Transitive cycle factorizations and prime parking functions*

Dongsu Kim†
Department of Mathematics
KAIST, Daejeon 305-701, Korea
dskim@math.kaist.ac.kr

and

Seunghyun Seo
Department of Mathematics
KAIST, Daejeon 305-701, Korea
shseo@math.kaist.ac.kr

June 25, 2003

Abstract

Minimal transitive cycle factorizations and parking functions are related very closely. Using the correspondence between them, we find a bijection between minimal transitive factorizations of permutations of type \((1, n - 1)\) and prime parking functions of length \(n\).

Keywords: Minimal transitive factorization, parking function, noncrossing partition

1 Introduction

For any natural number \(n\), let \(S_n\) denote the group of all permutations of \([n] = \{1, 2, \ldots, n\}\). A cycle of length \(n\) can be expressed as a product of \(n - 1\) transpositions. The total number of different such expressions is \(n^{n-2}\). In general, we may count different ways to form a given permutation as a product of transpositions with some interesting conditions. For example, we may require that the transpositions in the product generate \(S_n\) and the minimal number of transpositions are used. This case has been considered in various literature [1, 5, 8, 16]. Clearly the count depends only on the cycle type of the permutation.

An integer partition of \(n\) is a weakly increasing sequence of positive integers, which sums to \(n\). Given an integer partition \(\lambda = (\lambda_1, \ldots, \lambda_l)\) of \(n\), let \(\lambda^\circ\) denote the permutation \((1 2 \cdots \lambda_1)(\lambda_1 + 1 \cdots \lambda_1 + \lambda_2)\cdots(n - \lambda_l + 1 \cdots n)\) of \([n]\) in the cycle notation. The cycle type of \(\lambda^\circ\) is \(\lambda\). Let \(\mathcal{F}_\lambda\) be the set of all \(m\)-tuples of transpositions \((\sigma_1, \ldots, \sigma_m)\) such that

(f1) \(\sigma_1 \cdots \sigma_m = \lambda^\circ\),

(f2) \(\{\sigma_1, \ldots, \sigma_m\}\) generates \(S_n\).

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†Corresponding author
(3) \( m = n + l - 2 \), i.e., \( m \) is minimal subject to (f1), (f2).

Elements of \( \mathcal{F}_\lambda \) are called minimal transitive factorizations of \( \lambda^2 \) (or simply \( \lambda \)). These factorizations are connected with branched covers of the sphere, that is treated in [1, 3, 7, 11].

Goulden and Jackson [8] proved that

\[
|\mathcal{F}_\lambda| = (n + l - 2)! n^{l-3} \prod_{i=1}^l \frac{\lambda_i^\lambda_i}{(\lambda_i - 1)!},
\]

which was originally suggested by Hurwitz [11]. They proved it for arbitrary \( \lambda \), but the proof was not combinatorial. Recently, Bousquet-Mélou and Schaeffer [3] enumerated the transitive factorizations by interpreting them as planar constellations and Eulerian trees, and obtained the formula (1) by using the principle of Inclusion–Exclusion. In case \( \lambda = (n) \), there have been several combinatorial proofs [2, 6, 9, 10]. For any other \( \lambda \), there has been no simple combinatorial proof yet. When \( \lambda = (1, n-1) \), the expression (1) reduces to

\[
|\mathcal{F}_{(1,n-1)}| = (n-1)^n.
\]

The aim of this paper is to give a combinatorial proof of (2), using prime parking functions.

2 Preliminaries

Let \( \pi \) be a permutation and \( \sigma = (wz) \), with \( w < z \), a transposition in \( S_n \). If \( w \) and \( z \) appear in the same cycle in \( \pi \), then this cycle is cut into two cycles in the product \( \sigma \pi \); otherwise, the cycles containing \( w \) and \( z \) are joined into one cycle in \( \sigma \pi \). We call \( \sigma \) a cut of \( \pi \) in the former case and a join in the latter. For an element \( (\sigma_1, \ldots, \sigma_m) \) in \( \mathcal{F}_\lambda \), where \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is an integer partition of \( n \), \( \sigma_i \) is called a cut (resp. a join), if \( \sigma_i \) is a cut (resp. join) of \( \sigma_{i+1} \cdots \sigma_m \). Note that, from Goulden and Jackson [8, Proposition 2.1], there are exactly \( l - 1 \) cuts and \( n - 1 \) joins in these factorizations.

