EXPLOITING REGENERATIVE STRUCTURE TO ESTIMATE FINITE TIME AVERAGES VIA SIMULATION: SUPPLEMENTARY MATERIAL

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This paper contains additional details about the simulation experiments discussed in the main paper.

Abstract from the Main Paper: We propose nonstandard simulation estimators of expected time averages over finite intervals $[0, t]$, seeking to enhance estimation efficiency. We make three key assumptions: (i) the underlying stochastic process has regenerative structure, (ii) the time average approaches a known limit as time t increases and (iii) time 0 is a regeneration time. To exploit those properties, we propose a *residual-cycle estimator*, based on data from the regenerative cycle in progress at time t, using only the data *after* time t. We prove that the residual-cycle estimator is unbiased and more efficient than the standard estimator for all sufficiently large t. Since the relative efficiency increases in t, the method is ideally suited to use when applying simulation to study the rate of convergence to the known limit. We also consider two other simulation techniques to be used with the residual-cycle estimator. The first involves *overlapping cycles*, paralleling the technique of overlapping batch means in steady-state estimation; multiple observations are taken from each replication, starting a new observation each time the initial regenerative state is revisited. The other technique is *splitting*, which involves independent replications of the terminal period after time t, for each simulation up to time t. We demonstrate that these alternative estimators provide efficiency improvement by conducting simulations of queueing models.

Categories and Subject Descriptors: G.3 [**Mathematics of Computing**]: Probability and Statistics—*renewal theory*; I.6.6 [**Computing Methodologies**]: Simulation and Modelling—*simulation output analysis*

General Terms: Algorithms, Experimentation, Performance, Theory

Additional Key Words and Phrases: efficiency improvement, variance reduction, regenerative processes, time averages

1. INTRODUCTION

This document serves as a supplement to our main paper, with the same title. There are four sections. The first three sections provide additional detail about the $M/G/1/0$ experiment, described in Sections 8 and 9 of the main paper, while the last section describes the $M/G/5/0$ experiment, discussed in Section 10 of the main paper, Kang et al. [2006].

2. FIGURES FOR $M/G/1/0$

This section contains additional figures describing simulation results for the $M/G/1/0$ model, which is discussed in Sections 8 and 9 of the main paper. The $M/G/1/0$ model has a Poisson arrival process, IID service times with a general distribution, which we assume has finite mean μ^{-1}, a single servers and no extra waiting room. Thus there are only two states.

The figures here plot the cpu-time efficiencies for each of the seventeen alternative estimators considered in this experiment, based on the data in the 9 tables in another part of this supplement. As in

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Figures 1 and 2 in the main paper, we plot the estimated logarithm of the efficiency versus both the estimated value of $\alpha(t)$ and the estimated value of $-\log_{10}(1-(\alpha(t)/\alpha))$.

For the overlapping-cycle modification of the standard estimator, we see nothing good happening in the first four figures. But, after that, we see strong evidence of increasing positive efficiency for large t, as discussed in the paper. Especially revealing are the second log-log plots, where we see linearity after a short initial period.

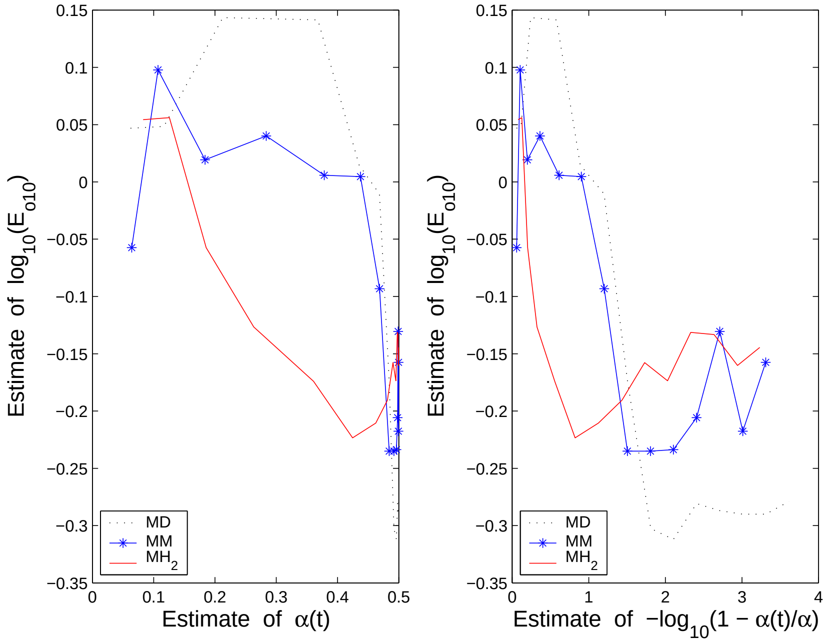

Fig. 1. Figures of $\mathcal{E}_{o10}$.

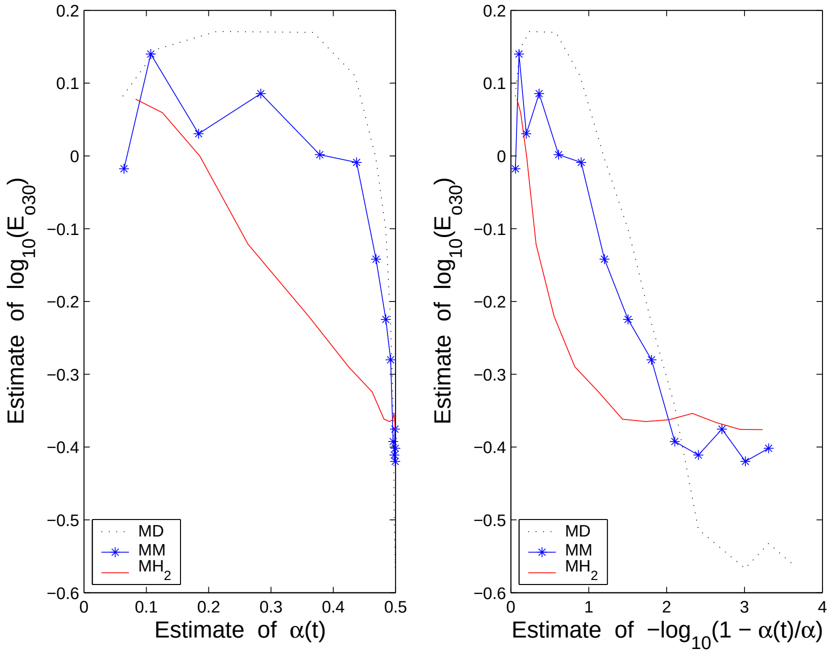

Fig. 2. Figures of $\mathcal{E}_{o30}$.

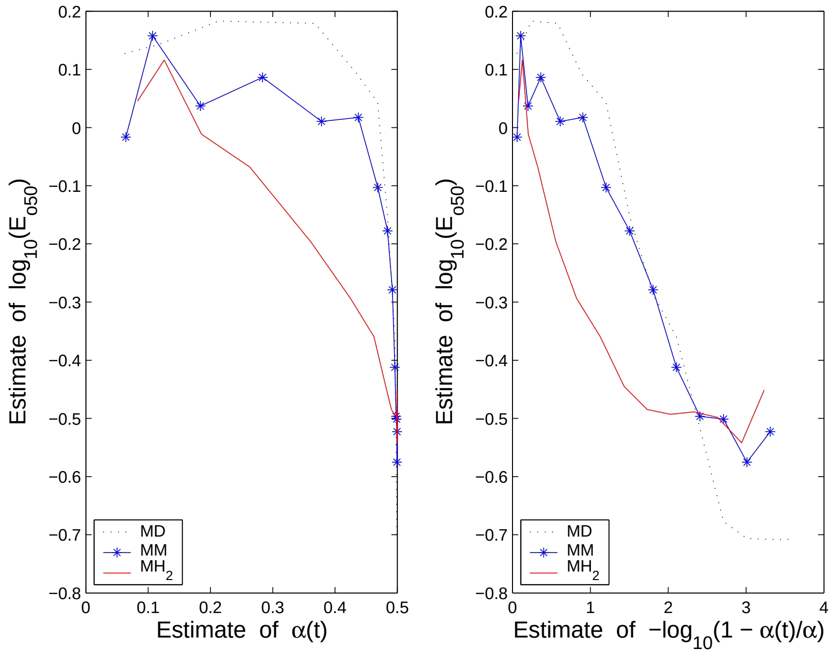

Fig. 3. Figures of $\mathcal{E}_{o50}$.

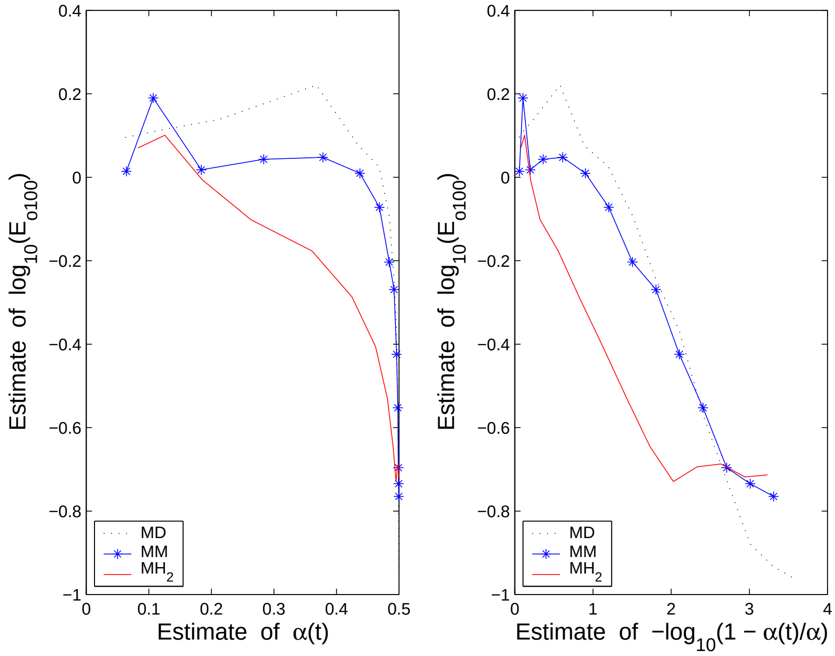

Fig. 4. Figures of $\mathcal{E}_{o100}$.

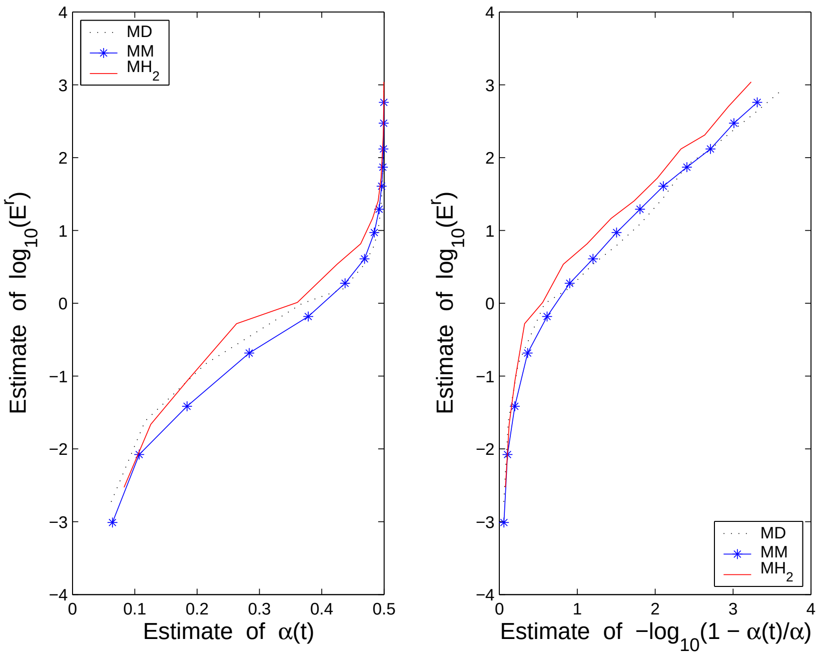

Fig. 5. Figures of $\mathcal{E}^r$.

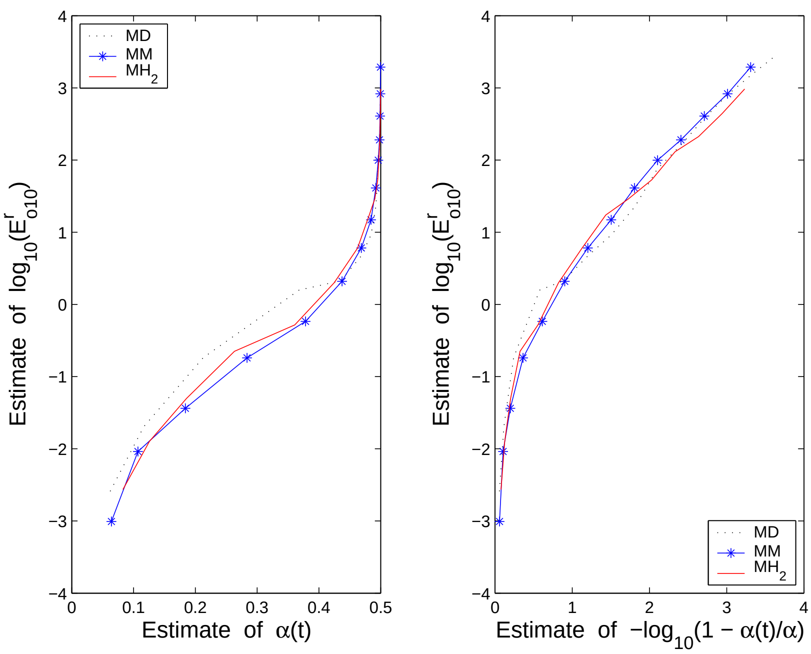

Fig. 6. Figures of $\mathcal{E}^r_{o10}$.

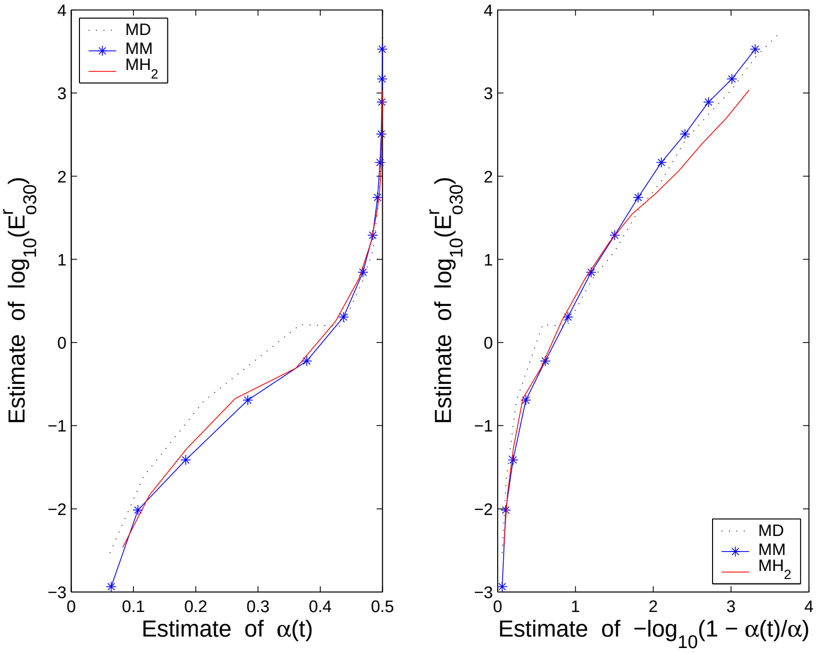

Fig. 7. Figures of $\mathcal{E}^r_{o30}$.

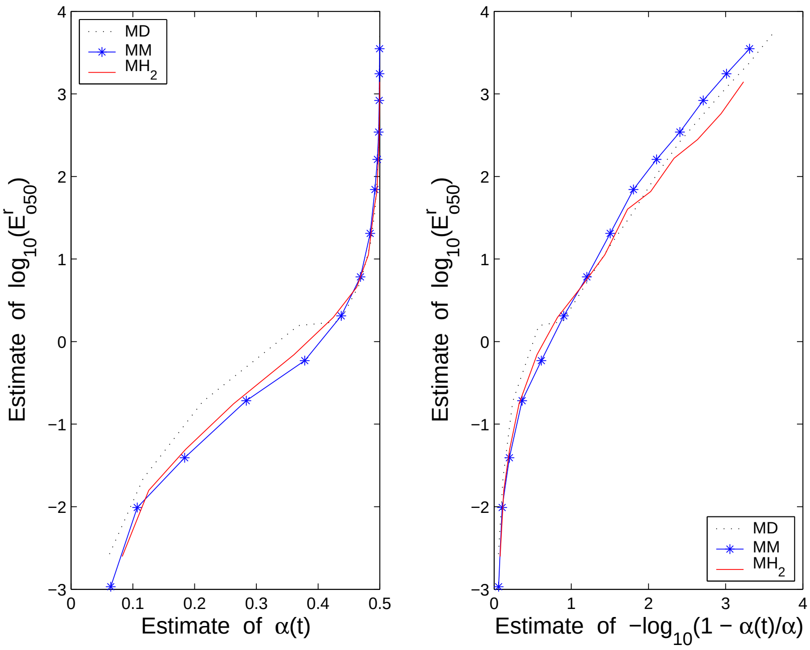

Fig. 8. Figures of $\mathcal{E}^r_{o50}$.

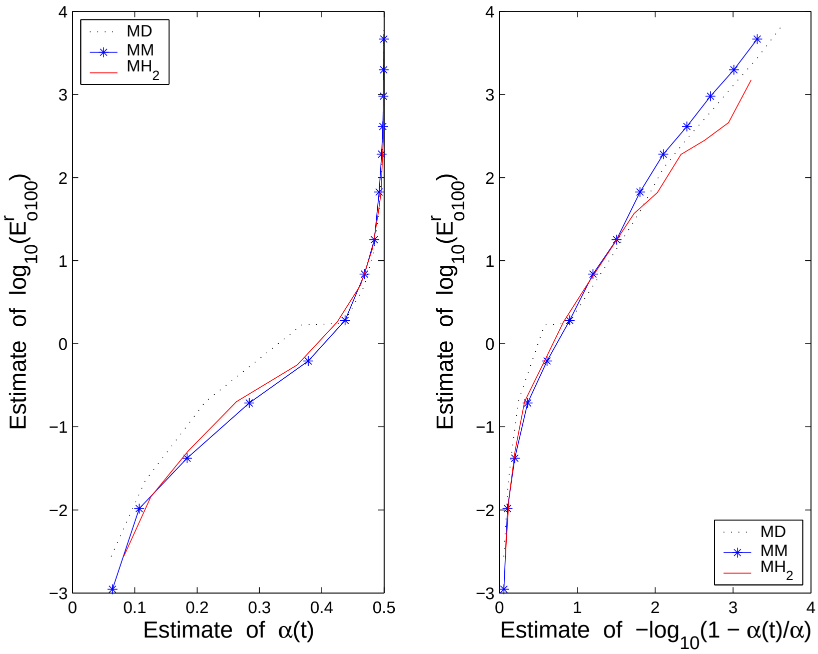

Fig. 9. Figures of $\mathcal{E}^r_{o100}$.

