the vicinity of some neighbouring critical speeds. As an illustration, four types of dFRFs, for the
cracked flexible rotor with the anisotropy effect of 23%, calculated by the modal expansion formula are displayed in Figs. 9 (a)–(d) for n-dFRFs and r-dFRFs with the degree of asymmetry varied from 10 to 50%.

Figure 9 (a) shows the n-dFRFs of index 0, where the peaks related to strong modes, $\lambda_{0(0)}^F$, $\lambda_{0(1)}^F$, and $\lambda_{2(0)}^F$, clearly appear, but other weak modes are hardly detectable. Figure 9 (c) shows the r-dFRFs of index -1. Here the modal peaks associated with the eigenvalues, $\lambda_{0(0)}^R$, $\lambda_{1(1)}^R = \lambda_{0(0)}^R + j2\Omega$, $\lambda_{1(1)}^R = \lambda_{0(0)}^R + j2\Omega$, $\lambda_{2(0)}^R = \lambda_{0(0)}^R + j2\Omega$, are clearly observed and the magnitude of r-dFRFs of index –1 is highly sensitive to the degree of asymmetry due to crack development on the shaft. So it can be used as a sensitive indicator for severity of crack growth in the shaft. Figure 9 (b) shows the r-dFRFs of index 0 which are an sensitive indicator for detection of anisotropy, the modal peaks related to natural frequencies of $\lambda_{0(0)}^R$, $\lambda_{1(1)}^R$, $\lambda_{2(0)}^R$, are clearly observed in the r-dFRFs of index 0. Figure 9 (d) shows the coupled effect of asymmetry and anisotropy of the rotor system. If there exists no anisotropy, the dFRFs shown in Figs. 9 (c) and (d) vanish.

5. CONCLUDING REMARKS
This paper presents a generalized modal analysis method for asymmetric rotor systems with anisotropic stators employing the modulated coordinates. An equivalent, infinite order time-invariant equation of motion in the modulated complex stationary coordinates is derived. The proposed method provides a complete, infinite order modal solution in the stationary coordinates for asymmetrical rotor systems with anisotropic stators. The characteristics of eigenvalues and eigenvectors are theoretically investigated by using the equivalent, infinite order time-invariant equation of motion and then four types of dFRFs are derived for practical use. It is found that the normal dFRF of modulation index 0 reflects the system symmetric properties, clearly indicating the presence of the strong and weak modes. On the other hand, the reverse dFRF of modulation index –1 (0) can be used as a good indicator for detection of the asymmetry (anisotropy), irrespective of the anisotropy (asymmetry). The normal dFRF of modulation index –1 is found to be sensitive to coupling of the asymmetry and anisotropy. It tends to vanish when either the asymmetry or anisotropy of the rotor system is not present. Finally, the analytical findings are demonstrated with a simple analysis rotor model and a flexible open cracked rotor model supported by anisotropic bearings.

ACKNOWLEDGEMENT
This work has been financially supported by Agency for Defense Development (TECD-413-001115).

REFERENCES
Table 1 Numerical data for the simple analysis rotor model

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<th>Parameter</th>
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Table 2 Simulation cases with varying degrees of anisotropy and asymmetry

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<thead>
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<td>δ (asymmetry)</td>
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<tr>
<td></td>
<td>Δ (anisotropy)</td>
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<td>4</td>
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Table 3 Specifications of the numerical model

Mesh Data
- # of elements = 26
- # of disks = 2
- # of bearings = 2

Shaft
- Length = 51 cm
- Diameter = 1.2 cm
- Density = 7806 kg/m³
- Young’s modulus = 2.08 x 10¹¹ N/m²

Disk
- Location, m
- Mass, kg
- Pol. Inertia, kg m²
- Dia. Inertia, kg m²
- Location, m
- Mass, kg
- Pol. Inertia, kg m²
- Dia. Inertia, kg m²
- 0.21
- 1.236
- 1.2 x 10⁻³
- 6.8 x 10⁻⁴
- 0.476
- 0.857
- 0.9 x 10⁻³
- 3.5 x 10⁻⁴

Bearings
- Node Number
- Stiffness, GN/m
- Damping, kN s/m
- 4, 21
- kᵧᵧ = 0.5, kzz = 0.8
- cᵧᵧ = czz = 4.5
- (identical)
- kᵧᵧ = kzz = 0
- cᵧᵧ = czz = 0

Crack
- Node = 12
- Crack depth (a/D) = 48%
Fig. 5 Comparison of r-dFRFs, $H_{2p}(j\omega)$, for detection of asymmetry.

Fig. 6 Comparison of n-dFRFs of modulation index -1, $H_{2p}(j\omega)$.

Fig. 7 Flexible rotor configuration.

Fig. 8 Whirl speed chart for the flexible asymmetrical rotor with anisotropic stator.

Fig. 9 (a) n-dFRFs and (b) r-dFRFs of order ‘0’, and, (c) r-dFRFs and (d) n-dFRFs of order ‘-1’ of the flexible rotor, crack depth ratio 10 ~ 50 %, sensor node= 13; exciter node = 25; crack node = 12, at 600 rpm with 23% anisotropy effect. (modal damping ratio 0.02 imposed solely for plotting convenience).