When \( \lambda = (n) \), a minimal transitive factorization has \( n - 1 \) transpositions which generate \( S_n \).

A circle chord diagram, first introduced in [10], is a circle drawn on the plane with \( n \) points on it, labelled \( 1, \ldots, n \) clockwise, and with \( n - 1 \) chords on these \( n \) points, numbered \( 1, \ldots, n - 1 \) distinctly. There is a natural injection \( \Gamma \) from factorizations to circle chord diagrams: \( \Gamma \) maps a factorization \( (\sigma_1, \ldots, \sigma_{n-1}) \in \mathcal{F}_{(n)} \), where \( \sigma_i = (w_i z_i) \), to the circle chord diagram in which the chords joining points \( w_i \) and \( z_i \) are numbered \( i \), for \( i = 1, \ldots, n - 1 \). The circle chord diagrams corresponding to the factorizations are characterized below.

**Proposition 1** The following three properties (c1), (c2), (c3) completely characterize the image set of \( \Gamma \), \( \Gamma(\mathcal{F}_{(n)}) \).

(c1) The chords form a tree on \( [n] \).

(c2) The chords meet only at the end points.

(c3) The numbers on the chords of the boundary of any region increase clockwise, starting immediately after the unique arc.

**Proof.** See Theorem 2.2 in [10].
Let $C_n$ be the set of all circle chord diagrams which satisfy the above conditions (c1), (c2) and (c3). Clearly, $\Gamma$ is a bijection from $F_n$ onto $C_n$. For example, the circle chord diagram illustrated in Figure 1 is $\Gamma(F)$, with

$$F = ((2,3), (4,5), (3,6), (3,5), (1,6), (6,8), (8,9), (6,7)).$$

Note that there are 9 regions and each region satisfies (c3) in the proposition.

Using the planarity of circle chord diagrams, Goulden and Yong [10] found a bijection between $C_n$ and the set $T_n$ of all labelled trees on $[n]$.

A partition of $[n]$ is a family of nonempty, pairwise disjoint sets $B_1, \ldots, B_k$, called blocks, whose union is $[n]$. The collection of all partitions of $[n]$ will be denoted by $\Pi_n$. A partition of $[n]$ is called noncrossing when it has the property that for any four elements $a < b < c < d$, if $a$ and $c$ are in the same block and $b$ is in another, then $b$ and $d$ are not in the same block. The collection of all noncrossing partitions of $[n]$ will be denoted by $NC_n$. The collection $\Pi_n$ can be viewed as a partially ordered set (poset) with the following order (called refinement order): for $\mu, \nu \in \Pi_n$, we have $\mu \leq \nu$ if each block of $\mu$ is contained in a block of $\nu$. The set $NC_n$ is also regarded as a poset with the refinement order. The Hasse diagram of $NC_4$ is shown in Figure 2. In fact, the poset $NC_n$ has many nice properties. See [13] for extensive studies of noncrossing partitions.

A parking function of length $n$ is a sequence $(x_1, \ldots, x_n)$ of positive integers such that the increasing rearrangement $y_1 \leq \cdots \leq y_n$ of $x_1, \ldots, x_n$ satisfies $y_i \leq i$ for all $i$. Let $P_n$ denote the set of all parking functions of length $n$. It is well known that $|P_{n-1}| = n^{n-2}$. For example, the set $P_3$ is as follows: (An array $abc$ means $(a,b,c)$.)

$$P_3 = \{111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213, 221, 231, 311, 312, 321\}$$

Parking functions are related to labelled trees, lattice paths, noncrossing partitions and hyperplane arrangements. For further information see [13] and [14, pp. 94]. Combining the results of Goulden and Yong [10] and Stanley [15, Theorem 3.1], we can define a combinatorial bijection $\Phi : F_n \to P_{n-1}$ by

$$\Phi((w_1 z_1), \ldots, (w_{n-1} z_{n-1})) = (w_1, \ldots, w_{n-1}).$$

We will use $\Phi$ to establish our main result later. Since the mapping $\Phi$ is not shown explicitly in [10, 15], we present an explicit description of $\Phi$ in the next section.
3 Bijection \( \Phi \)