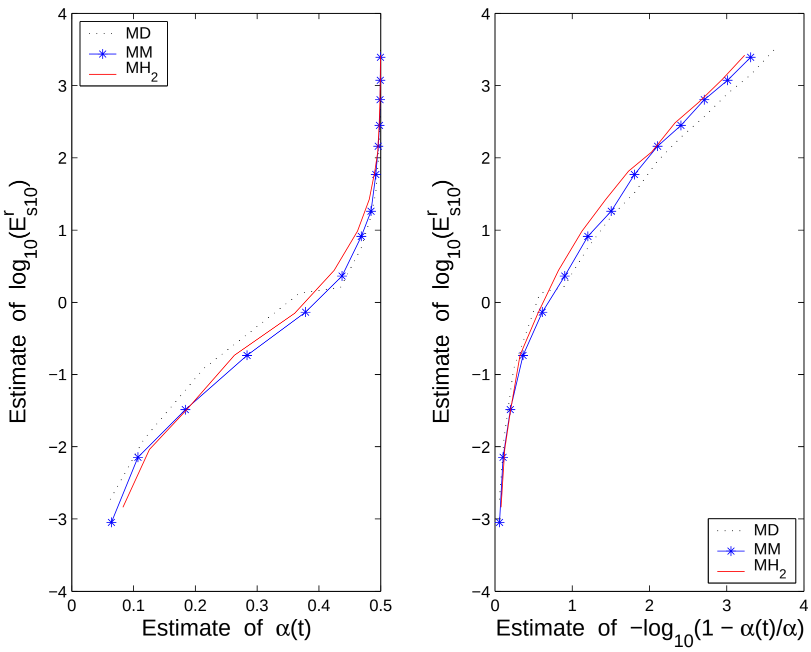

Fig. 10. Figures of $\mathcal{E}^r_{s10}$.

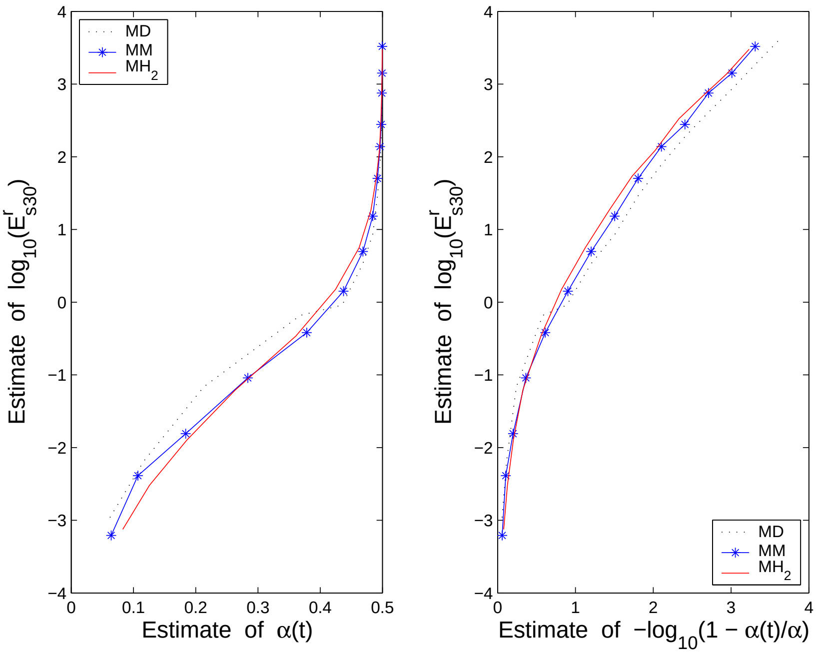

Fig. 11. Figures of $\mathcal{E}^r_{s30}$.

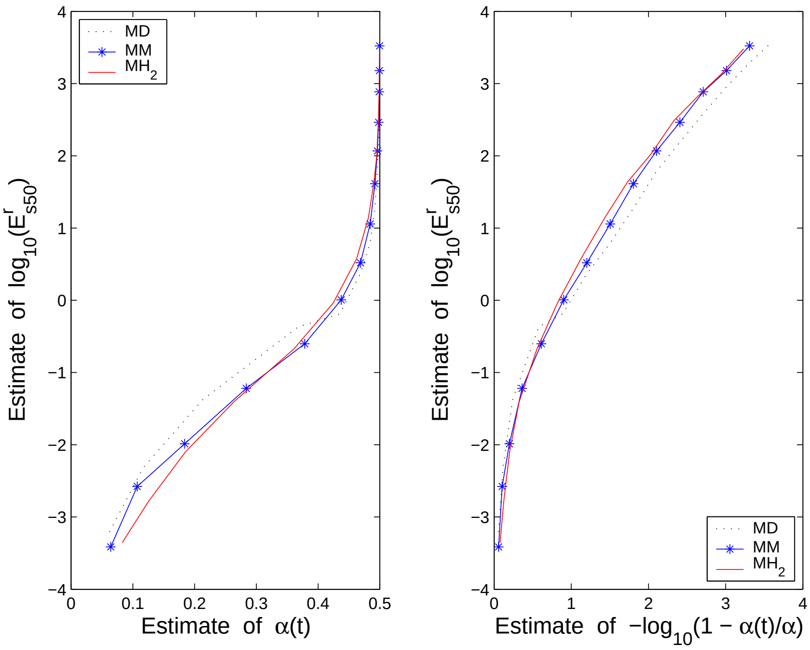

Fig. 12. Figures of $\mathcal{E}^r_{s50}$.

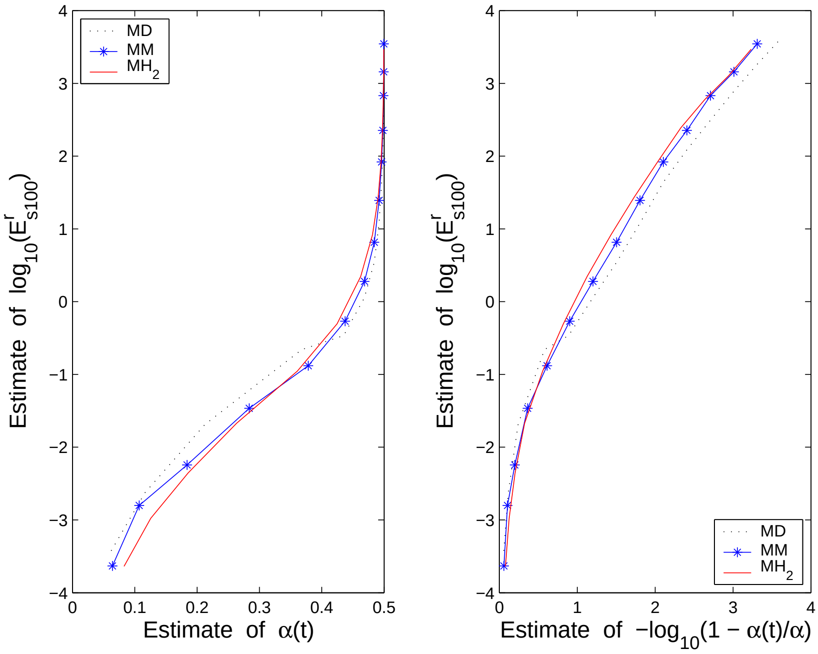

Fig. 13. Figures of $\mathcal{E}^r_{s100}$.

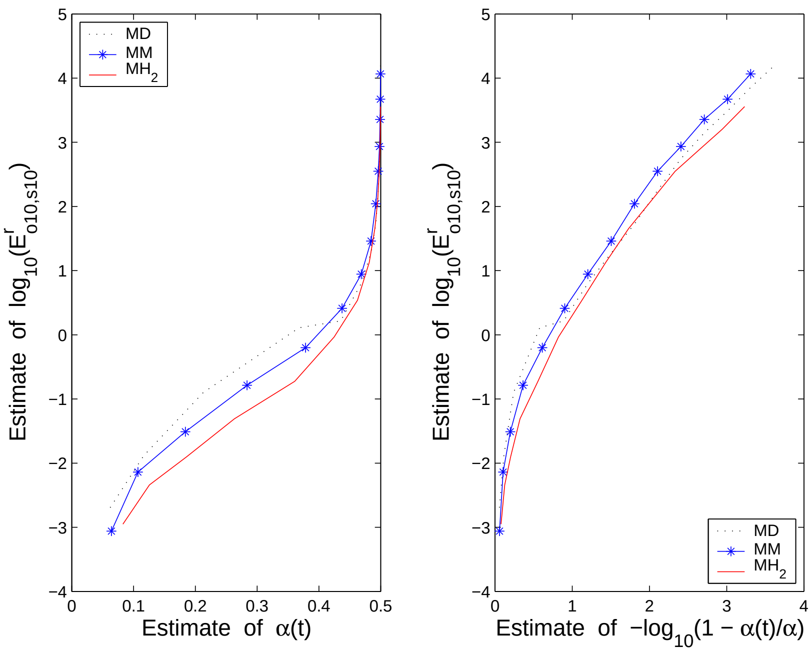

Fig. 14. Figures of $\mathcal{E}^r_{o10,s10}$.

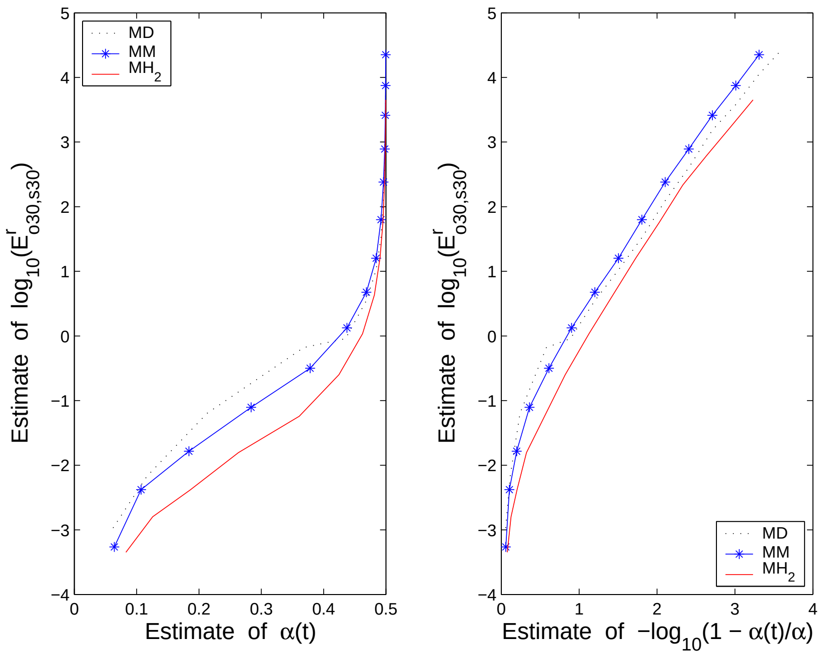

Fig. 15. Figures of $\mathcal{E}^r_{o30,s30}$.

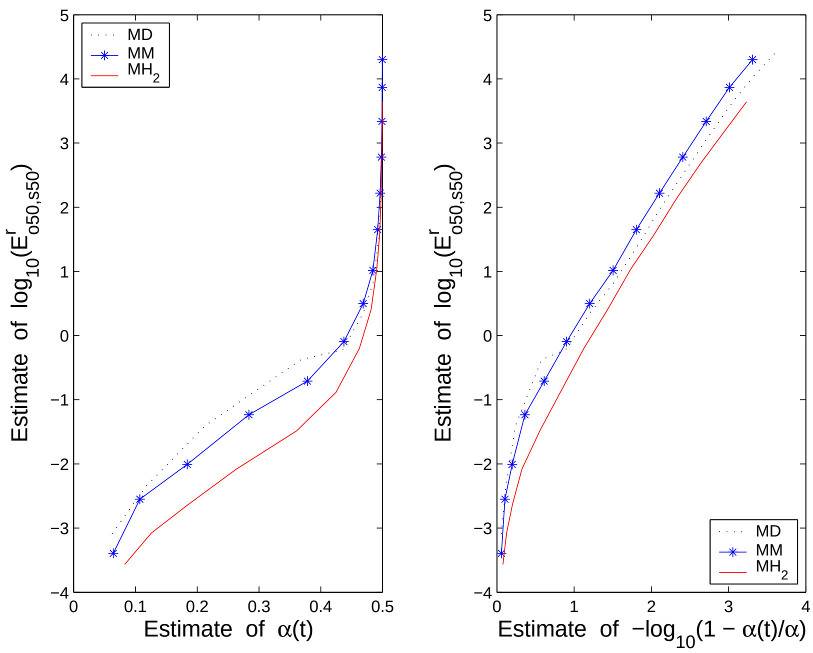

Fig. 16. Figures of $\mathcal{E}^r_{o50,s50}$.

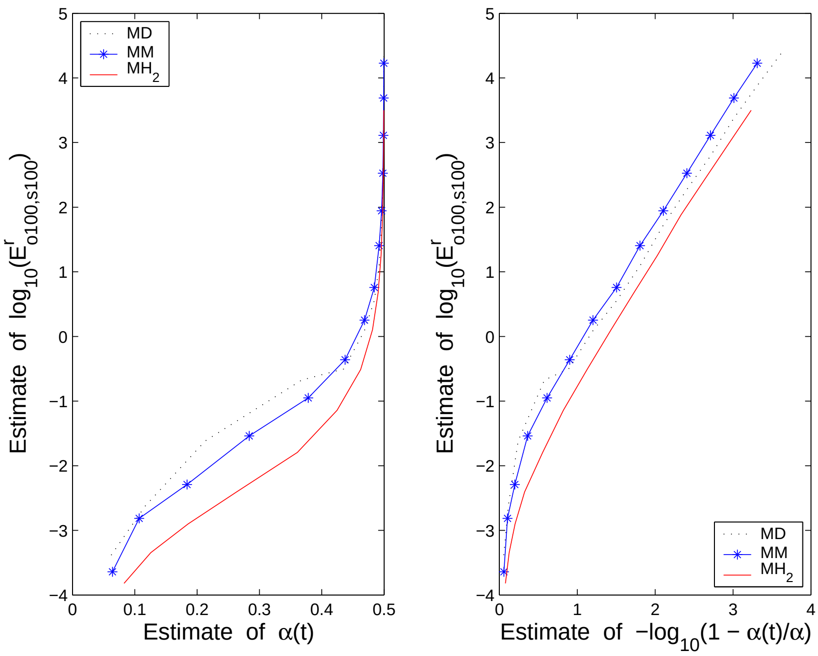

Fig. 17. Figures of $\mathcal{E}^{r}_{o100,s100}$.

3. TABLES FOR $M/G/1/0$

This section contains additional tables describing simulation results for the $M/G/1/0$ model, which is discussed in Sections 8 and 9 of the main paper.

3.1 An Initial Experiment with Three Values of λ

In the first three tables, Tables I, II and III, we display results for the $M/M/1/0$ model for three different values of λ: 0.5, 1.0 and 1.5. Since these results are similar, we focus on the special case of $\lambda = 1.0$. In these tables, we give both exact values, computed from the explicit formulas given in Section 8 of the paper, and simulation estimates based on $n = 1000$ independent replications. We give the exact values of $\alpha(t)$ plus the standard and residual-cycle estimators. We give the estimated halfwidths of 95% confidence intervals, for the standard and residual-cycle estimators. We given the estimated variance ratio $\widehat{Var}(\hat{\alpha_n}(t))/\widehat{Var}(\hat{\alpha_n^r}(t))$ and the exact variance ratio $Var(\alpha_n(t))/Var(\alpha_n^r(t))$. In this case the efficiency ratio is the exact value of the run-length efficiency $\widehat{\mathcal{F}}^r(t)$. In all three tables we see efficiency gains for the larger times.

We also give three figures to go with these first three tables. In those figures, the limiting value α and the actual exact value $\alpha(t)$ are plotted by asterisks. The standard and residual-cycle estimators are plotted along with their 95% confidence intervals. We can see that the confidence intervals are considerably tighter for the residual-cycle estimator at the larger times (on the right).

Table I. $n = 1000$, $\lambda = 0.5$, $\mu = 1$, $\alpha = 0.3333$.

t	$\alpha(t)$	$\hat{\alpha_n}(t)$	$\hat{\alpha_n^r}(t)$	HW_n	HW_n^r	$\frac{\widehat{Var}(\hat{\alpha_n}(t))}{\widehat{Var}(\hat{\alpha_n^r}(t))}$	$\frac{Var(\alpha_n(t))}{Var(\alpha_n^r(t))}$	Efficiency ratio
5.0	0.2889	0.2864	0.2850	0.0127	0.0114	1.2469	1.3536	1.1792
16.25	0.3197	0.3267	0.3201	0.0083	0.0035	5.6352	5.3235	5.8909
27.5	0.3253	0.3219	0.3278	0.0062	0.0020	9.1588	9.2941	10.8891
38.75	0.3276	0.3262	0.3274	0.0055	0.0014	14.2777	13.2647	14.0711
50.0	0.3289	0.3272	0.3294	0.0048	0.0011	17.8734	17.2353	18.2941

Table II. $n = 1000$, $\lambda = 1$, $\mu = 1$, $\alpha = 0.5$.

t	$\alpha(t)$	$\hat{\alpha_n}(t)$	$\hat{\alpha_n^r}(t)$	HW_n	HW_n^r	$\frac{\widehat{Var}(\hat{\alpha_n}(t))}{\widehat{Var}(\hat{\alpha_n^r}(t))}$	$\frac{Var(\alpha_n(t))}{Var(\alpha_n^r(t))}$	Efficiency ratio
5.0	0.4500	0.4536	0.4452	0.0123	0.0082	2.2434	2.4286	2.2492
16.25	0.4846	0.4869	0.4826	0.0073	0.0027	7.5462	8.8571	7.6373
27.5	0.4909	0.4911	0.4912	0.0058	0.0015	14.9590	15.2857	15.1693
38.75	0.4935	0.4932	0.4935	0.0049	0.0011	21.3758	21.7143	22.1580
50.0	0.4950	0.4940	0.4953	0.0043	0.0008	26.8597	28.1429	27.6243

Table III. $n = 1000$, $\lambda = 1.5$, $\mu = 1$, $\alpha = 0.6$.

t	$\alpha(t)$	$\hat{\alpha_n}(t)$	$\hat{\alpha_n^r}(t)$	HW_n	HW_n^r	$\frac{\widehat{Var}(\hat{\alpha_n}(t))}{\widehat{Var}(\hat{\alpha_n^r}(t))}$	$\frac{Var(\alpha_n(t))}{Var(\alpha_n^r(t))}$	Efficiency ratio
5.0	0.5520	0.5514	0.5546	0.0112	0.0062	3.2275	3.2927	3.5853
16.25	0.5852	0.5846	0.5847	0.0065	0.0020	10.7545	11.5244	11.1586
27.5	0.5913	0.5904	0.5908	0.0053	0.0011	21.1933	19.7561	21.3128
38.75	0.5938	0.5937	0.5942	0.0044	0.0009	25.1803	27.9878	23.5075
50.0	0.5952	0.5928	0.5949	0.0038	0.0007	32.9431	36.2195	31.5837

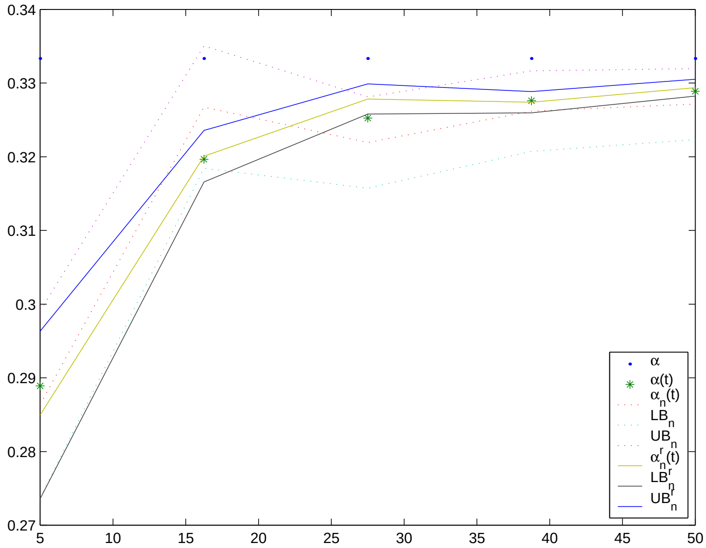

Fig. 18. $n = 1000$, $\lambda = 0.5$, $\mu = 1$, $\alpha = 0.3333$.