Let \( \mathcal{M}_n \) be the set of all maximal chains in \( \mathcal{NC}_n \). We define a mapping \( \Sigma : C_n \rightarrow \mathcal{M}_n \) through the following process. Let \( C \) be a circle chord diagram in \( C_n \). Then \( C = \Gamma( (w_1 z_1), \ldots, (w_{n-1} z_{n-1}) ) \) for some factorization \( ( (w_1 z_1), \ldots, (w_{n-1} z_{n-1}) ) \in \mathcal{F}(n) \). Let \( m = [\mu_0 < \mu_1 < \cdots < \mu_{n-1}] \) be a maximal chain in \( \mathcal{M}_n \) such that \( \mu_0 = 0, \mu_{n-1} = 1 \), and while \( i \) changes from \( n-1 \) to \( 1 \), \( \mu_i \) is refined to \( \mu_{i-1} \) by ‘splitting’ the blocks in \( \mu_i \) with \((a, b)\) in \( \mathcal{NC}_n \). We define the base, middle, left, and right and left-right sets of \((a, b)\) as follows (Figure 3):

\[
B(\mu, \nu) = \tilde{B} \cap [w_i + 1, z_i] \quad \text{or} \quad B = \tilde{B} \cap (\{n\} \setminus [w_i + 1, z_i]),
\]

where \([a, b]\) denotes the set \(\{a, a+1, \ldots, b\}\). Since \( C \) satisfies (c2), the chords \( (w_1 z_1), \ldots, (w_{n-1} z_{n-1}) \) do not cross in \( C \), and so each \( \mu_i \) is an element of \( \mathcal{NC}_n \) with \( bk(\mu_{i-1}) \leq bk(\mu_i) + 1 \), where \( bk(\mu) \) is the number of blocks in \( \mu \). Moreover, \( w_i \) and \( z_i \) are in the same block in \( \mu_i \), which implies \( bk(\mu_{i-1}) \geq bk(\mu_i) + 1 \); if they are not in the same block, then for some \( j > i \), the chord \( (w_j z_j) \) crosses or coincides with \( (w_i z_i) \), contradicting to the condition (c2). Thus \( \mu_i \) covers \( \mu_{i-1} \) for each \( i \) and \( m \) is really in \( \mathcal{M}_n \). Set \( \Sigma(C) = m \).

Now let us find the inverse of \( \Sigma \). Suppose that \( \mu, \nu \in \mathcal{NC}_n \), where \( \nu \) covers \( \mu \). Then there exist \( A \in \nu \) and \( A', A'' \in \mu \) satisfying \( A = A' \cup A'' \), with \( \min A' > \min A'' \). We define the base, middle, left, right and left-right sets of \((\mu, \nu)\), denoted by \( B, M, L, R \) and \( LR \) respectively, as follows (Figure 3):

\[
B(\mu, \nu) = A, \quad M(\mu, \nu) = A',
\]

\[
L(\mu, \nu) = \{i \in A : i < \min A'\},
\]

\[
R(\mu, \nu) = \{i \in A : i > \max A'\},
\]

\[
LR(\mu, \nu) = L(\mu, \nu) \cup R(\mu, \nu) = A''.
\]

Note that \( L(\mu, \nu) \) and \( M(\mu, \nu) \) are not empty.

For \( \mu, \nu \in \mathcal{NC}_n \) such that \( \nu \) covers \( \mu \), define \( \Delta_1(\mu, \nu) \) and \( \Delta_2(\mu, \nu) \) by

\[
\Delta_1(\mu, \nu) = \max L(\mu, \nu), \quad \Delta_2(\mu, \nu) = \max M(\mu, \nu),
\]
and define $\Delta(\mu, \nu)$ to be the transposition $(\Delta_1(\mu, \nu) \Delta_2(\mu, \nu))$ in $\mathcal{S}_n$.

We now define $\Omega : \mathcal{M}_n \to \mathcal{F}_n$ as follows: Given a maximal chain $m = [\mu_0 < \mu_1 < \cdots < \mu_{n-1}]$,

$$\Omega(m) = (\Delta(\mu_0, \mu_1), \ldots, \Delta(\mu_{n-2}, \mu_{n-1})),$$

It easily follows from construction that $\Omega$ is well-defined and injective.