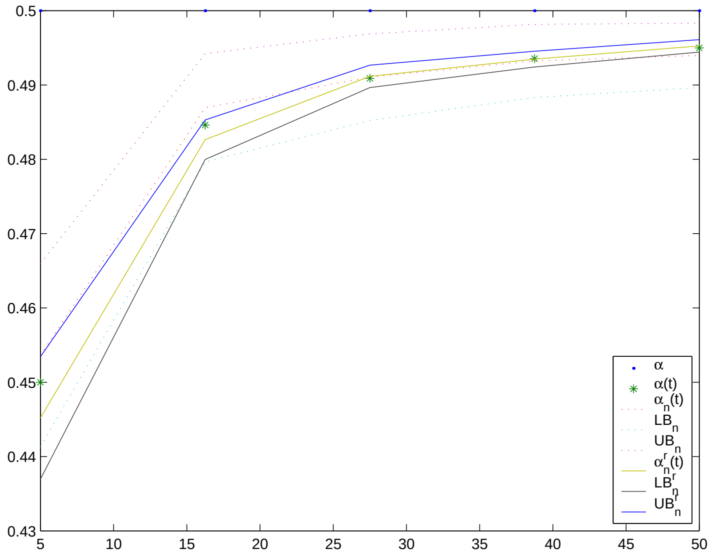

Fig. 19. $n = 1000$, $\lambda = 1$, $\mu = 1$, $\alpha = 0.5$.

3.2 The Main Experiment

There are then nine more tables, three each for each of the three service-time distributions: D, M and H_2, each taken to have mean $\mu^{-1} = 1$. The specific H_2 probability density was

$$g(x) = \frac{0.8}{17.0} \times 0.1e^{-0.1x} + \frac{16.2}{17.0} \times 1.8e^{-1.8x}, \quad x \geq 0 ,$$

which has SCV $= 9.0$ (i.e., squared coefficient of variation, defined as the variance divided by the square of the mean).

We performed simulation experiments for *six* different estimators: (1) standard, (2) standard plus overlapping cycles, (3) residual-cycle, (4) residual-cycle plus overlapping cycles, (5) residual-cycle plus splitting, and (6) residual-cycle plus both overlapping cycles and splitting. For each model and estimator, we considered 14 different time points t, changing in powers of 2. For $M/D/1/0$ and $M/M/1/0$, the times were t^k for $-3 \leq k \leq 10$. Since $\alpha(t)$ approached α much more slowly with the highly variable H_2 service-time distribution, we used times t^k for $0 \leq k \leq 13$ for the $M/H_2/1/0$ model. For both

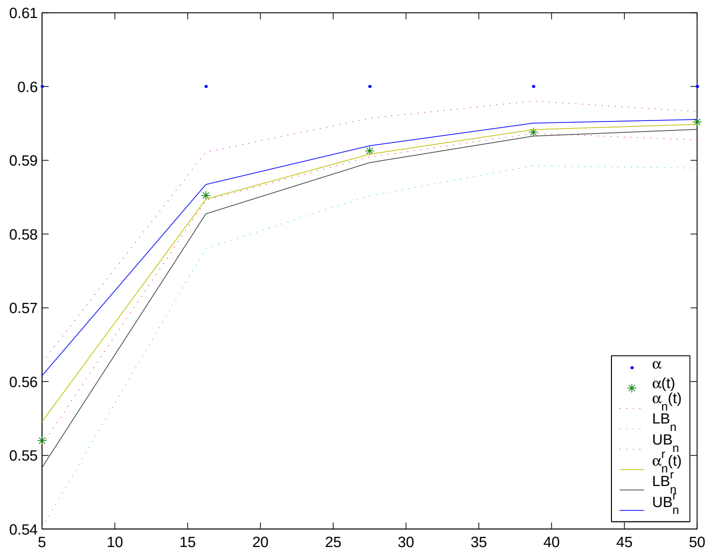

Fig. 20. $n = 1000$, $\lambda = 1.5$, $\mu = 1$, $\alpha = 0.6$.

overlapping (m) and splitting (p), we considered 5 levels: 1 (not used), 10, 30, 50 and 100. When we combined all three techniques, we used the same number of splittings as overlapping cycles. That produced $18 \times 14 = 252$ separate simulation experiments. In each separate simulation experiment we used $n = 1000$ independent replications. Thus we performed $252 \times 1000 = 252,000$ simulation runs.

3.3 Results

We now describe the results of the experiment. Here we repeat the discussion in Section 9 of the main paper. However, there is more information here than in the main paper. In particular, here we display results for D service, which were omitted before. Also we have about twice as many results in each table. For each service-time distribution, the first table gives the values of the simulation estimates of $\alpha(t)$. Then remaining two tables for each service-time distribution describe the results for 14 different time points. All the variance ratios and efficiencies are estimated, thus they should properly have hats, as in the four tables in the main paper; we did not correct them here.

The tables clearly show that the overlapping cycles does not help the standard estimator. The efficiencies $\widehat{\mathcal{E}}_{o,m}$ and $\widehat{\mathcal{F}}_{o,m}$ are consistently less than or equal to 1 or only slightly above 1. Hence we conclude that approach has little to offer.

The situation looks much better when we consider the residual-cycle estimators, but even then the performance is not uniformly good. The time t evidently must exceed a *threshold* for there to be any efficiency benefit. That threshold seems to occur when $0.40 \le \alpha(t) \le 0.45$, when $\alpha(t)$ is within between 20% and 10% of the limit $\alpha = 0.50$. When $\alpha(t) \ge 0.45$, we see consistent efficiency gains from the residual-cycle estimator and its overlapping-cycle and splitting refinements. When $\alpha(t) < 0.40$, the standard estimator is consistently more efficient. The observed threshold is only slightly greater than the mean cycle length $E[\tau_1] = 2.0$ for M service, but considerably greater for H_2 service, and somewhat less for D service. Our limited experimentation suggests that the threshold might be approximately at the mean of the equilibrium residual-lifetime, $E[T_+(\infty)] = E[\tau_1^2]/2E[\tau_1] = E[\tau_1](SCV(\tau_1) + 1)/2$, which equals 1 for D, 2 for M and 10 for H_2, but that remains to be investigated.

We see dramatic improvement as t increases. From the perspective of the variance ratio, especially $\widehat{\mathcal{V}}^r_{o,s}$ for the combined estimator, the variance reduction looks very impressive for larger values of t, e.g., $\widehat{\mathcal{V}}^r_{o,100,s,100} = 326,000$ for the $M/M/1/0$ model with $t = 1024$. However, the story is less impressive when we look at the efficiencies, because the variance reduction is obtained at the expense of some computational effort. Nevertheless, $\widehat{\mathcal{F}}^r_{o,30,s,30} = 35,000$ for the $M/M/1/0$ model with $t = 1024$. However, for the largest values of t, we might already be confident that $\alpha(t)$ will be close to the limit α. It may be best to judge the performance by looking at cases in which $\alpha(t) \approx 0.495$, which is within 1% of the

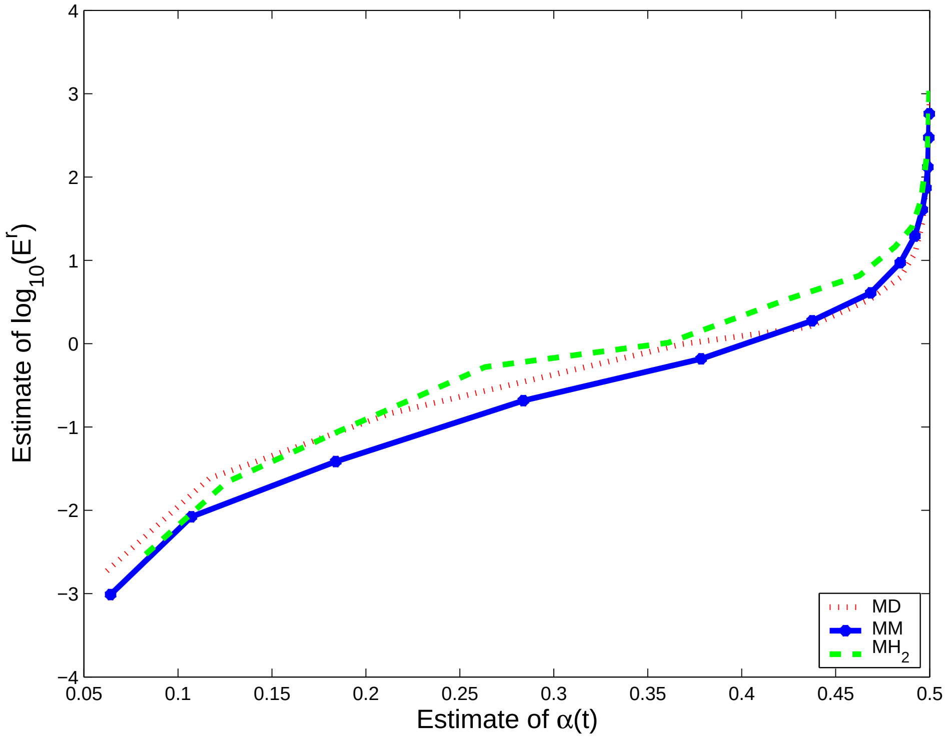

Fig. 21. The logarithm (base 10) of the cpu-time efficiency of the residual-cycle estimator, $\widehat{\mathcal{E}}^r(t)$, as a function of the estimated value $\hat{\alpha}^r(t)$.

limit. Then the efficiency gains can be considered from the cases $t = 64$ for $M/M/1/0$ and $t = 512$ for $M/H_2/1/0$; the maximum efficiency gains, considering only $\widehat{\mathcal{E}}$ is 354 for M service and 125 for H_2 service.

Consistent with our theoretical analysis of splitting in Sections 6 and 8, we see that the benefits of splitting is not monotone in the number p of splittings for each t, and the optimal level of splitting, $p^*(t)$ evidently is increasing in t; $p = 10$ seems best for small t, while $p = 30$ seems best for medium t, and $p = 100$ seems best for large t. In contrast, the residual-cycle plus overlapping-cycle efficiencies $\widehat{\mathcal{E}}^r_{o,m}$ and $\widehat{\mathcal{F}}^r_{o,m}$ seem to increase in m when t is not too small.

3.3.0.1 *Uniformity of Effectiveness.*. The tables show that the efficiency gains for the H_2 service time occur for much larger values of t than for the other two service-time distributions. However, the time-dependent mean $\alpha(t)$ approaches the common limit $\alpha = 0.5$ much more slowly for the H_2 service-time distribution. We find a fascinating uniformity of effectiveness over time if instead of looking at the efficiency as a function of time, we look at the efficiency as a function of $\alpha(t)$.

We also found it informative to plot the logarithm of the efficiency, with base 10. The logarithm with base 10 shows the order of magnitude effect; a value k corresponds to 10^k or k orders of magnitude improvement. To see the uniformity, we plotted the logarithm (base 10) of all the efficiency functions $\widehat{\mathcal{E}}(t)$ versus $\alpha(t)$ for the time points t considered.

By uniformity of effectiveness we mean that the three curves for the three service times tend to fall on top of each other. They all show steady improvement as t increases (with the exception of the overlapping-cycle modification of the standard estimator). A sample of two of those plots are shown here in Figures 21 and 22 below: for $\widehat{\mathcal{E}}^r(t)$ and $\widehat{\mathcal{E}}^r_{o,30,s,30}(t)$, respectively. More figures are shown in another part of this supplement.

We found that it is even more revealing to use a log scale on both axes; i.e., to plot the logarithm of the efficiency against $-\log_{10}(1 - (\hat{\alpha}^r_{o,30,s,30}(t)/\alpha))$, with all logarithms to base 10. We then see a striking *linear relationship* when t is sufficiently large or, more precisely, when $\hat{\alpha}^r_{o,30,s,30}(t)$ is sufficiently close to $\alpha = 0.5$. We illustrate in Figure 23. The efficiency curves for the other estimators are similar.

The values 1 and 2 on the horizontal x axis of Figure 23 mean that $\alpha(t)$ differs from the limit α by

$10^{-1}\% = 10\%$ and $10^{-2}\% = 1\%$, respectively, while the values 1 and 2 on the vertical y axis mean that the estimated efficiency $\widehat{\mathcal{E}}^r_{o,30,s,30}(t)$ is $10^1 = 10$ and $10^2 = 100$, respectively. By definition, the threshold for efficiency gain occurs at 0 on the y axis. The linear relationship might be said to begin when $-\log_{10}(1 - (\hat{\alpha}^r_{o,30,s,30}(t)/\alpha)) = 0.5$, but certainly has begun when $-\log_{10}(1 - (\hat{\alpha}^r_{o,30,s,30}(t)/\alpha)) = 1.0$, when $\alpha(t)$ is within 10% of α.

3.3.0.2 *Implemented Efficiency Versus Run-Length Efficiency.*. The experiments nicely illustrate differences between different notions of "efficiency." First, we expect to see that the run-length and cpu-time efficiencies will be less than the variance ratios, because the variance reduction is usually obtained at some run-length and cpu-time cost, but sometimes that cost is small. To illustrate, first consider the largest time, $t = 1024$ for the $M/M/1/0$ model in Table 0**??**. Since a mean cycle time $E[\tau_1] = 2.0$ is negligible compared to $t = 1024$, it should not be surprising that $\widehat{\mathcal{F}}^r \approx \widehat{\mathcal{E}}^r \approx \widehat{\mathcal{V}}^r \approx 570$. In this case, the variance reduction is fully realized without run-length or cpu-time cost.

But in other cases, we expect to see a degradation of efficiency gain as we go from the the variance ratio to the run-length and cpu-time efficiency measures. For example, for that same time $t = 1024$ with M service, when we turn to the overlapping-cycle residual-cycle estimator $\hat{\alpha}^r_o$, we see that there is a bit more run-length cost, because now the m overlapping cycles make up a larger portion of the time $t = 1024$. Accordingly, we are not surprised to see that $\widehat{\mathcal{F}}^r_{o,100} \approx 24,300 < 29,100 \approx \widehat{\mathcal{V}}^r_o$.

The experiments also nicely illustrate differences between the "theoretical" run-length efficiency and the "implemented" cpu-time efficiency. Even though $\widehat{\mathcal{F}}^r_{o,100} \approx 24,300$, the cpu-time efficiency is only $\widehat{\mathcal{E}}^r_{o,100} \approx 4,600$, a substantial difference. That occurs because it requires extra computation to implement the refined overlapping-cycle estimator. The much higher run-length efficiency indicates what might possibly be gained from a more efficient implementation, but the cpu-time efficiency is what was actually realized. The main point is to recognize that it is important to substantiate efficiency gains through actual implementation.

There other interesting subtle differences as well. Consider the overlapping-cycle modification of the standard estimator, $\hat{\alpha}_{o,r}$ for M service at the smallest time point, $t = 0.125$. The variance ratios

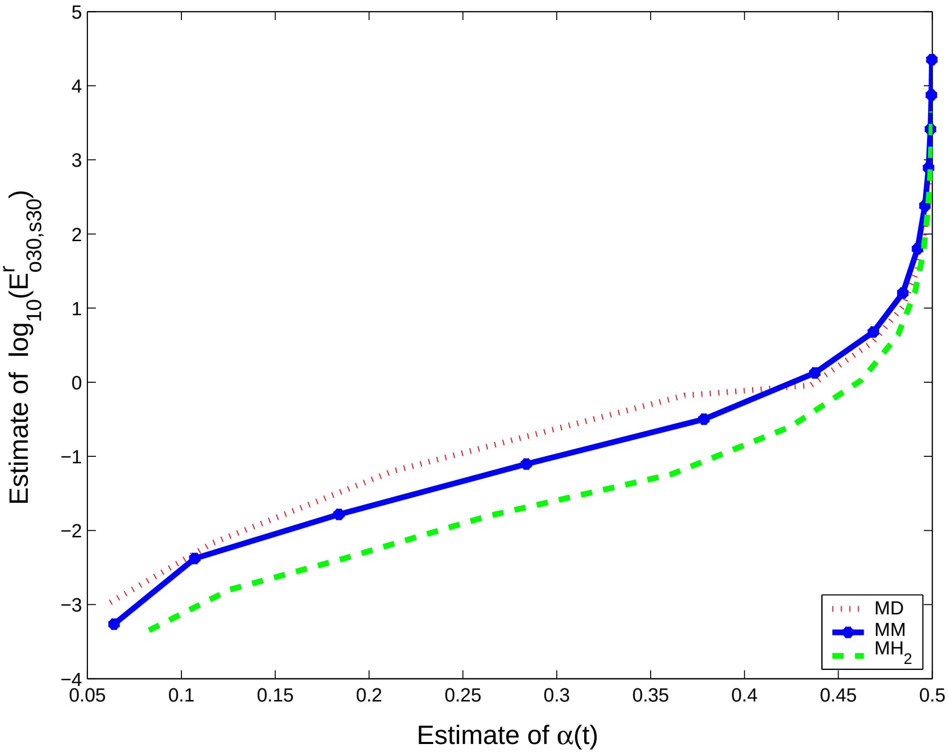

Fig. 22. The logarithm (base 10) of the cpu-time efficiency of the combined estimator, $\widehat{\mathcal{E}}^r_{o,30,s,30}(t)$, as a function of the estimated value $\hat{\alpha}^r_{o,30,s,30}(t)$.

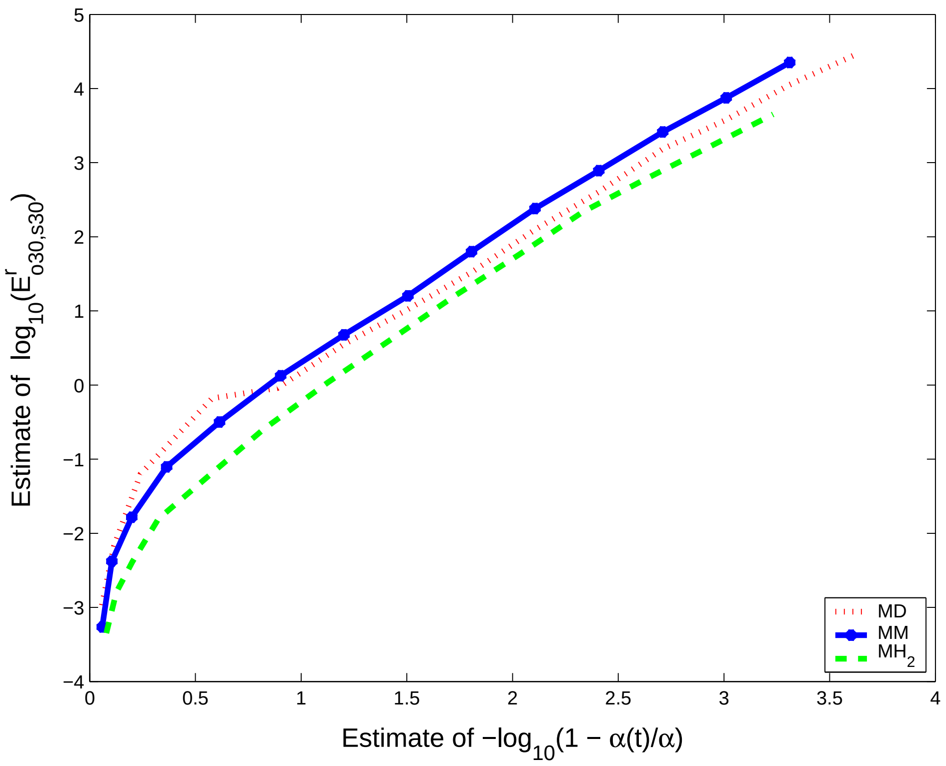

Fig. 23. The logarithm (base 10) of the cpu-time efficiency of the combined estimator, $\widehat{\mathcal{E}}^r_{o,30,s,30}(t)$, as a function of minus the logarithm (base 10) of the estimated $1-(\hat{\alpha}^r_{o,30,s,30}(t)/\alpha)$.

$\widehat{\mathcal{V}}_{o,m}$ show variance gains just slightly less than would occur with m multiple independent replications. However, the run-length efficiencies are very very low: $\widehat{\mathcal{F}}_{o,m} \approx 0.05$. That can be understood by recognizing that we waste a lot of time generating full cycles. The time t is only $1/16$ of the mean cycle time $E[\tau_1] = 2.0$, so we should expect the run-length efficiency loss we see.

What is startling to see, however, is that the cpu-time efficiencies show no such degradation: $\widehat{\mathcal{E}}_{o,m} \approx 1.0$. At first that suggests an error, but upon reflection it is not hard to understand. From the cpu-time perspective, there is relatively little waste during a cycle, because a full cycle is generated by generating only two random variables: the initial interarrival time and then the following service time. Every cycle, no matter how short, will require generating at least one of these variables.

Table IV. The estimates for $M/D/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	0.1250	0.2500	0.5000	1.0	2.0	4.0	8.0
$\hat{\alpha}(t)$	0.0549	0.1098	0.2271	0.3774	0.4433	0.4654	0.4885
$\hat{\alpha}_{o10}(t)$	0.0609	0.1184	0.2153	0.3678	0.4413	0.4664	0.4853
$\hat{\alpha}_{o30}(t)$	0.0599	0.1170	0.2139	0.3662	0.4354	0.4693	0.4840
$\hat{\alpha}_{o50}(t)$	0.0596	0.1142	0.2125	0.3684	0.4340	0.4701	0.4844
$\hat{\alpha}_{o100}(t)$	0.0598	0.1158	0.2113	0.3682	0.4351	0.4681	0.4836
$\hat{\alpha}^r(t)$	0.0978	0.1262	0.2417	0.3727	0.4304	0.4654	0.4836
$\hat{\alpha}^r_{o10}(t)$	0.0872	0.1141	0.2210	0.3713	0.4358	0.4696	0.4851
$\hat{\alpha}^r_{o30}(t)$	0.0884	0.1222	0.2070	0.3685	0.4377	0.4689	0.4848
$\hat{\alpha}^r_{o50}(t)$	0.0641	0.1036	0.2123	0.3676	0.4357	0.4683	0.4841
$\hat{\alpha}^r_{o100}(t)$	0.0573	0.1072	0.2112	0.3690	0.4344	0.4684	0.4840
$\hat{\alpha}^r_{s10}(t)$	0.1475	0.1158	0.2129	0.3644	0.4351	0.4685	0.4839
$\hat{\alpha}^r_{s30}(t)$	0.0960	0.0608	0.1965	0.3611	0.4372	0.4716	0.4847
$\hat{\alpha}^r_{s50}(t)$	0.0186	0.1214	0.2228	0.3626	0.4362	0.4678	0.4854
$\hat{\alpha}^r_{s100}(t)$	0.0468	0.0867	0.1990	0.3616	0.4428	0.4690	0.4846
$\hat{\alpha}^r_{o10,s10}(t)$	0.0712	0.1275	0.2146	0.3674	0.4357	0.4687	0.4845
$\hat{\alpha}^r_{o30,s30}(t)$	0.0535	0.1203	0.2096	0.3651	0.4359	0.4689	0.4844
$\hat{\alpha}^r_{o50,s50}(t)$	0.0712	0.1169	0.2135	0.3675	0.4350	0.4690	0.4845
$\hat{\alpha}^r_{o100,s100}(t)$	0.0622	0.1167	0.2129	0.3681	0.4354	0.4685	0.4844

t	16.0	32.0	64.0	128.0	256.0	512.0	1024.0
$\hat{\alpha}(t)$	0.4898	0.4966	0.4967	0.5001	0.4987	0.4990	0.5004
$\hat{\alpha}_{o10}(t)$	0.4931	0.4958	0.5000	0.4980	0.5003	0.4998	0.5000
$\hat{\alpha}_{o30}(t)$	0.4935	0.4965	0.4986	0.4992	0.5004	0.4996	0.5005
$\hat{\alpha}_{o50}(t)$	0.4905	0.4965	0.4961	0.4999	0.4994	0.5000	0.4994
$\hat{\alpha}_{o100}(t)$	0.4937	0.4965	0.4988	0.4986	0.4998	0.5004	0.5003
$\hat{\alpha}^r(t)$	0.4924	0.4960	0.4976	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o10}(t)$	0.4929	0.4962	0.4982	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o30}(t)$	0.4921	0.4961	0.4980	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o50}(t)$	0.4923	0.4961	0.4980	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o100}(t)$	0.4922	0.4961	0.4981	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{s10}(t)$	0.4919	0.4961	0.4981	0.4991	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{s30}(t)$	0.4928	0.4960	0.4980	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{s50}(t)$	0.4926	0.4961	0.4979	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{s100}(t)$	0.4925	0.4962	0.4981	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o10,s10}(t)$	0.4922	0.4962	0.4981	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o30,s30}(t)$	0.4922	0.4961	0.4981	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o50,s50}(t)$	0.4922	0.4961	0.4980	0.4990	0.4995	0.4998	0.4999
$\hat{\alpha}^r_{o100,s100}(t)$	0.4922	0.4961	0.4980	0.4990	0.4995	0.4998	0.4999

Table V. The efficiencies for $M/D/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	0.1250	0.2500	0.5000	1.0	2.0	4.0	8.0
$\hat{\alpha}^r_{o100,s100}(t)$	0.06223	0.11669	0.21287	0.36813	0.43538	0.46846	0.48444
$HW_{n,1}$	0.0115	0.0154	0.0203	0.0224	0.0143	0.0110	7.68e-003
$\mathcal{V}_{o10}$	8.9	8.8	10.4	9.7	6.4	4.8	2.6
$\mathcal{V}_{o30}$	27.6	27.6	33.3	29.6	21.1	13.6	7.6
$\mathcal{V}_{o50}$	46.0	50.8	53.7	52.8	34.0	23.5	11.2
$\mathcal{V}_{o100}$	89.3	91.5	103.2	104.3	64.9	42.3	24.5
$\mathcal{V}^r$	2.15e-003	0.0161	0.1453	0.9868	1.4	3.9	6.5
$\mathcal{V}^r_{o10}$	0.0214	0.1716	1.4	11.9	11.6	24.0	37.5
$\mathcal{V}^r_{o30}$	0.0674	0.5580	4.5	34.9	28.5	75.8	131.0
$\mathcal{V}^r_{o50}$	0.1076	0.8605	7.5	54.7	51.3	116.9	244.8
$\mathcal{V}^r_{o100}$	0.2127	1.7	15.4	118.4	102.1	216.5	431.9
$\mathcal{V}^r_{s10}$	0.0117	0.0713	0.5376	4.6	5.3	13.4	26.0
$\mathcal{V}^r_{s30}$	0.0179	0.0918	0.7007	5.3	6.8	16.9	32.7
$\mathcal{V}^r_{s50}$	0.0178	0.1082	0.7337	5.4	7.1	17.4	34.8
$\mathcal{V}^r_{s100}$	0.0206	0.1083	0.7740	5.5	8.1	18.4	36.1
$\mathcal{V}^r_{o10,s10}$	0.1159	0.7044	5.4	38.7	48.2	126.0	242.8
$\mathcal{V}^r_{o30,s30}$	0.5120	2.9	21.1	152.7	191.6	501.1	979.0
$\mathcal{V}^r_{o50,s50}$	1.1	5.2	35.4	259.0	330.0	838.6	1577.5
$\mathcal{V}^r_{o100,s100}$	2.1	11.0	81.0	500.7	668.9	1788.5	3379.4
$\mathcal{E}_{o10}$	1.1	1.1	1.4	1.4	1.0	0.9761	0.6696
$\mathcal{E}_{o30}$	1.2	1.4	1.5	1.5	1.3	0.9923	0.7918
$\mathcal{E}_{o50}$	1.3	1.4	1.5	1.5	1.2	1.1	0.7020
$\mathcal{E}_{o100}$	1.2	1.3	1.4	1.7	1.2	1.1	0.8087
$\mathcal{E}^r$	1.88e-003	0.0241	0.1453	0.9868	1.6	3.4	6.3
$\mathcal{E}^r_{o10}$	2.58e-003	0.0203	0.1832	1.6	2.1	4.7	9.2
$\mathcal{E}^r_{o30}$	2.95e-003	0.0248	0.1974	1.6	1.6	5.5	12.5
$\mathcal{E}^r_{o50}$	2.67e-003	0.0218	0.1861	1.6	1.8	5.2	14.6
$\mathcal{E}^r_{o100}$	2.76e-003	0.0224	0.1989	1.7	1.8	4.9	13.9
$\mathcal{E}^r_{s10}$	1.86e-003	0.0127	0.1195	1.3	1.6	5.7	15.5
$\mathcal{E}^r_{s30}$	1.09e-003	6.52e-003	0.0684	0.6578	0.9125	3.4	8.5
$\mathcal{E}^r_{s50}$	6.24e-004	4.80e-003	0.0419	0.4289	0.6494	2.3	6.2
$\mathcal{E}^r_{s100}$	3.78e-004	2.39e-003	0.0206	0.2190	0.3416	1.1	3.5
$\mathcal{E}^r_{o10,s10}$	2.03e-003	0.0132	0.1262	1.3	1.6	6.3	18.9
$\mathcal{E}^r_{o30,s30}$	1.07e-003	6.78e-003	0.0634	0.6626	0.9076	3.5	10.6
$\mathcal{E}^r_{o50,s50}$	8.08e-004	4.43e-003	0.0394	0.4222	0.5982	2.2	6.5
$\mathcal{E}^r_{o100,s100}$	4.13e-004	2.37e-003	0.0241	0.2119	0.3145	1.2	3.6
$\mathcal{F}_{o10}$	0.0610	0.1196	0.2837	0.5092	0.6439	0.8646	0.8037
$\mathcal{F}_{o30}$	0.0593	0.1183	0.2861	0.4983	0.7014	0.8814	0.9173
$\mathcal{F}_{o50}$	0.0584	0.1296	0.2729	0.5336	0.6782	0.9220	0.8443
$\mathcal{F}_{o100}$	0.0563	0.1155	0.2594	0.5242	0.6486	0.8368	0.9498
$\mathcal{F}^r$	1.34e-004	2.01e-003	0.0355	0.4847	0.8930	3.0	5.7
$\mathcal{F}^r_{o10}$	1.33e-004	2.15e-003	0.0342	0.5912	1.1	4.1	11.0
$\mathcal{F}^r_{o30}$	1.40e-004	2.32e-003	0.0375	0.5816	0.9279	4.8	15.6
$\mathcal{F}^r_{o50}$	1.34e-004	2.16e-003	0.0375	0.5461	1.0	4.5	18.3
$\mathcal{F}^r_{o100}$	1.33e-004	2.14e-003	0.0385	0.5909	1.0	4.3	16.7
$\mathcal{F}^r_{s10}$	7.65e-005	1.00e-003	0.0173	0.4271	0.7208	3.2	10.1
$\mathcal{F}^r_{s30}$	3.94e-005	4.43e-004	7.84e-003	0.1753	0.3324	1.6	5.7
$\mathcal{F}^r_{s50}$	2.39e-005	3.08e-004	4.79e-003	0.1069	0.2192	1.1	3.8
$\mathcal{F}^r_{s100}$	1.38e-005	1.56e-004	2.64e-003	0.0575	0.1257	0.5781	2.1
$\mathcal{F}^r_{o10,s10}$	7.66e-005	9.88e-004	0.0173	0.3535	0.7166	3.7	13.8
$\mathcal{F}^r_{o30,s30}$	3.79e-005	4.61e-004	7.75e-003	0.1665	0.3295	1.7	6.8
$\mathcal{F}^r_{o50,s50}$	2.81e-005	2.95e-004	4.70e-003	0.1023	0.2061	1.1	4.0
$\mathcal{F}^r_{o100,s100}$	1.43e-005	1.57e-004	2.70e-003	0.0496	0.1052	0.5689	2.1

Table VI. The efficiencies for $M/D/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	16.0	32.0	64.0	128.0	256.0	512.0	1024.0
$\hat{\alpha}^r_{o100,s100}(t)$	0.49224	0.49607	0.49804	0.49903	0.49951	0.49976	0.49988
$HW_{n,1}$	5.44e-003	3.98e-003	2.80e-003	1.95e-003	1.37e-003	9.62e-004	6.80e-004
$\mathcal{V}_{o10}$	1.6	1.3	1.2	1.1	1.0	1.0	1.0
$\mathcal{V}_{o30}$	4.1	2.6	1.4	1.2	1.0	1.1	0.9938
$\mathcal{V}_{o50}$	6.0	3.8	2.1	1.3	1.1	1.1	1.0
$\mathcal{V}_{o100}$	12.7	6.9	3.5	2.1	1.4	1.1	1.0
$\mathcal{V}^r$	12.6	28.1	82.9	129.9	245.1	432.3	825.0
$\mathcal{V}^r_{o10}$	73.2	209.4	358.0	806.7	1597.5	2891.8	5288.5
$\mathcal{V}^r_{o30}$	286.7	505.9	1244.6	2298.2	4262.3	9738.1	18747.2
$\mathcal{V}^r_{o50}$	456.4	941.8	1866.1	3481.7	6780.3	12865.7	28652.3
$\mathcal{V}^r_{o100}$	921.9	2134.0	3792.8	6837.7	13463.2	26476.5	61907.9
$\mathcal{V}^r_{s10}$	52.5	112.9	226.2	397.5	834.0	1493.6	3304.2
$\mathcal{V}^r_{s30}$	68.5	138.4	295.4	516.6	976.6	2020.0	4280.1
$\mathcal{V}^r_{s50}$	70.8	152.5	283.8	561.1	1127.2	2174.7	4223.5
$\mathcal{V}^r_{s100}$	73.6	164.3	318.4	561.9	1099.3	2240.6	4720.9
$\mathcal{V}^r_{o10,s10}$	475.0	1020.1	2034.6	4020.0	7348.9	15149.3	31381.1
$\mathcal{V}^r_{o30,s30}$	1860.2	3902.2	7304.4	16698.9	27487.1	59564.2	124632.2
$\mathcal{V}^r_{o50,s50}$	3462.3	7254.0	14427.1	27986.8	56515.1	106212.3	200746.4
$\mathcal{V}^r_{o100,s100}$	6927.2	15826.2	30398.3	53868.1	114977.9	221634.2	449371.0
$\mathcal{E}_{o10}$	0.4982	0.4874	0.5238	0.5164	0.5128	0.5128	0.5261
$\mathcal{E}_{o30}$	0.5866	0.4527	0.3070	0.2883	0.2713	0.2935	0.2751
$\mathcal{E}_{o50}$	0.5156	0.4370	0.3041	0.2103	0.1967	0.1959	0.1959
$\mathcal{E}_{o100}$	0.5676	0.4282	0.2695	0.1882	0.1322	0.1166	0.1082
$\mathcal{E}^r$	12.3	28.1	79.5	130.1	241.0	436.9	827.3
$\mathcal{E}^r_{o10}$	22.5	76.3	161.4	378.6	799.8	1485.4	2698.7
$\mathcal{E}^r_{o30}$	38.6	87.7	264.3	555.9	1105.3	2672.5	5164.5
$\mathcal{E}^r_{o50}$	37.8	104.8	264.4	567.2	1192.8	2386.5	5414.4
$\mathcal{E}^r_{o100}$	39.7	129.3	293.4	610.6	1308.7	2737.4	6497.6
$\mathcal{E}^r_{s10}$	33.2	90.4	192.6	370.8	771.2	1411.6	3126.5
$\mathcal{E}^r_{s30}$	29.5	78.1	188.7	408.4	849.9	1850.4	4017.5
$\mathcal{E}^r_{s50}$	19.1	61.1	154.3	383.1	899.0	1932.7	3871.6
$\mathcal{E}^r_{s100}$	12.2	43.9	122.0	311.4	774.5	1827.8	4134.1
$\mathcal{E}^r_{o10,s10}$	53.9	177.6	557.0	1404.2	3058.4	7158.7	15606.9
$\mathcal{E}^r_{o30,s30}$	33.6	123.8	402.2	1527.5	3777.7	11226.6	28123.6
$\mathcal{E}^r_{o50,s50}$	23.9	90.7	325.7	1119.2	3718.7	10576.0	26321.2
$\mathcal{E}^r_{o100,s100}$	12.6	52.9	187.9	619.1	2372.4	7580.8	23732.7
$\mathcal{F}_{o10}$	0.7609	0.8578	0.9074	0.9355	0.9733	0.9735	1.0
$\mathcal{F}_{o30}$	0.8830	0.9309	0.7350	0.8181	0.8543	1.0	0.9407
$\mathcal{F}_{o50}$	0.8369	0.9416	0.8305	0.7230	0.8034	0.8862	0.9446
$\mathcal{F}_{o100}$	0.9498	0.9553	0.8444	0.8131	0.7627	0.8098	0.8579
$\mathcal{F}^r$	11.7	27.0	81.4	128.7	244.0	431.2	824.0
$\mathcal{F}^r_{o10}$	33.1	130.2	275.5	702.3	1486.1	2786.8	5190.2
$\mathcal{F}^r_{o30}$	60.9	177.8	645.8	1572.1	3465.4	8734.8	17717.8
$\mathcal{F}^r_{o50}$	63.1	229.8	732.8	1963.5	4888.3	10778.8	26119.5
$\mathcal{F}^r_{o100}$	68.5	294.6	921.5	2674.8	7581.1	19064.1	51819.5
$\mathcal{F}^r_{s10}$	29.4	80.4	189.7	362.4	794.6	1457.9	3265.0
$\mathcal{F}^r_{s30}$	20.3	64.1	187.5	398.4	853.4	1881.2	4130.2
$\mathcal{F}^r_{s50}$	14.1	51.2	145.6	380.9	903.2	1937.2	3973.2
$\mathcal{F}^r_{s100}$	8.1	33.5	107.0	285.2	740.6	1801.2	4208.9
$\mathcal{F}^r_{o10,s10}$	51.7	199.2	665.5	1976.7	4843.0	12059.0	27803.6
$\mathcal{F}^r_{o30,s30}$	25.6	106.0	385.6	1685.5	5037.3	18420.9	58751.3
$\mathcal{F}^r_{o50,s50}$	17.4	72.7	285.4	1091.3	4232.6	14827.6	49186.1
$\mathcal{F}^r_{o100,s100}$	8.8	40.1	153.6	542.9	2290.6	8658.2	33778.4