**Proposition 2** The composition map $\Gamma \circ \Omega$ is the inverse of $\Sigma$. So the mappings $\Sigma$ and $\Omega$ are all bijections.

**Proof.** Consider the maximal chain

$$\Sigma(C) = [\mu_0 < \mu_1 < \cdots < \mu_{n-1}],$$

where $C = \Gamma( (w_1 z_1), \ldots, (w_{n-1} z_{n-1}) )$. Recall that $w_i$ and $z_i$ belong to the same block $A$ in $\mu_i$. The block $A$ is split into two blocks $A'$ and $A''$ in $\mu_{i-1}$, where

$$A' = A \cap [w_i + 1, z_i], \quad A'' = A \cap ([1, w_i] \cup [z_i + 1, n]).$$

On the other hand,

$$M(\mu_{i-1}, \mu_i) = A', \quad L(\mu_{i-1}, \mu_i) = A \cap [1, w_i].$$

Since $w_i, z_i$ are in $A$,

$$\max L(\mu_{i-1}, \mu_i) = w_i, \quad \max M(\mu_{i-1}, \mu_i) = z_i.$$

Thus $\Delta(\mu_{i-1}, \mu_i) = (w_i z_i)$ and

$$\Omega(\Sigma(C)) = ( (w_1 z_1), \ldots, (w_{n-1} z_{n-1}) ).$$

Applying the both sides of the above with $\Gamma$ yields that $(\Gamma \circ \Omega) \circ \Sigma$ is the identity map on $\mathcal{C}_n$. Thus $\Gamma \circ \Omega$ is surjective. On the other hand, since $\Omega$ is injective, $\Gamma \circ \Omega$ is the inverse of $\Sigma$. \qed
Given a maximal chain \( m = [\mu_0 < \cdots < \mu_{n-1}] \in \mathcal{M}_n \), define \( \Omega_1(m) \) by
\[
\Omega_1(m) = (\Delta_1(\mu_0, \mu_1), \ldots, \Delta_1(\mu_{n-2}, \mu_{n-1})).
\]

Stanley [15, Theorem 3.1] showed that the map \( \Omega_1 \) (in his notation, \( \Lambda \)) is a bijection between \( \mathcal{M}_n \) and \( \mathcal{P}_{n-1} \). Thus we have the following consequence.

**Theorem 1** The map \( \Phi : \mathcal{F}(n) \to \mathcal{P}_{n-1} \) defined by
\[
\Phi( (w_1 z_1), \ldots, (w_{n-1} z_{n-1}) ) = (w_1, \ldots, w_{n-1})
\]
is a bijection.

**Proof.** It is easy to show that
\[
\Phi = \Omega_1 \circ \Sigma \circ \Gamma.
\]
So \( \Phi \) is a bijection. \( \square \)

**Remark.** It is well known that \( |\mathcal{P}_{n-1}| = n^{n-2} \). So Theorem 1 gives a bijective proof of (1) for the case \( \lambda = (n) \). In fact, the resulting bijection \( \Phi \) is the same as given by Biane [2], who proved it without Stanley’s result.

In Figure 4, we give an example which summarizes all the consequences in this section. Note that all maps are bijective and all diagrams commute.

**4 Main Result**

We now proceed to construct a combinatorial proof of (2). Let \( (\tau_1, \ldots, \tau_m) \) be an element of \( \mathcal{F}_{(1,n-1)} \). Since \( \lambda \) has two parts, i.e. \( l = 2 \), we have \( m = n \) and there is only one cut among \( \tau_1, \ldots, \tau_{n-1} \) (Note that \( \tau_n \) cannot be a cut). Moreover, since \( \tau_1 \cdots \tau_n = (1)(2 \cdots n) \), the unique cut has the form \((1 \ell)\) for some \( \ell, 2 \leq \ell \leq n \). Let \( \mathcal{G}^r_{n,k} \) be the set of all minimal factorizations \((\tau_1, \ldots, \tau_n) \in \mathcal{F}_{(1,n-1)} \) which have a cut \((1 \ell)\) at the \( k \)-th position, i.e. \( \tau_k = (1 \ell) \). Set
\[
\mathcal{G}^r_n = \bigcup_{k=1}^{n-1} \mathcal{G}^r_{n,k}.
\]

Clearly,
\[
\mathcal{F}_{(1,n-1)} = \bigcup_{r=2}^{n} \mathcal{G}^r_n.
\]

We first observe that the cardinality of \( \mathcal{G}^r_{n,k} \) is independent of \( r \).