Table VII. The estimates for $M/M/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	0.1250	0.2500	0.5000	1.0	2.0	4.0	8.0
$\hat{\alpha}(t)$	0.0489	0.1069	0.1891	0.2997	0.3674	0.4294	0.4742
$\hat{\alpha}_{o10}(t)$	0.0579	0.1108	0.1850	0.2857	0.3723	0.4363	0.4643
$\hat{\alpha}_{o30}(t)$	0.0594	0.1063	0.1841	0.2825	0.3767	0.4419	0.4676
$\hat{\alpha}_{o50}(t)$	0.0566	0.1083	0.1842	0.2846	0.3812	0.4366	0.4689
$\hat{\alpha}_{o100}(t)$	0.0583	0.1060	0.1852	0.2850	0.3803	0.4370	0.4671
$\hat{\alpha}^r(t)$	-4.00e-003	0.0730	0.1858	0.2550	0.3771	0.4396	0.4687
$\hat{\alpha}^r_{o10}(t)$	0.0795	0.0408	0.1907	0.2805	0.3750	0.4341	0.4681
$\hat{\alpha}^r_{o30}(t)$	0.0328	0.1184	0.1934	0.2879	0.3746	0.4389	0.4693
$\hat{\alpha}^r_{o50}(t)$	0.0904	0.1019	0.1789	0.2847	0.3762	0.4382	0.4684
$\hat{\alpha}^r_{o100}(t)$	0.0576	0.1113	0.1749	0.2828	0.3769	0.4382	0.4685
$\hat{\alpha}^r_{s10}(t)$	7.57e-003	0.1382	0.1867	0.2843	0.3752	0.4355	0.4670
$\hat{\alpha}^r_{s30}(t)$	0.0556	0.0977	0.1936	0.2832	0.3738	0.4368	0.4672
$\hat{\alpha}^r_{s50}(t)$	0.0641	0.1038	0.1797	0.2943	0.3821	0.4395	0.4677
$\hat{\alpha}^r_{s100}(t)$	0.0579	0.1221	0.1734	0.2859	0.3768	0.4381	0.4690
$\hat{\alpha}^r_{o10,s10}(t)$	0.0537	0.1095	0.1806	0.2832	0.3769	0.4386	0.4688
$\hat{\alpha}^r_{o30,s30}(t)$	0.0563	0.1012	0.1837	0.2834	0.3780	0.4375	0.4684
$\hat{\alpha}^r_{o50,s50}(t)$	0.0544	0.1152	0.1866	0.2837	0.3780	0.4373	0.4686
$\hat{\alpha}^r_{o100,s100}(t)$	0.0642	0.1071	0.1839	0.2838	0.3784	0.4375	0.4686

t	16.0	32.0	64.0	128.0	256.0	512.0	1024.0
$\hat{\alpha}(t)$	0.4859	0.4932	0.4946	0.5000	0.4972	0.5001	0.4997
$\hat{\alpha}_{o10}(t)$	0.4829	0.4953	0.4963	0.4966	0.4985	0.4993	0.4991
$\hat{\alpha}_{o30}(t)$	0.4806	0.4933	0.4957	0.4990	0.4988	0.4990	0.4998
$\hat{\alpha}_{o50}(t)$	0.4840	0.4914	0.4946	0.4967	0.5010	0.5006	0.4999
$\hat{\alpha}_{o100}(t)$	0.4837	0.4931	0.4956	0.4984	0.4988	0.4996	0.4996
$\hat{\alpha}^r(t)$	0.4858	0.4915	0.4960	0.4981	0.4991	0.4995	0.4998
$\hat{\alpha}^r_{o10}(t)$	0.4846	0.4921	0.4958	0.4981	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o30}(t)$	0.4847	0.4922	0.4960	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o50}(t)$	0.4840	0.4922	0.4961	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o100}(t)$	0.4846	0.4922	0.4961	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{s10}(t)$	0.4847	0.4915	0.4963	0.4982	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{s30}(t)$	0.4847	0.4926	0.4963	0.4981	0.4991	0.4995	0.4998
$\hat{\alpha}^r_{s50}(t)$	0.4843	0.4920	0.4963	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{s100}(t)$	0.4849	0.4923	0.4962	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o10,s10}(t)$	0.4846	0.4921	0.4961	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o30,s30}(t)$	0.4845	0.4922	0.4961	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o50,s50}(t)$	0.4844	0.4921	0.4961	0.4980	0.4990	0.4995	0.4998
$\hat{\alpha}^r_{o100,s100}(t)$	0.4844	0.4922	0.4961	0.4980	0.4990	0.4995	0.4998

Table VIII. The efficiencies for $M/M/1/1 : n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	0.1250	0.2500	0.5000	1.0	2.0	4.0	8.0
$\hat{\alpha}^r_{o100,s100}(t)$	0.06419	0.10713	0.18395	0.28376	0.37839	0.43748	0.46863
$HW_{n,1}$	0.0106	0.0151	0.0177	0.0197	0.0171	0.0138	0.0103
$\mathcal{V}_{o10}$	7.7	9.2	9.2	8.6	6.5	4.6	2.8
$\mathcal{V}_{o30}$	23.9	29.0	27.7	27.8	18.0	11.8	6.4
$\mathcal{V}_{o50}$	39.7	49.7	46.0	44.9	30.0	20.3	11.2
$\mathcal{V}_{o100}$	84.4	108.0	87.7	82.3	64.5	39.1	23.1
$\mathcal{V}^r$	9.75e-004	6.97e-003	0.0462	0.2383	0.6580	1.9	4.1
$\mathcal{V}^r_{o10}$	8.89e-003	0.0703	0.3353	1.5	3.9	9.6	21.7
$\mathcal{V}^r_{o30}$	0.0304	0.2172	1.0	4.6	11.2	25.2	63.4
$\mathcal{V}^r_{o50}$	0.0459	0.3596	1.7	7.2	17.9	43.0	87.4
$\mathcal{V}^r_{o100}$	0.0951	0.7737	3.6	14.4	37.6	80.3	192.7
$\mathcal{V}^r_{s10}$	6.48e-003	0.0431	0.2153	0.9953	3.1	7.6	18.4
$\mathcal{V}^r_{s30}$	0.0128	0.0672	0.2931	1.3	4.2	10.4	23.8
$\mathcal{V}^r_{s50}$	0.0131	0.0715	0.3170	1.4	4.4	11.5	24.5
$\mathcal{V}^r_{s100}$	0.0157	0.0847	0.3438	1.6	4.5	12.4	26.5
$\mathcal{V}^r_{o10,s10}$	0.0627	0.4183	2.0	8.3	24.1	64.7	140.0
$\mathcal{V}^r_{o30,s30}$	0.3329	2.0	9.1	33.5	99.8	264.0	590.7
$\mathcal{V}^r_{o50,s50}$	0.6820	3.8	14.9	68.0	169.1	433.1	1039.4
$\mathcal{V}^r_{o100,s100}$	1.5	8.2	31.5	133.8	379.7	929.0	2302.3
$\mathcal{E}_{o10}$	0.8762	1.3	1.0	1.1	1.0	1.0	0.8067
$\mathcal{E}_{o30}$	0.9603	1.4	1.1	1.2	1.0	0.9798	0.7213
$\mathcal{E}_{o50}$	0.9627	1.4	1.1	1.2	1.0	1.0	0.7889
$\mathcal{E}_{o100}$	1.0	1.5	1.0	1.1	1.1	1.0	0.8473
$\mathcal{E}^r$	9.75e-004	8.36e-003	0.0385	0.2077	0.6580	1.9	4.1
$\mathcal{E}^r_{o10}$	9.85e-004	9.17e-003	0.0364	0.1821	0.5821	2.1	6.1
$\mathcal{E}^r_{o30}$	1.16e-003	9.64e-003	0.0387	0.2025	0.5998	2.0	7.0
$\mathcal{E}^r_{o50}$	1.07e-003	9.79e-003	0.0392	0.1925	0.5870	2.0	6.1
$\mathcal{E}^r_{o100}$	1.11e-003	0.0104	0.0420	0.1940	0.6211	1.9	6.9
$\mathcal{E}^r_{s10}$	9.00e-004	7.16e-003	0.0326	0.1840	0.7315	2.3	8.2
$\mathcal{E}^r_{s30}$	6.21e-004	4.11e-003	0.0156	0.0907	0.3815	1.4	5.0
$\mathcal{E}^r_{s50}$	3.86e-004	2.64e-003	0.0103	0.0603	0.2497	1.0	3.3
$\mathcal{E}^r_{s100}$	2.34e-004	1.58e-003	5.70e-003	0.0342	0.1317	0.5395	1.9
$\mathcal{E}^r_{o10,s10}$	8.73e-004	7.26e-003	0.0308	0.1633	0.6289	2.6	8.8
$\mathcal{E}^r_{o30,s30}$	5.44e-004	4.19e-003	0.0165	0.0788	0.3168	1.3	4.8
$\mathcal{E}^r_{o50,s50}$	4.05e-004	2.81e-003	9.83e-003	0.0584	0.1952	0.8033	3.1
$\mathcal{E}^r_{o100,s100}$	2.28e-004	1.53e-003	5.12e-003	0.0290	0.1119	0.4356	1.8
$\mathcal{F}_{o10}$	0.0532	0.1242	0.2485	0.4511	0.6559	0.8273	0.8714
$\mathcal{F}_{o30}$	0.0515	0.1242	0.2374	0.4691	0.6004	0.7682	0.7699
$\mathcal{F}_{o50}$	0.0507	0.1265	0.2333	0.4525	0.6029	0.7983	0.8424
$\mathcal{F}_{o100}$	0.0533	0.1360	0.2217	0.4141	0.6490	0.7716	0.8953
$\mathcal{F}^r$	5.99e-005	8.27e-004	0.0107	0.0959	0.3716	1.4	3.4
$\mathcal{F}^r_{o10}$	5.52e-005	8.83e-004	8.29e-003	0.0711	0.3576	1.6	6.3
$\mathcal{F}^r_{o30}$	6.35e-005	9.06e-004	8.54e-003	0.0757	0.3620	1.6	7.5
$\mathcal{F}^r_{o50}$	5.75e-005	8.97e-004	8.66e-003	0.0718	0.3535	1.7	6.5
$\mathcal{F}^r_{o100}$	5.96e-005	9.65e-004	9.07e-003	0.0721	0.3745	1.6	7.4
$\mathcal{F}^r_{s10}$	4.27e-005	5.92e-004	6.22e-003	0.0598	0.3649	1.6	6.4
$\mathcal{F}^r_{s30}$	2.83e-005	3.11e-004	2.88e-003	0.0275	0.1802	0.8451	3.6
$\mathcal{F}^r_{s50}$	1.73e-005	1.99e-004	1.88e-003	0.0177	0.1122	0.5796	2.4
$\mathcal{F}^r_{s100}$	1.04e-005	1.17e-004	1.03e-003	9.97e-003	0.0594	0.3198	1.3
$\mathcal{F}^r_{o10,s10}$	4.16e-005	5.77e-004	5.95e-003	0.0517	0.3062	1.6	6.9
$\mathcal{F}^r_{o30,s30}$	2.44e-005	3.14e-004	2.99e-003	0.0236	0.1447	0.7720	3.5
$\mathcal{F}^r_{o50,s50}$	1.80e-005	2.09e-004	1.76e-003	0.0172	0.0889	0.4589	2.2
$\mathcal{F}^r_{o100,s100}$	1.01e-005	1.13e-004	9.33e-004	8.51e-003	0.0500	0.2469	1.2

Table IX. The efficiencies for $M/M/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	16.0	32.0	64.0	128.0	256.0	512.0	1024.0
$\hat{\alpha}^r_{o100,s100}(t)$	0.48437	0.49218	0.49608	0.49805	0.49903	0.49951	0.49976
$HW_{n,1}$	7.50e-003	5.53e-003	3.94e-003	2.72e-003	1.97e-003	1.34e-003	9.80e-004
$\mathcal{V}_{o10}$	1.6	1.2	1.1	1.1	1.2	0.9684	1.1
$\mathcal{V}_{o30}$	3.7	2.3	1.5	1.2	1.2	1.0	1.0
$\mathcal{V}_{o50}$	6.5	3.5	2.0	1.4	1.2	0.9915	1.1
$\mathcal{V}_{o100}$	11.7	6.6	3.5	2.1	1.4	1.2	1.0
$\mathcal{V}^r$	9.3	19.0	39.9	73.1	130.2	292.8	567.3
$\mathcal{V}^r_{o10}$	41.5	88.1	188.4	328.6	667.5	1326.3	3085.5
$\mathcal{V}^r_{o30}$	124.9	246.2	520.8	997.8	2167.3	3918.3	8682.6
$\mathcal{V}^r_{o50}$	204.3	467.6	844.1	1553.7	3284.5	6550.0	12711.1
$\mathcal{V}^r_{o100}$	349.0	834.5	1814.3	3248.3	6583.2	12808.2	29094.5
$\mathcal{V}^r_{s10}$	37.5	78.7	174.3	316.9	666.0	1243.5	2587.2
$\mathcal{V}^r_{s30}$	51.0	106.6	221.9	411.2	870.8	1568.4	3504.7
$\mathcal{V}^r_{s50}$	54.4	118.8	236.4	454.2	969.3	1772.3	3608.8
$\mathcal{V}^r_{s100}$	56.5	119.8	253.3	476.2	1042.1	1905.0	3934.9
$\mathcal{V}^r_{o10,s10}$	287.9	646.5	1363.0	2399.1	4823.3	8886.1	19671.0
$\mathcal{V}^r_{o30,s30}$	1174.5	2459.2	5268.9	9910.2	19407.2	38196.4	83507.7
$\mathcal{V}^r_{o50,s50}$	2057.4	4621.1	9361.8	18885.0	37271.4	75825.7	137370.2
$\mathcal{V}^r_{o100,s100}$	4440.1	10185.2	18866.9	38161.5	77404.4	160977.2	326297.8
$\mathcal{E}_{o10}$	0.5821	0.5821	0.5838	0.6226	0.7404	0.6059	0.6959
$\mathcal{E}_{o30}$	0.5962	0.5248	0.4052	0.3883	0.4213	0.3805	0.3966
$\mathcal{E}_{o50}$	0.6643	0.5262	0.3872	0.3189	0.3154	0.2660	0.3002
$\mathcal{E}_{o100}$	0.6263	0.5382	0.3766	0.2802	0.2014	0.1843	0.1719
$\mathcal{E}^r$	9.3	19.6	40.5	74.0	131.2	296.7	573.7
$\mathcal{E}^r_{o10}$	14.8	41.1	99.6	190.8	406.1	828.7	1935.4
$\mathcal{E}^r_{o30}$	19.5	55.4	146.1	321.2	777.7	1475.8	3362.7
$\mathcal{E}^r_{o50}$	20.4	69.4	160.7	345.6	831.6	1749.5	3521.7
$\mathcal{E}^r_{o100}$	17.9	67.0	191.0	411.4	953.3	1978.3	4629.0
$\mathcal{E}^r_{s10}$	18.2	58.8	144.7	280.1	636.5	1184.9	2464.0
$\mathcal{E}^r_{s30}$	15.3	50.3	137.8	278.7	752.7	1418.1	3301.3
$\mathcal{E}^r_{s50}$	11.4	41.0	116.8	291.3	767.1	1512.7	3318.8
$\mathcal{E}^r_{s100}$	6.6	24.6	83.1	226.1	676.1	1437.8	3484.3
$\mathcal{E}^r_{o10,s10}$	28.8	110.5	353.7	862.5	2267.9	4706.5	11666.1
$\mathcal{E}^r_{o30,s30}$	16.0	63.0	240.2	779.1	2594.8	7476.1	22412.0
$\mathcal{E}^r_{o50,s50}$	10.4	44.6	165.2	605.9	2175.9	7364.6	19997.2
$\mathcal{E}^r_{o100,s100}$	5.7	25.4	87.8	335.0	1290.6	4892.9	16906.9
$\mathcal{F}_{o10}$	0.7590	0.7958	0.8544	0.9575	1.1	0.9354	1.1
$\mathcal{F}_{o30}$	0.8097	0.8152	0.8035	0.8141	0.9520	0.9010	0.9712
$\mathcal{F}_{o50}$	0.9172	0.8585	0.7969	0.7895	0.8944	0.8326	0.9807
$\mathcal{F}_{o100}$	0.8695	0.9185	0.8580	0.8430	0.7717	0.8410	0.8792
$\mathcal{F}^r$	8.5	18.1	39.0	72.2	129.4	292.0	566.5
$\mathcal{F}^r_{o10}$	18.7	54.6	144.9	285.2	620.1	1277.9	3027.9
$\mathcal{F}^r_{o30}$	26.4	86.1	269.1	681.7	1759.0	3511.3	8204.8
$\mathcal{F}^r_{o50}$	28.3	114.0	330.7	873.6	2363.6	5487.8	11582.2
$\mathcal{F}^r_{o100}$	25.9	115.3	441.4	1267.3	3702.3	9221.3	24344.9
$\mathcal{F}^r_{s10}$	19.2	53.6	140.9	282.9	629.5	1208.4	2549.6
$\mathcal{F}^r_{s30}$	13.3	44.0	130.0	303.6	740.0	1441.9	3355.3
$\mathcal{F}^r_{s50}$	9.6	35.7	108.2	288.0	750.0	1542.8	3362.8
$\mathcal{F}^r_{s100}$	5.4	21.1	75.3	219.7	657.4	1476.8	3428.0
$\mathcal{F}^r_{o10,s10}$	26.9	110.8	399.4	1086.8	3006.2	6831.4	17085.7
$\mathcal{F}^r_{o30,s30}$	13.6	56.4	236.4	848.9	3056.8	10402.2	35830.8
$\mathcal{F}^r_{o50,s50}$	8.7	38.9	156.4	620.7	2364.2	9043.3	29329.4
$\mathcal{F}^r_{o100,s100}$	4.7	21.6	80.0	322.1	1294.2	5292.4	20802.6

Table X. The estimates for $M/H_2/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	1.0	2.0	4.0	8.0	16.0	32.0	64.0
$\hat{\alpha}(t)$	0.0927	0.1426	0.1844	0.2591	0.3615	0.4224	0.4593
$\hat{\alpha}_{o10}(t)$	0.0852	0.1274	0.1805	0.2624	0.3499	0.4278	0.4707
$\hat{\alpha}_{o30}(t)$	0.0867	0.1255	0.1827	0.2601	0.3687	0.4222	0.4670
$\hat{\alpha}_{o50}(t)$	0.0866	0.1238	0.1805	0.2654	0.3635	0.4282	0.4631
$\hat{\alpha}_{o100}(t)$	0.0867	0.1258	0.1831	0.2671	0.3635	0.4255	0.4691
$\hat{\alpha}^r(t)$	0.0487	0.1637	0.2104	0.2811	0.3592	0.4321	0.4614
$\hat{\alpha}^r_{o10}(t)$	0.0568	0.0868	0.1804	0.2730	0.3574	0.4260	0.4646
$\hat{\alpha}^r_{o30}(t)$	0.1070	0.1379	0.1934	0.2743	0.3534	0.4274	0.4639
$\hat{\alpha}^r_{o50}(t)$	0.0692	0.1437	0.1852	0.2633	0.3613	0.4262	0.4629
$\hat{\alpha}^r_{o100}(t)$	0.0764	0.1235	0.1841	0.2656	0.3573	0.4252	0.4631
$\hat{\alpha}^r_{s10}(t)$	0.0885	0.1301	0.1765	0.2682	0.3551	0.4278	0.4642
$\hat{\alpha}^r_{s30}(t)$	0.1355	0.0901	0.1771	0.2606	0.3713	0.4237	0.4644
$\hat{\alpha}^r_{s50}(t)$	0.0672	0.0632	0.1704	0.2738	0.3621	0.4258	0.4634
$\hat{\alpha}^r_{s100}(t)$	0.1052	0.1121	0.1829	0.2560	0.3615	0.4292	0.4668
$\hat{\alpha}^r_{o10,s10}(t)$	0.0750	0.1384	0.1816	0.2768	0.3538	0.4283	0.4623
$\hat{\alpha}^r_{o30,s30}(t)$	0.0941	0.1215	0.1759	0.2618	0.3584	0.4256	0.4634
$\hat{\alpha}^r_{o50,s50}(t)$	0.0818	0.1212	0.1811	0.2683	0.3604	0.4263	0.4631
$\hat{\alpha}^r_{o100,s100}(t)$	0.0828	0.1258	0.1858	0.2634	0.3609	0.4246	0.4626

t	128.0	256.0	512.0	1024.0	2048.0	4096.0	8192.0
$\hat{\alpha}(t)$	0.4896	0.4881	0.4963	0.4966	0.4976	0.5001	0.4999
$\hat{\alpha}_{o10}(t)$	0.4841	0.4918	0.4994	0.4977	0.4987	0.4993	0.5003
$\hat{\alpha}_{o30}(t)$	0.4759	0.4893	0.4979	0.4978	0.4989	0.4989	0.5001
$\hat{\alpha}_{o50}(t)$	0.4859	0.4938	0.4965	0.4987	0.4989	0.4989	0.4993
$\hat{\alpha}_{o100}(t)$	0.4860	0.4902	0.4926	0.4941	0.4977	0.4978	0.5004
$\hat{\alpha}^r(t)$	0.4821	0.4905	0.4954	0.4978	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o10}(t)$	0.4838	0.4909	0.4953	0.4977	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o30}(t)$	0.4825	0.4910	0.4953	0.4977	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o50}(t)$	0.4807	0.4912	0.4953	0.4977	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o100}(t)$	0.4811	0.4908	0.4955	0.4978	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{s10}(t)$	0.4823	0.4913	0.4954	0.4975	0.4989	0.4994	0.4997
$\hat{\alpha}^r_{s30}(t)$	0.4809	0.4908	0.4952	0.4978	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{s50}(t)$	0.4824	0.4910	0.4951	0.4976	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{s100}(t)$	0.4823	0.4907	0.4955	0.4977	0.4989	0.4994	0.4997
$\hat{\alpha}^r_{o10,s10}(t)$	0.4820	0.4910	0.4954	0.4977	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o30,s30}(t)$	0.4816	0.4906	0.4953	0.4977	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o50,s50}(t)$	0.4816	0.4907	0.4953	0.4977	0.4988	0.4994	0.4997
$\hat{\alpha}^r_{o100,s100}(t)$	0.4816	0.4907	0.4953	0.4977	0.4988	0.4994	0.4997

Table XI. The efficiencies for $M/H_2/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	1.0	2.0	4.0	8.0	16.0	32.0	64.0
$\hat{\alpha}^r_{o100,s100}(t)$	0.08281	0.12582	0.18582	0.26338	0.36094	0.42457	0.46257
$HW_{n,1}$	0.0110	0.0142	0.0162	0.0174	0.0174	0.0145	0.0112
$\mathcal{V}_{o10}$	7.6	5.4	3.5	2.1	1.6	1.2	1.1
$\mathcal{V}_{o30}$	21.3	15.9	9.0	5.2	3.2	2.0	1.5
$\mathcal{V}_{o50}$	34.5	28.1	14.7	8.8	5.1	3.0	2.0
$\mathcal{V}_{o100}$	68.2	54.3	29.6	16.2	9.9	5.6	3.1
$\mathcal{V}^r$	3.29e-003	0.0235	0.0894	0.2986	1.1	3.2	6.5
$\mathcal{V}^r_{o10}$	0.0182	0.0663	0.1949	0.6395	1.3	3.7	10.4
$\mathcal{V}^r_{o30}$	0.0610	0.2026	0.4771	1.3	2.8	7.4	18.9
$\mathcal{V}^r_{o50}$	0.0775	0.3394	0.7934	1.9	5.6	11.8	21.4
$\mathcal{V}^r_{o100}$	0.1751	0.6435	1.6	3.9	8.7	20.1	42.4
$\mathcal{V}^r_{s10}$	0.0129	0.0565	0.1937	0.7497	2.6	7.6	17.9
$\mathcal{V}^r_{s30}$	0.0183	0.0606	0.2152	0.8079	3.2	8.6	20.1
$\mathcal{V}^r_{s50}$	0.0176	0.0589	0.2257	0.8690	3.2	8.7	20.4
$\mathcal{V}^r_{s100}$	0.0190	0.0699	0.2352	0.8519	3.3	9.0	21.8
$\mathcal{V}^r_{o10,s10}$	0.0921	0.3037	0.6501	1.8	5.5	15.9	34.6
$\mathcal{V}^r_{o30,s30}$	0.3256	0.9594	1.9	5.2	14.7	38.2	88.9
$\mathcal{V}^r_{o50,s50}$	0.5490	1.4	3.1	7.6	23.1	55.1	143.5
$\mathcal{V}^r_{o100,s100}$	1.2	3.0	6.7	14.8	45.6	122.9	282.1
$\mathcal{E}_{o10}$	1.1	1.1	0.8762	0.7472	0.6697	0.5978	0.6159
$\mathcal{E}_{o30}$	1.2	1.1	1.0000	0.7562	0.6016	0.5132	0.4741
$\mathcal{E}_{o50}$	1.1	1.3	0.9739	0.8550	0.6367	0.5090	0.4370
$\mathcal{E}_{o100}$	1.2	1.3	0.9856	0.7910	0.6657	0.5172	0.3930
$\mathcal{E}^r$	2.96e-003	0.0218	0.0894	0.5241	1.0	3.4	6.6
$\mathcal{E}^r_{o10}$	2.73e-003	0.0126	0.0498	0.2245	0.5209	2.0	6.0
$\mathcal{E}^r_{o30}$	3.43e-003	0.0146	0.0530	0.2112	0.4914	1.8	5.9
$\mathcal{E}^r_{o50}$	2.49e-003	0.0158	0.0496	0.1767	0.6921	2.0	4.6
$\mathcal{E}^r_{o100}$	2.80e-003	0.0144	0.0511	0.2012	0.5582	1.8	5.1
$\mathcal{E}^r_{s10}$	1.45e-003	9.18e-003	0.0323	0.1853	0.7062	2.8	9.7
$\mathcal{E}^r_{s30}$	7.51e-004	3.03e-003	0.0127	0.0616	0.3426	1.5	5.6
$\mathcal{E}^r_{s50}$	4.41e-004	1.66e-003	8.06e-003	0.0390	0.2125	0.9187	3.8
$\mathcal{E}^r_{s100}$	2.30e-004	1.06e-003	4.44e-003	0.0213	0.1112	0.4949	2.2
$\mathcal{E}^r_{o10,s10}$	1.12e-003	4.58e-003	0.0125	0.0492	0.1880	0.9231	3.5
$\mathcal{E}^r_{o30,s30}$	4.49e-004	1.60e-003	4.12e-003	0.0157	0.0571	0.2516	1.1
$\mathcal{E}^r_{o50,s50}$	2.71e-004	8.36e-004	2.33e-003	8.28e-003	0.0328	0.1305	0.6297
$\mathcal{E}^r_{o100,s100}$	1.51e-004	4.53e-004	1.26e-003	3.93e-003	0.0160	0.0719	0.3079
$\mathcal{F}_{o10}$	0.4042	0.5372	0.6238	0.6748	0.7678	0.7994	0.8306
$\mathcal{F}_{o30}$	0.3604	0.5222	0.5758	0.6389	0.6720	0.7218	0.7927
$\mathcal{F}_{o50}$	0.3543	0.5672	0.5856	0.6655	0.7200	0.7431	0.7920
$\mathcal{F}_{o100}$	0.3425	0.5383	0.5844	0.6307	0.7322	0.7768	0.7626
$\mathcal{F}^r$	9.08e-004	9.47e-003	0.0463	0.1869	0.7882	2.7	6.0
$\mathcal{F}^r_{o10}$	8.36e-004	5.64e-003	0.0301	0.1664	0.5141	2.1	7.6
$\mathcal{F}^r_{o30}$	1.00e-003	6.42e-003	0.0287	0.1512	0.5540	2.5	9.4
$\mathcal{F}^r_{o50}$	7.63e-004	6.67e-003	0.0300	0.1378	0.7484	2.8	8.2
$\mathcal{F}^r_{o100}$	8.66e-004	6.42e-003	0.0308	0.1470	0.6374	2.7	10.1
$\mathcal{F}^r_{s10}$	4.77e-004	3.44e-003	0.0175	0.1053	0.5676	2.8	9.5
$\mathcal{F}^r_{s30}$	2.41e-004	1.22e-003	6.94e-003	0.0409	0.2916	1.3	5.6
$\mathcal{F}^r_{s50}$	1.35e-004	7.06e-004	4.38e-003	0.0278	0.1737	0.8695	3.8
$\mathcal{F}^r_{s100}$	7.41e-005	4.40e-004	2.35e-003	0.0132	0.0935	0.4887	2.3
$\mathcal{F}^r_{o10,s10}$	3.61e-004	2.03e-003	6.79e-003	0.0306	0.1595	0.9079	3.6
$\mathcal{F}^r_{o30,s30}$	1.42e-004	6.88e-004	2.20e-003	9.39e-003	0.0478	0.2398	1.1
$\mathcal{F}^r_{o50,s50}$	8.55e-005	3.54e-004	1.25e-003	5.00e-003	0.0271	0.1245	0.6468
$\mathcal{F}^r_{o100,s100}$	4.75e-005	1.91e-004	6.80e-004	2.40e-003	0.0133	0.0681	0.3140

Table XII. The efficiencies for $M/H_2/1/1$: $n = 1000$, $\lambda = 1.000000$, $\mu = 1.000000$, $\alpha = 0.500000$.

t	128.0	256.0	512.0	1024.0	2048.0	4096.0	8192.0
$\hat{\alpha}^r_{o100,s100}(t)$	0.48162	0.49070	0.49533	0.49767	0.49885	0.49943	0.49971
$HW_{n,1}$	8.30e-003	5.93e-003	4.09e-003	3.04e-003	2.11e-003	1.46e-003	1.06e-003
$\mathcal{V}_{o10}$	1.0	1.1	0.9978	1.1	1.1	1.0	1.0
$\mathcal{V}_{o30}$	1.2	1.1	1.0	1.0	0.9883	0.9616	0.9382
$\mathcal{V}_{o50}$	1.4	1.1	1.1	1.0	0.9947	0.8953	1.1
$\mathcal{V}_{o100}$	2.0	1.4	1.0	1.1	1.1	1.0	1.0
$\mathcal{V}^r$	14.4	25.8	52.9	132.4	203.5	505.0	1064.3
$\mathcal{V}^r_{o10}$	27.4	47.2	80.1	190.7	310.0	644.4	1383.0
$\mathcal{V}^r_{o30}$	44.0	88.8	149.2	274.7	575.5	1158.3	2424.6
$\mathcal{V}^r_{o50}$	44.9	141.2	223.1	533.2	894.4	1807.7	4276.8
$\mathcal{V}^r_{o100}$	95.9	223.0	377.1	1030.2	1513.8	2389.3	7584.2
$\mathcal{V}^r_{s10}$	38.5	82.0	146.6	325.4	635.0	1237.4	2587.5
$\mathcal{V}^r_{s30}$	44.8	91.5	175.4	393.8	745.2	1449.4	3013.2
$\mathcal{V}^r_{s50}$	46.5	96.8	174.6	399.8	767.9	1463.8	3062.6
$\mathcal{V}^r_{s100}$	48.1	94.2	186.3	401.4	781.6	1512.7	3109.4
$\mathcal{V}^r_{o10,s10}$	80.5	168.8	330.4	719.6	1307.9	2584.3	5354.5
$\mathcal{V}^r_{o30,s30}$	196.1	395.2	765.0	1711.5	3121.3	6220.2	12958.0
$\mathcal{V}^r_{o50,s50}$	311.8	692.7	1259.6	2608.8	5274.2	10402.5	21494.7
$\mathcal{V}^r_{o100,s100}$	619.6	1221.5	2361.2	5172.2	9855.1	20206.7	39968.9
$\mathcal{E}_{o10}$	0.6448	0.6955	0.6705	0.7389	0.7358	0.6915	0.7170
$\mathcal{E}_{o30}$	0.4348	0.4316	0.4341	0.4429	0.4302	0.4209	0.4207
$\mathcal{E}_{o50}$	0.3588	0.3276	0.3214	0.3247	0.3174	0.2872	0.3539
$\mathcal{E}_{o100}$	0.2946	0.2261	0.1866	0.2023	0.2056	0.1914	0.1935
$\mathcal{E}^r$	14.6	25.4	52.4	131.4	204.5	506.7	1093.4
$\mathcal{E}^r_{o10}$	17.3	29.1	53.2	130.3	211.5	441.2	966.0
$\mathcal{E}^r_{o30}$	15.9	35.1	61.4	116.8	250.0	502.3	1087.3
$\mathcal{E}^r_{o50}$	11.2	40.2	65.5	165.9	279.3	577.9	1400.1
$\mathcal{E}^r_{o100}$	14.0	36.9	66.0	189.8	281.0	456.7	1493.4
$\mathcal{E}^r_{s10}$	26.5	65.5	120.7	304.5	584.5	1212.7	2623.9
$\mathcal{E}^r_{s30}$	18.4	54.3	123.6	333.6	683.6	1374.0	3010.9
$\mathcal{E}^r_{s50}$	13.9	43.4	104.2	307.1	659.8	1349.1	3018.6
$\mathcal{E}^r_{s100}$	8.3	27.1	82.5	246.4	596.5	1268.0	2948.1
$\mathcal{E}^r_{o10,s10}$	13.5	45.0	125.4	352.3	751.1	1599.6	3606.1
$\mathcal{E}^r_{o30,s30}$	4.3	16.2	57.0	217.3	617.9	1701.0	4499.3
$\mathcal{E}^r_{o50,s50}$	2.5	10.7	36.4	139.2	477.9	1495.1	4388.1
$\mathcal{E}^r_{o100,s100}$	1.3	4.8	18.0	75.8	269.5	954.8	3158.0
$\mathcal{F}_{o10}$	0.8920	1.0	0.9636	1.1	1.1	0.9980	1.0
$\mathcal{F}_{o30}$	0.8292	0.8764	0.9339	0.9791	0.9608	0.9482	0.9315
$\mathcal{F}_{o50}$	0.7877	0.8229	0.8915	0.9403	0.9494	0.8746	1.1
$\mathcal{F}_{o100}$	0.7723	0.7695	0.7531	0.9087	0.9869	0.9675	0.9885
$\mathcal{F}^r$	13.8	25.3	52.3	131.7	203.0	504.4	1063.6
$\mathcal{F}^r_{o10}$	23.2	43.1	76.6	186.5	306.6	640.7	1379.1
$\mathcal{F}^r_{o30}$	29.3	71.3	132.9	258.7	557.9	1140.5	2405.6
$\mathcal{F}^r_{o50}$	24.8	100.6	185.7	484.6	850.9	1762.5	4223.3
$\mathcal{F}^r_{o100}$	36.9	124.2	269.4	860.7	1375.9	2275.6	7398.8
$\mathcal{F}^r_{s10}$	26.8	67.6	131.8	307.1	617.8	1220.3	2569.6
$\mathcal{F}^r_{s30}$	18.9	54.8	130.8	338.3	686.4	1389.4	2951.5
$\mathcal{F}^r_{s50}$	14.7	46.1	110.2	311.0	671.8	1366.6	2962.2
$\mathcal{F}^r_{s100}$	9.0	29.0	89.0	256.8	611.0	1330.6	2903.6
$\mathcal{F}^r_{o10,s10}$	15.6	54.9	161.3	471.7	1030.1	2278.9	5016.9
$\mathcal{F}^r_{o30,s30}$	4.9	18.9	69.6	289.1	899.6	2771.2	7997.3
$\mathcal{F}^r_{o50,s50}$	2.8	12.3	43.5	174.3	665.8	2332.3	7865.6
$\mathcal{F}^r_{o100,s100}$	1.4	5.4	20.9	90.9	342.0	1365.8	4984.7

4. FIGURES EVALUATING OVERLAPPING CYCLES

This section of the supplement contains additional figures describing the simulation results for the $M/G/1/0$ model, which is discussed in Sections 8 and 9 of the main paper. This particular component focuses on the effectiveness of the overlapping-cycle refinment of the residual-cycle estimator.