**Lemma 1** Let \( \theta \) be the permutation \((1)(2 \cdots n)\). For any \( r \) and \( s \), \( 2 \leq r \leq s \leq n \), the mapping \( \Theta_{s-r} : \mathcal{G}^r_{n,k} \to \mathcal{G}^s_{n,k} \) defined by
\[
\Theta_{s-r}(\tau_1, \ldots, \tau_n) = (\theta^{s-r} \tau_1 \theta^{r-s}, \ldots, \theta^{r-s} \tau_n \theta^{s-r})
\]
is a bijection. Moreover, \( \Theta_{s-r} \) can be extended to the bijection between \( \mathcal{G}^r_n \) and \( \mathcal{G}^s_n \).
Figure 4: Bijections between $\mathcal{F}_{(n)}$, $\mathcal{C}_n$, $\mathcal{M}_n$ and $\mathcal{P}_{n-1}$. 
Proof. Clearly it is a bijection since $\theta^{r-s} \tau_i \theta^{r-s}$ fixes 1 and changes $r$ into $s$ in $\tau_i$. □

A parking function $(x_1, \ldots, x_n)$ is said to be prime, if the increasing rearrangement $y_1 \leq \cdots \leq y_n$ of $x_1, \ldots, x_n$ satisfies $y_i < i$ for $i = 2, \ldots, n$. Let $\mathcal{Q}_n$ denote the set of prime parking functions $Q = (x_1, \ldots, x_n)$ of length $n$ and define $\mathcal{Q}_{n,k}$ to be the set of all $Q \in \mathcal{Q}_n$ in which $x_k = 1$ is the leftmost occurrence of 1. Clearly, we have

$$\mathcal{Q}_n = \bigcup_{k=1}^{n-1} \mathcal{Q}_{n,k}.$$ 

It is clear that the prime parking functions satisfy the following:

**Lemma 2** $(x_1, \ldots, x_n) \in \mathcal{Q}_{n,k}$ if and only if $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathcal{P}_{n-1}$ with $x_i \neq 1$ for $i = 1, 2, \ldots, k - 1$.

We will eventually show that $\mathcal{G}_{n,k}^2$ and $\mathcal{Q}_{n,k}$ have the same cardinality.

Let $\mathcal{A}_{n,k}$ be the set consisting of all $(\sigma_1, \ldots, \sigma_{n-1}) \in \mathcal{F}(n)$ such that

(a1) 1 occurs in $\sigma_k, \ldots, \sigma_{n-1}$.

(a2) 2 does not occur in $\sigma_1, \ldots, \sigma_{k-1}$.

**Lemma 3** The map $\Psi_1 : \mathcal{G}_{n,k}^2 \rightarrow \mathcal{A}_{n,k}$ defined by

$$\Psi_1(\tau_1, \ldots, \tau_n) = ((1 2)\tau_1(1 2), \ldots, (1 2)\tau_{k-1}(1 2), \tau_{k+1}, \ldots, \tau_n),$$

is a bijection.

Proof. Set $\tilde{\tau}_1 = (1 2)\tau_1(1 2)$. Assume that $(\tau_1, \ldots, \tau_n) \in \mathcal{G}_{n,k}^2$. Since $(\tau_1, \ldots, \tau_n) \in \mathcal{F}_{(1,n-1)}$, $\tau_1 \cdots \tau_n = (1)(2 \cdots n)$ and $\tau_k = (1 2)$. Multiplying both sides with $(1 2)$ yields $\tilde{\tau}_1 \cdots \tilde{\tau}_{k-1} \tau_{k+1} \cdots \tau_n = (1 \cdots n)$. This implies that $(\tilde{\tau}_1, \ldots, \tilde{\tau}_{k-1}, \tau_{k+1}, \ldots, \tau_n) \in \mathcal{F}(n)$. In addition, since $\tau_k = (1 2)$ is a cut, $\tau_{k+1} \cdots \tau_n$ doesn’t fix 1 and so there exists a $j$ greater than $k$ such that $\tau_j$ contains 1. Moreover, since $\tau_k$ is the unique cut in $(\tau_1, \ldots, \tau_n)$, 1 doesn’t appear in any of $\tau_1, \ldots, \tau_{k-1}$ (otherwise, 1 cannot be a fixed point in $\tau_1 \cdots \tau_n$). Thus $\tilde{\tau}_1, \ldots, \tilde{\tau}_{k-1}$ do not have 2.