Any point (x, n) in each plot implies that a cycle of length x was generated and this cycle contains n *residual runs* under the overlapping scheme. Bigger n implies high dependency among the runs in one overlapping estimator, which spoils the effectiveness of overlapping idea.

The first three figures - Figure 24, Figure 25, and Figure 26) - are comparisons of various overlapping numbers for each possible service-time distribution: D, M and H_2. The other four figures - Figure 27, Figure 28, Figure 29 and Figure 30) - have points for the different service-time distributions, for given levels m of overlapping. The results show why overlapping cycles does not help the residual-cycle estimator so well in $M/H_2/1/0$.

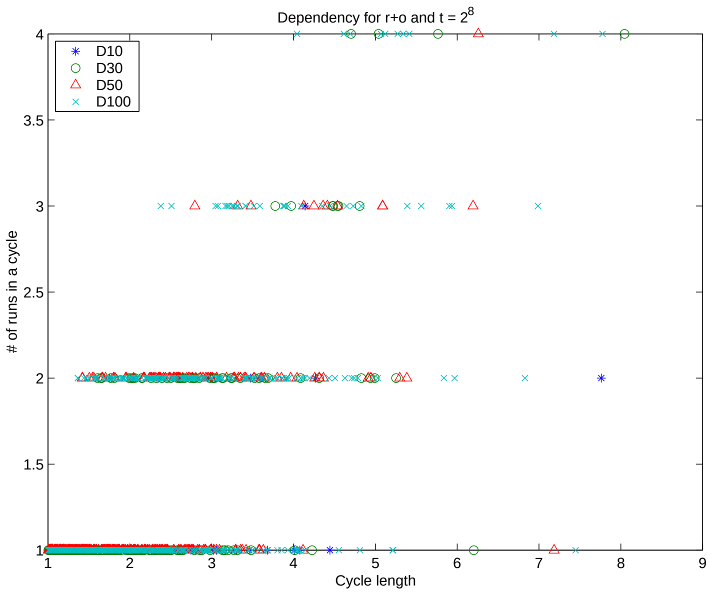

Fig. 24. Cycle length vs. # of runs in a cycle : $M/D/1$ case.

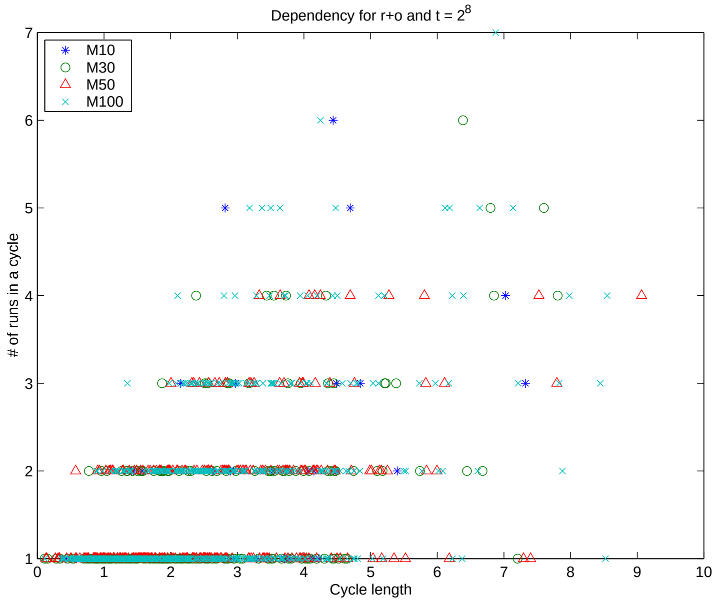

Fig. 25. Cycle length vs. # of runs in a cycle : $M/M/1$ case.

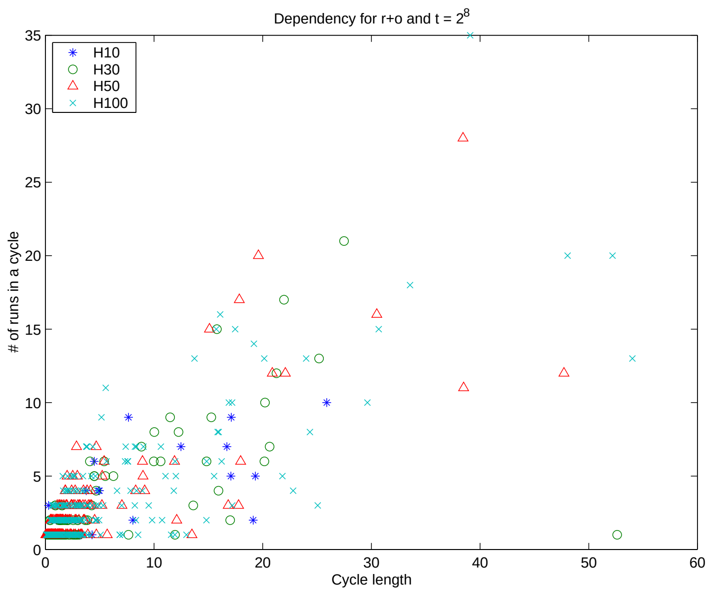

Fig. 26. Cycle length vs. # of runs in a cycle : $M/H_2/1$ case.

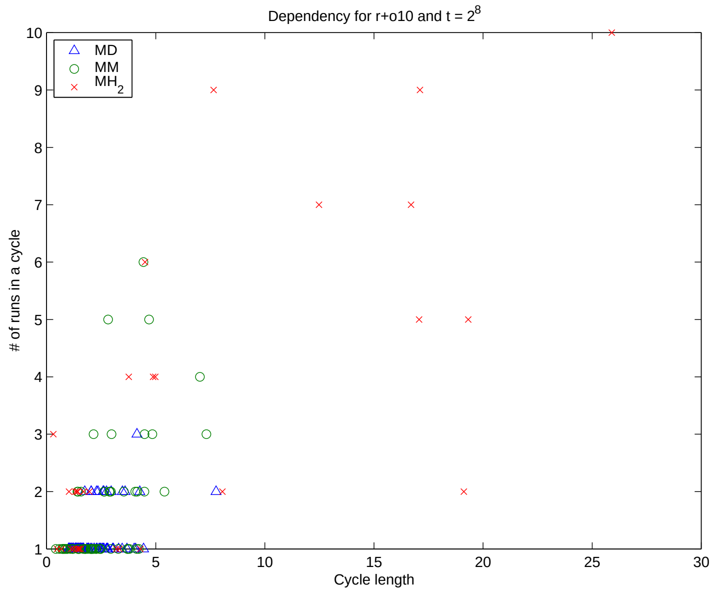

Fig. 27. r+o10

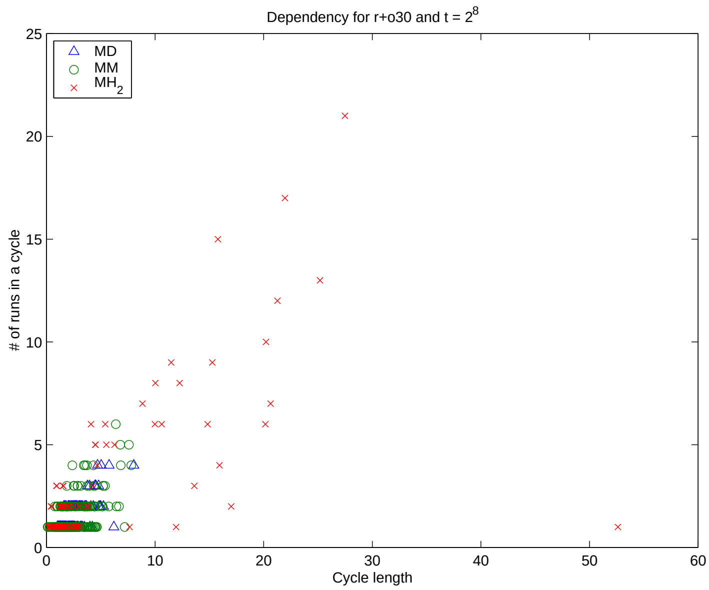

Fig. 28. r+o30

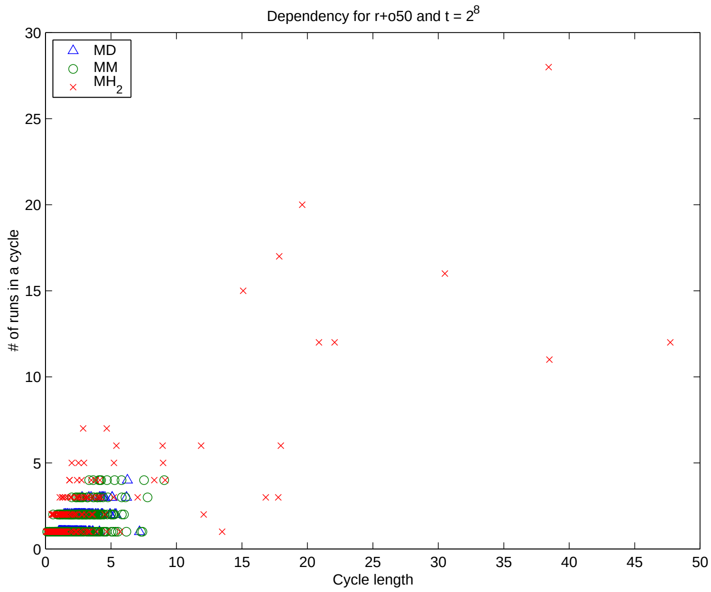

Fig. 29. r+o50

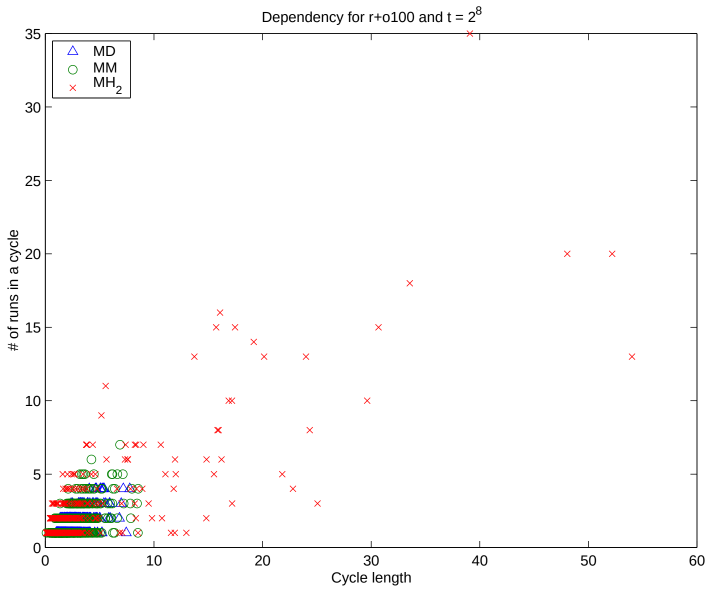

Fig. 30. r+o100

5. EXPERIMENTS FOR THE $M/G/5/0$ MODEL

It is evident that the results for the $M/G/1/0$ model are roughly indicative of what will happen for other single-server $GI/G/1/r$ models, provided that either the traffic intensity $\rho = \lambda/\mu$ is not large (e.g., $\rho < 0.7$) or the traffic intensity is moderate (e.g., $\rho < 1.5$) and r is relatively small. Then the busy cycles will not be long. But the busy cycles will be much more variable, so we can expect $\alpha(t)$ to approach α more slowly.

As we indicated in Section 2 of the main paper, the situation can be much more complicated with multiserver $GI/G/s/r$ queues, when the regeneration epochs are the times arrivals come to an empty system, because the busy cycles can become very long. For large s, the empty state is visited so rarely that the residual-cycle estimator and its refinements are not promising, and we did not consider them.

To consider a second manageable case, we considered the $M/G/s/0$ loss model with $s = 5$ and $\mu = 1$. The material here expands upon Section 10 of the main paper. As before, we let $X(t)$ be the number of customers in the system at time t, and we focus on the time average, with the function f in equation (1.1) of the main paper being the identity map. As before, arrivals to an empty system are the designated regeneration epochs.

5.1 Overview

For this $M/G/5/0$ model, the intervals τ_i between successive regeneration points are busy cycles. It is instructive to start to see how α and the mean busy cycle $E[\tau_1]$ depend on the arrival rate λ. Table XIII contains these results for λ varying from 1 to 10. The mean busy period $E[\tau_1]$ depends only on λ, μ and the steady-state (truncated Poisson) distribution π, which has the insensitivity property, so both $E[\tau_1]$ and α depend on the service-time distribution only through its mean. In particular, with I an idle period,

$$\pi_0 = \frac{E[I]}{E[\tau_1]} = \frac{\lambda^{-1}}{E[\tau_1]} \ . \qquad (1)$$

5.1.0.3 *The Experiment..* We performed simulations for the three service-time distributions - D, M and H_2 - with $\mu = 1$ and for three values of λ - 1, 3 and 5. We choose the time horizons to roughly satisfy $1 \leq -\log(1 - \alpha(t)/\alpha) \leq 4$. For these experiments, we again considered $n = 1000$ replications.

5.1.0.4 *The Results..* Paralleling Figure 23, for each possible value of λ, we plotted the logarithm (base (10) of the observed cpu-time efficiency $\widehat{\mathcal{E}}^r$ versus $-\log_{10}(1 - (\hat{\alpha}^r(t)/\alpha))$ for the three service-time distributions. We present the plot for $\lambda = 5$ in Figure 31. Tables and plots for all other cases appear below.

Once again we see, first, that there are substantial efficiency gains for large t, second, that the three curves for D, M and H_2 service fall on top of each other and, third, that the relationship between these two quantities is again approximately linear.

5.2 Additional Details

This final Subsection contains additional figures and tables describing simulation results for the $M/G/5/0$ model, discussed above. Paralleling Figures 1 and 2 in the main paper, for each possible

Table XIII. The mean busy cycle, $E[\tau_1]$, and the steady-state limiting time average number of customers in the system, α, in the $M/G/5/0$ queue as a function of λ.

λ	$E[\tau_1]$	α
1	2.716667	0.996933
2	3.633333	1.926606
3	6.133333	2.669837
4	10.716667	3.203733
5	18.283333	3.575661
6	29.966667	3.837597
7	47.109524	4.026964
8	71.258333	4.167934
9	104.161111	4.275828
10	147.766667	4.360478

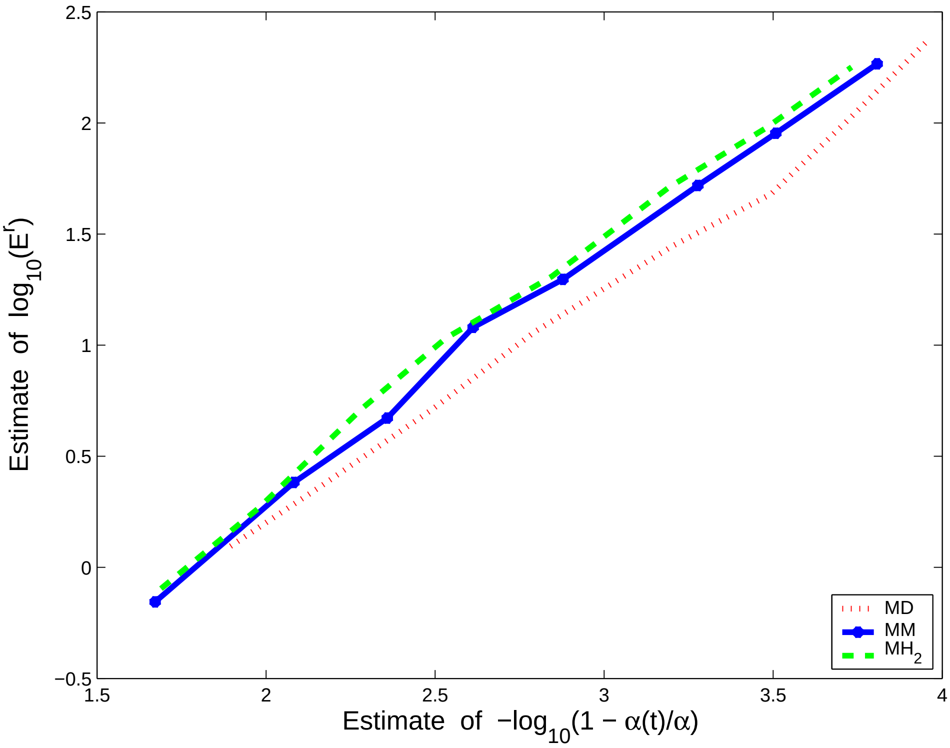

Fig. 31. The logarithm (base 10) of the cpu-time efficiency of the residual-cycle estimator, $\widehat{\mathcal{E}}^r(t)$, as a function of minus the logarithm (base 10) of the estimated $1-(\hat{\alpha}^r(t)/\alpha)$ for the $M/G/5/0$ model with $\lambda = 5$, based on $n = 1000$ replications.

value of λ, we plotted the logarithm (base (10) of the observed cpu-time efficiency $\widehat{\mathcal{E}}^r$ versus both the estimated value $\hat{\alpha}^r(t)$ and $-\log_{10}(1-(\hat{\alpha}^r(t)/\alpha))$ for the three service-time distributions. That is done for the three values of λ: 1, 3 and 5 in Figures 32, 33 and 34, respectively.

Once again we see, first, that there are substantial efficiency gains for large t, second, that the three curves for D, M and H_2 service fall on top of each other and, third, that there is a strong linear relationship, revealed in the log-log plots.

We also present the numerical results in Tables 2-10. There are 9 tables for all combinations of 3 arrival rates and 3 service-time distributions. In those tables, we give the values of the two estimators - standard $\hat{\alpha}(t)$ and residual-cycle $\hat{\alpha}^r(t)$ - and the halfwidths of their 95% confidence intervals. We give estimated values for the variance ratio $\mathcal{V}^r$, the cpu-time efficiency $\mathcal{E}^r$ and the run-length efficiency $\mathcal{F}^r$. As with the $M/G/1/0$ model, we see efficiency gains for larger times. In this case we do not describe results for very small times, for which again the standard estimator is superior.

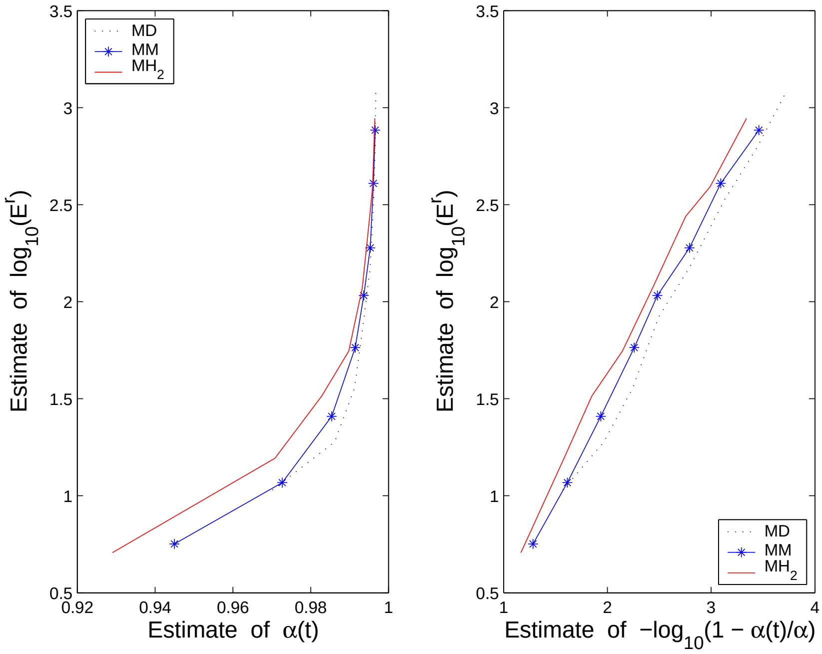

Fig. 32. Figures of $\mathcal{E}^r$ for $\lambda = 1$.

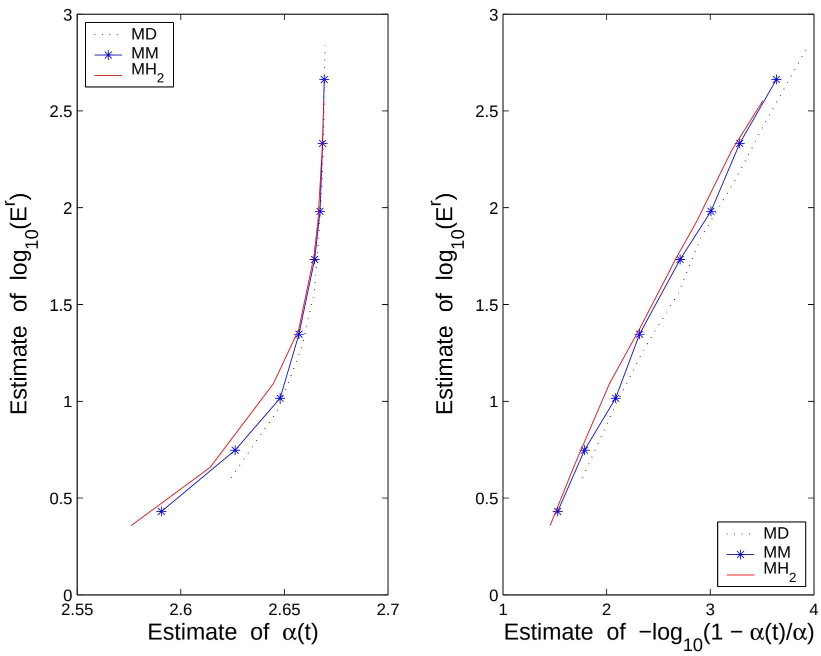

Fig. 33. Figures of $\mathcal{E}^r$ for $\lambda = 3$.

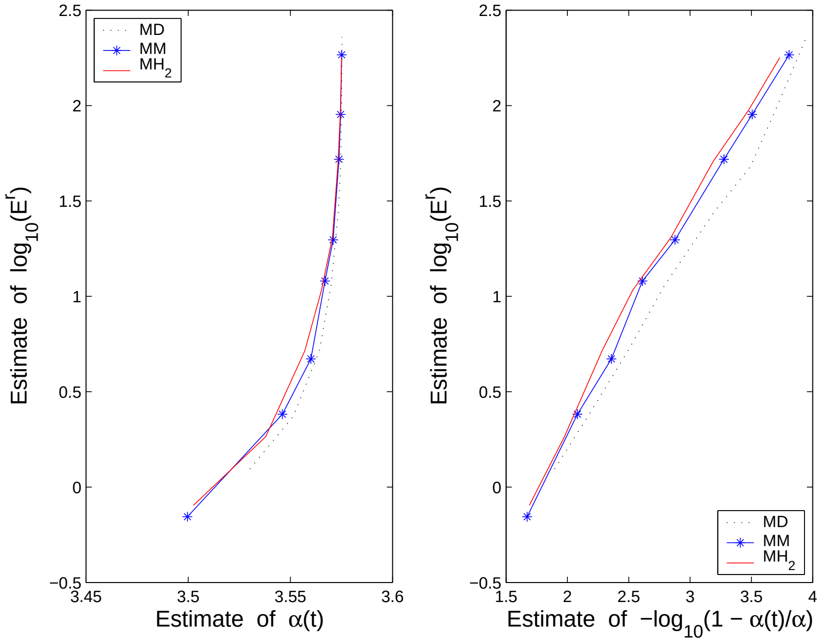

Fig. 34. Figures of $\mathcal{E}^r$ for $\lambda = 5$.

Table XIV. $M/D/5/0$: $n = 1000$, $\lambda = 1.0$, $\mu = 1.0$, $\alpha = 0.996933$

t	20.0	40.0	80.0	160.0
$\hat{\alpha}(t)$	0.97321	0.98507	0.98809	0.99306
$\hat{\alpha}^r(t)$	0.97012	0.98617	0.99126	0.99373
HW_n	0.0137	9.66e-003	6.57e-003	4.75e-003
HW_n^r	4.17e-003	2.19e-003	1.10e-003	5.47e-004
$\mathcal{V}^r$	10.8	19.5	35.4	75.5
$\mathcal{E}^r$	10.7	19.0	35.9	83.1
$\mathcal{F}^r$	10.0	18.7	34.7	74.7
t	320.0	640.0	1280.0	2560.0
$\hat{\alpha}(t)$	0.99454	0.99629	0.99535	0.99697
$\hat{\alpha}^r(t)$	0.99532	0.99620	0.99657	0.99674
HW_n	3.40e-003	2.43e-003	1.74e-003	1.19e-003
HW_n^r	2.77e-004	1.33e-004	7.17e-005	3.46e-005
$\mathcal{V}^r$	150.0	336.7	591.5	1194.2
$\mathcal{E}^r$	150.0	337.7	626.2	1202.9
$\mathcal{F}^r$	149.2	335.9	590.7	1193.4

Table XV. $M/D/5/0$: $n = 1000$, $\lambda = 3.0$, $\mu = 1.0$, $\alpha = 2.669837$

t	25.0	50.0	100.0	200.0
$\hat{\alpha}(t)$	2.61644	2.65007	2.65960	2.66395
$\hat{\alpha}^r(t)$	2.62425	2.64659	2.65821	2.66442
HW_n	0.0145	0.0105	7.41e-003	5.23e-003
HW_n^r	6.55e-003	3.39e-003	1.69e-003	8.65e-004
$\mathcal{V}^r$	4.9	9.6	19.2	36.6
$\mathcal{E}^r$	4.0	8.9	18.6	36.3
$\mathcal{F}^r$	4.1	8.7	18.3	35.7
t	400.0	800.0	1600.0	3200.0
$\hat{\alpha}(t)$	2.66792	2.66718	2.66909	2.66939
$\hat{\alpha}^r(t)$	2.66657	2.66830	2.66903	2.66954
HW_n	3.71e-003	2.49e-003	1.90e-003	1.32e-003
HW_n^r	4.45e-004	2.13e-004	1.17e-004	5.05e-005
$\mathcal{V}^r$	69.4	136.3	265.9	681.3
$\mathcal{E}^r$	70.6	138.5	268.1	690.9
$\mathcal{F}^r$	68.5	135.5	265.0	680.1

Table XVI. $M/D/5/0$: $n = 1000$, $\lambda = 5.0$, $\mu = 1.0$, $\alpha = 3.575661$

t	30.0	60.0	120.0	240.0
$\hat{\alpha}(t)$	3.52927	3.55014	3.56301	3.57258
$\hat{\alpha}^r(t)$	3.53014	3.55129	3.56415	3.56991
HW_n	0.0100	6.83e-003	5.00e-003	3.42e-003
HW_n^r	7.97e-003	3.83e-003	2.07e-003	9.76e-004
$\mathcal{V}^r$	1.6	3.2	5.8	12.2
$\mathcal{E}^r$	1.2	2.3	5.1	11.5
$\mathcal{F}^r$	0.9880	2.5	5.1	11.4
t	480.0	960.0	1920.0	3840.0
$\hat{\alpha}(t)$	3.57174	3.57301	3.57395	3.57568
$\hat{\alpha}^r(t)$	3.57340	3.57452	3.57498	3.57526
HW_n	2.56e-003	1.72e-003	1.24e-003	8.85e-004
HW_n^r	4.82e-004	2.51e-004	1.23e-004	5.87e-005
$\mathcal{V}^r$	28.3	46.9	101.9	227.5
$\mathcal{E}^r$	27.8	48.0	102.8	228.8
$\mathcal{F}^r$	27.3	46.1	101.0	226.5

Table XVII. $M/M/5/0$: $n = 1000$, $\lambda = 1.0$, $\mu = 1.0$, $\alpha = 0.996933$

t	20.0	40.0	80.0	160.0
$\hat{\alpha}(t)$	0.94141	0.95732	0.98177	0.99213
$\hat{\alpha}^r(t)$	0.94497	0.97270	0.98541	0.99144
HW_n	0.0185	0.0131	9.78e-003	6.92e-003
HW_n^r	7.67e-003	3.77e-003	1.95e-003	9.09e-004
$\mathcal{V}^r$	5.8	12.1	25.1	58.0
$\mathcal{E}^r$	5.6	11.7	25.7	58.0
$\mathcal{F}^r$	5.2	11.5	24.4	57.2
t	320.0	640.0	1280.0	2560.0
$\hat{\alpha}(t)$	0.99108	0.99452	0.99613	0.99602
$\hat{\alpha}^r(t)$	0.99364	0.99532	0.99613	0.99659
HW_n	4.75e-003	3.35e-003	2.39e-003	1.67e-003
HW_n^r	4.70e-004	2.46e-004	1.20e-004	6.07e-005
$\mathcal{V}^r$	102.2	184.9	396.8	755.3
$\mathcal{E}^r$	107.6	189.7	407.1	765.1
$\mathcal{F}^r$	101.5	184.3	396.1	754.7

Table XVIII. $M/M/5/0:\ n = 1000,\ \lambda = 3.0,\ \mu = 1.0,\ \alpha = 2.669837$

t	25.0	50.0	100.0	200.0
$\hat{\alpha}(t)$	2.57656	2.63333	2.64417	2.66365
$\hat{\alpha}^r(t)$	2.59067	2.62624	2.64793	2.65699
HW_n	0.0215	0.0148	0.0101	7.23e-003
HW_n^r	0.0116	5.88e-003	3.03e-003	1.51e-003
$\mathcal{V}^r$	3.4	6.3	11.1	22.8
$\mathcal{E}^r$	2.7	5.6	10.4	22.2
$\mathcal{F}^r$	2.7	5.4	10.3	22.0
t	400.0	800.0	1600.0	3200.0
$\hat{\alpha}(t)$	2.66670	2.66551	2.66891	2.67014
$\hat{\alpha}^r(t)$	2.66461	2.66720	2.66845	2.66922
HW_n	5.04e-003	3.62e-003	2.59e-003	1.85e-003
HW_n^r	6.80e-004	3.70e-004	1.77e-004	8.67e-005
$\mathcal{V}^r$	54.9	95.8	213.5	455.3
$\mathcal{E}^r$	54.1	95.8	215.0	459.3
$\mathcal{F}^r$	53.8	94.8	212.5	454.2

Table XIX. $M/M/5/0:\ n = 1000,\ \lambda = 5.0,\ \mu = 1.0,\ \alpha = 3.575661$

t	30.0	60.0	120.0	240.0
$\hat{\alpha}(t)$	3.50047	3.53525	3.55761	3.56793
$\hat{\alpha}^r(t)$	3.49960	3.54608	3.56001	3.56693
HW_n	0.0133	9.98e-003	6.65e-003	4.85e-003
HW_n^r	0.0119	5.48e-003	2.82e-003	1.34e-003
$\mathcal{V}^r$	1.2	3.3	5.6	13.1
$\mathcal{E}^r$	0.6992	2.4	4.7	12.0
$\mathcal{F}^r$	0.6953	2.5	4.7	12.0
t	480.0	960.0	1920.0	3840.0
$\hat{\alpha}(t)$	3.56857	3.57318	3.57525	3.57509
$\hat{\alpha}^r(t)$	3.57093	3.57377	3.57455	3.57510
HW_n	3.41e-003	2.40e-003	1.71e-003	1.23e-003
HW_n^r	7.52e-004	3.29e-004	1.81e-004	9.08e-005
$\mathcal{V}^r$	20.5	53.1	89.5	183.3
$\mathcal{E}^r$	19.8	52.3	89.9	184.5
$\mathcal{F}^r$	19.6	51.9	88.5	182.2

Table XX. $M/H_2/5/0:\ n = 1000,\ \lambda = 1.0,\ \mu = 1.0,\ \alpha = 0.996933$

t	160.0	320.0	640.0	1280.0
$\hat{\alpha}(t)$	0.91750	0.97392	0.97596	0.98957
$\hat{\alpha}^r(t)$	0.92905	0.97083	0.98285	0.98978
HW_n	0.0201	0.0147	0.0108	7.18e-003
HW_n^r	8.64e-003	3.66e-003	1.89e-003	9.64e-004
$\mathcal{V}^r$	5.4	16.1	32.8	55.4
$\mathcal{E}^r$	5.1	15.6	32.7	55.7
$\mathcal{F}^r$	5.0	15.5	32.2	54.9
t	2560.0	5120.0	10240.0	20480.0
$\hat{\alpha}(t)$	0.99431	0.99672	0.99551	0.99599
$\hat{\alpha}^r(t)$	0.99327	0.99518	0.99591	0.99648
HW_n	5.37e-003	3.93e-003	2.65e-003	1.88e-003
HW_n^r	4.98e-004	2.38e-004	1.35e-004	6.38e-005
$\mathcal{V}^r$	116.2	271.7	386.9	871.0
$\mathcal{E}^r$	117.7	275.2	391.4	881.7
$\mathcal{F}^r$	115.6	271.0	386.4	870.4

Table XXI. $M/H_2/5/0:\ n = 1000,\ \lambda = 3.0,\ \mu = 1.0,\ \alpha = 2.669837$

t	200.0	400.0	800.0	1600.0
$\hat{\alpha}(t)$	2.56323	2.61381	2.65061	2.65995
$\hat{\alpha}^r(t)$	2.57609	2.61417	2.64471	2.65687
HW_n	0.0221	0.0160	0.0115	7.81e-003
HW_n^r	0.0128	6.99e-003	3.15e-003	1.59e-003
$\mathcal{V}^r$	3.0	5.2	13.2	24.1
$\mathcal{E}^r$	2.3	4.6	12.3	23.5
$\mathcal{F}^r$	2.3	4.5	12.2	23.1
t	3200.0	6400.0	12800.0	25600.0
$\hat{\alpha}(t)$	2.66182	2.66888	2.66698	2.66887
$\hat{\alpha}^r(t)$	2.66421	2.66621	2.66817	2.66901
HW_n	5.78e-003	3.98e-003	2.87e-003	1.96e-003
HW_n^r	7.73e-004	4.34e-004	2.04e-004	1.05e-004
$\mathcal{V}^r$	55.9	84.5	197.3	350.9
$\mathcal{E}^r$	55.6	84.6	196.8	356.5
$\mathcal{F}^r$	54.8	83.6	196.3	350.1

Table XXII. $M/H_2/5/0:\ n = 1000,\ \lambda = 5.0,\ \mu = 1.0,\ \alpha = 3.575661$

t	240.0	480.0	960.0	1920.0
$\hat{\alpha}(t)$	3.49216	3.53693	3.55220	3.56514
$\hat{\alpha}^r(t)$	3.50256	3.53789	3.55702	3.56516
HW_n	0.0146	0.0102	7.67e-003	5.30e-003
HW_n^r	0.0124	6.32e-003	3.09e-003	1.55e-003
$\mathcal{V}^r$	1.4	2.6	6.1	11.7
$\mathcal{E}^r$	0.8022	1.8	5.2	10.8
$\mathcal{F}^r$	0.8013	1.9	5.2	10.6
t	3840.0	7680.0	15360.0	30720.0
$\hat{\alpha}(t)$	3.57230	3.57445	3.57420	3.57545
$\hat{\alpha}^r(t)$	3.57056	3.57336	3.57447	3.57500
HW_n	3.70e-003	2.69e-003	1.87e-003	1.30e-003
HW_n^r	8.05e-004	3.73e-004	1.92e-004	9.76e-005
$\mathcal{V}^r$	21.1	51.9	94.7	176.8
$\mathcal{E}^r$	20.4	51.2	94.6	178.2
$\mathcal{F}^r$	20.2	50.7	93.6	175.7

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