Hence $(\tilde{\tau}_1, \ldots, \tilde{\tau}_{k-1}, \tau_{k+1}, \ldots, \tau_n)$ is in $\mathcal{A}_{n,k}$.

We now show that $\Psi_1$ has the inverse. Define the map $\Psi_1' : \mathcal{A}_{n,k} \rightarrow \mathcal{G}_{n,k}^2$ by

$$\Psi_1'(\sigma_1, \ldots, \sigma_{n-1}) = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{k-1}, (1 2)\sigma_k, \ldots, \sigma_{n-1}).$$

We first prove that $\Psi_1'$ is well-defined. Since $(\sigma_1, \ldots, \sigma_{n-1}) \in \mathcal{F}(n)$, we have $\sigma_1 \cdots \sigma_{n-1} = (1 \cdots n)$. Multiplying both sides with $(1 2)$ yields $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{k-1} (1 2)\sigma_k \cdots \sigma_{n-1} = (1)(2 \cdots n)$. This implies that $F = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{k-1}, (1 2)\sigma_k, \ldots, \sigma_{n-1}) \in \mathcal{F}_{(1,n-1)}$. Since $(\sigma_1, \ldots, \sigma_{n-1}) \in \mathcal{F}(n)$ has no cut, none of $\sigma_k, \ldots, \sigma_{n-1}$ is a cut in $F$. We now show that $(1 2)$ is a cut in $F$. Suppose that it is a join in $F$. The unique cut of $F$ is $\tilde{\sigma}_j$ for some $j$ less than $k$. So 1 is a fixed point of $\tilde{\sigma}_j \cdots \tilde{\sigma}_{k-1} (1 2)\sigma_k \cdots \sigma_{n-1}$. But from the condition (a1), 1 appears at least twice among $\tilde{\sigma}_{j+1}, \ldots, \tilde{\sigma}_{k-1}, (1 2)\sigma_k, \ldots, \sigma_{n-1}$ and 1 is contained in a cycle of length at least 3 in $\tilde{\sigma}_{j+1} \cdots \tilde{\sigma}_{k-1} (1 2)\sigma_k \cdots \sigma_{n-1}$. So $\tilde{\sigma}_j$ is of the form $(1 m)$ and 2 appears in $\sigma_j$, contradicting to the condition (a2). Therefore, the $k$-th transposition $(1 2)$ is a cut and we have $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{k-1}, (1 2)\sigma_k, \ldots, \sigma_{n-1}) \in \mathcal{G}_{n,k}^2$.

Clearly, $\Psi_1 \circ \Psi_1'$ is the identity map and so $\Psi_1$ is a bijection. □

We now show that $\mathcal{A}_{n,k}$ and $\mathcal{Q}_{n,k}$ are in 1-1 correspondence.
Lemma 4 The map \( \Psi_2 : \mathcal{A}_{n,k} \rightarrow \mathcal{Q}_{n,k} \) defined by
\[
\Psi_2(\sigma_1, \ldots, \sigma_{n-1}) = (\tilde{a}_1, \ldots, \tilde{a}_{k-1}, 1, a_k, \ldots, a_{n-1}),
\]
where \( \sigma_i = (a_i, b_i) \) and \( \tilde{a}_i \) is the image of \( a_i \) under the transposition \((1\ 2)\), is a bijection.

Proof. Let \((\sigma_1, \ldots, \sigma_{n-1})\) be an element of \( \mathcal{A}_{n,k} \). Then \((a_1, \ldots, a_{n-1}) \in \mathcal{P}_{n-1} \) with \( 2 \notin \{a_1, \ldots, a_{k-1}\} \) and \( 1 \notin \{a_{k+1}, \ldots, a_{n-1}\} \), which imply \((\tilde{a}_1, \tilde{a}_{k-1}, a_k, \ldots, a_{n-1}) \in \mathcal{P}_{n-1} \) with \( 1 \notin \{\tilde{a}_1, \ldots, \tilde{a}_{k-1}\} \). So by Lemma 2, \((\tilde{a}_1, \ldots, \tilde{a}_{k-1}, 1, a_k, \ldots, a_{n-1}) \in \mathcal{Q}_{n,k} \).

The map \( \Psi_2 \) has the inverse. It can be shown easily that the map \( \Psi_2 : \mathcal{Q}_{n,k} \rightarrow \mathcal{A}_{n,k} \) defined by
\[
\Psi_2'(x_1, \ldots, x_{k-1}, 1, x_k+1, \ldots, x_n) = \Phi^{-1}((\tilde{x}_1, \ldots, \tilde{x}_{k-1}, \tilde{x}_{k+1}, \ldots, x_n)
\]
is the inverse of \( \Psi_2 \).

Hence \( \Psi_2 \) is a bijection. \( \Box \)

Theorem 2 The mapping \( \Psi : \mathcal{G}_{n,k}^2 \rightarrow \mathcal{Q}_{n,k} \) defined by
\[
\Psi((a_1 b_1), \ldots, (a_n b_n)) = (a_1, \ldots, a_n),
\]
is a bijection. Moreover, \( \Psi \) can be extended to a bijection between \( \mathcal{G}_{n,k}^2 \) and \( \mathcal{Q}_n \).

Proof. Since \((a_i, b_i) \neq (1\ 2)\) for \( i = 1, 2, \ldots, k - 1 \), we clearly have \( \Psi = \Psi_2 \circ \Psi_1 \), where \( \Psi_1 \) and \( \Psi_2 \) are the bijections defined in Lemmas 3 and 4.

Since \( \mathcal{G}_{n,k}^2 = \bigcup_{i=1}^{k-1} \mathcal{G}_{n,k}^2 \) and \( \mathcal{Q}_n = \bigcup_{i=1}^{n-1} \mathcal{Q}_{n,k} \), \( \Psi \) can be defined as a mapping from \( \mathcal{G}_{n,k}^2 \) to \( \mathcal{Q}_n \).

Now we are ready to finish a combinatorial proof of (2).

Corollary 1 The mapping \( \Theta : \mathcal{Q}_n \times [n-1] \rightarrow \mathcal{F}_{(1,n-1)} \) defined by
\[
\Theta(Q, t) = \Theta_{t-1}(\Psi^{-1}(Q)),
\]
is a bijection.

Proof. \( \Theta_{t-1} \) and \( \Psi^{-1} \) are bijections and \( \Theta_{t-1}(\mathcal{G}_n^2) = \mathcal{G}_n^{t+1} \) for \( t = 1, \ldots, n-1 \). Since
\[
\mathcal{F}_{(1,n-1)} = \bigcup_{r=2}^n \mathcal{G}_n^r,
\]
\( \Theta \) is a bijection. (See Figure 5.) \( \Box \)

Since \( |\mathcal{Q}_n| = (n-1)^{n-1} \) ([14, pp. 95]), Corollary 1 is a combinatorial proof of \( |\mathcal{F}_{(1,n-1)}| = (n-1)^n \).

5 Remark

With a variation of the depth-first-search, we are able to find a bijection between \( \mathcal{Q}_{n,k} \) and the set \( \mathcal{R}_{n,k} \) of rooted labelled trees on \([n]\) such that the root \( k \) is a local minimum of the tree. Chauve, Dulucq and Guibert [4] enumerated the cardinality of \( \mathcal{R}_{n,k} \) with a bijective proof. Their result is:
\[
|\mathcal{R}_{n,k}| = (n-k) n^{n-k-1} (n-1)^{k-2}.
\]
So we are able to count not only \( \mathcal{F}_{(1,n-1)} \) but also \( \mathcal{G}_{n,k}^2, \ k = 1, 2, \ldots, n-1 \), which are refinements of \( \mathcal{F}_{(1,n-1)} \). Moreover, by this refinement, our approach can be extended to handle the case \( \lambda = (2, n-2) \).
Figure 5: The map \( \Theta : Q_n \times [n-1] \rightarrow \mathcal{F}_{(1,n-1)} \)

Acknowledgment

We are thankful to an anonymous referee and Mireille Bousquet-Mélou for bringing [3] to our attention. This work was partially supported by KOSEF: R03-2001-00003.

References


