Impact of sequential disorder on the scaling behavior of airplane boarding time

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(Received 8 February 2013; published 16 May 2013)

Airplane boarding process is an example where disorder properties of the system are relevant to the emergence of universality classes. Based on a simple model, we present a systematic analysis of finite-size effects in boarding time, and propose a comprehensive view of the role of sequential disorder in the scaling behavior of boarding time against the plane size. Using numerical simulations and mathematical arguments, we find how the scaling behavior depends on the number of seat columns and the range of sequential disorder. Our results show that new scaling exponents can arise as disorder is localized to varying extents.

DOI: 10.1103/PhysRevE.87.052803 PACS number(s): 89.75.Da, 89.40.Dd, 02.50.−r, 05.40.−a

I. INTRODUCTION

How disorder properties affect the scaling behavior of a characteristic time scale is a problem studied in various areas of physics. For example, dynamic exponents of surface growth models [1] depend on the geometry of substrate and the type of disorder. In complex networks, consensus time of opinion dynamics [2] and first-passage time of random walk [3] scale differently with the system size depending on the level of structural heterogeneity. Those studies reveal the subtle interplay between dynamics and disorder that gives rise to different universality classes of scaling behaviors. Expanding the list of universality classes and clarifying their origins has been established as a general framework frequently employed by physicists.

Airplane boarding process provides another interesting example where this framework can be applied. The average time required for all passengers to get seated, or *average boarding time* $\langle T\rangle$, may scale differently with the plane size N depending on sequential disorder of passengers, which is controlled by the airline's boarding policy and each individual's gate arrival time. However, most studies of boarding process were limited to the practical problem of reducing boarding time at a fixed plane size [4–10], which provides only fragmented knowledge about boarding time in prescribed situations. To understand the general nature of boarding process, we should examine its scaling properties, which came to be studied only very recently. Analytical studies by Bachmat *et al.* [11–14] argued for $\langle T\rangle \sim N^{1/2}$ on the basis of mathematical results about longest monotonic subsequences in random permutations of real number pairs [15,16]. Meanwhile, a numerical study by Frette and Hemmer [17] based on a simple boarding model (see Fig. 1) reported $\langle T\rangle \sim N^{0.69}$. The conflict between the two exponents has been discussed by subsequent numerical works, which supported $\langle T\rangle \sim N^{1/2}$ as the true asymptotic behavior after the finite-size effect is systematically filtered out [18,19].

In this paper, we revisit the original mathematical theorem [15,16] that led to the prediction of $N^{1/2}$ scaling. A physical interpretation of its conditions reveals that previous studies [11–14,17–19] were limited to the cases with a broad range of disorder in the passenger sequence, and that different scaling behaviors are possible if the range of sequential disorder is localized to various extents. This leads us to a more comprehensive picture of the scaling behaviors of $\langle T\rangle$.

This paper is organized as follows. In Sec. II, we generalize the boarding model proposed by [17], incorporating additional parameters for airplane structure and sequential disorder. In Sec. IV, we make analytical predictions on possible scaling behaviors in special cases of fully reserved planes. Our predictions are numerically confirmed in Sec. IV, where they are also verified for more complicated situations, taking into account factors such as vacancy effect and general localized disorder. Finally, we conclude with a summary of our findings in Sec. V.

II. MODEL

A. Boarding process

We consider an airplane with N rows and c columns of seats along a one-dimensional aisle. Each seat is labeled with a row index r if it is in the rth row from the front. The aisle is discretized into N sites, each of which cannot hold more than one passenger at once. At $t = 0$, cN passengers enter the plane from the front in a sequence $\{r_1, r_2, \ldots, r_{cN}\}$, where r_i denotes the row index of the ith passenger to enter. Passengers' positions are synchronously updated. Every passenger moves along the aisle front-to-back, instantly crossing successive empty sites until blocked by an occupied site or reaching the row of its assigned seat. If the latter is the case, sitting down takes one time step for every passenger. Continuing the process, all passengers would get seated at $t = T$, which we call boarding time (see Fig. 1).

B. Sequential disorder

Although T is completely determined by the initial passenger sequence, the entry order of each passenger is not strictly

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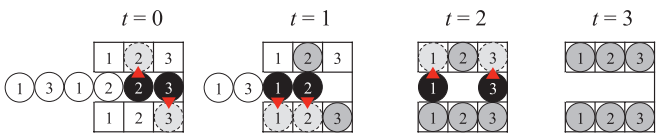

FIG. 1. (Color online) Snapshots of boarding in an airplane with $N = 3$ rows and $c = 2$ columns of seats, where each passenger (circle) is labeled by the row index of its seat. At each time step, passengers are waiting in line (white), sitting down (black), or staying seated (gray).

controlled in normal situations. Thus, it makes more sense to deal with $\langle T \rangle$ for an ensemble of initial passenger sequences.

The ensemble of sequences is primarily constrained by boarding policies. Here we focus on back-to-front policies with equal-sized boarding groups. According to such policies, passengers are divided into boarding groups of m rows each (m is a divisor of N), so that the nth group to enter the plane consists of passengers whose row indices satisfy $N - nm + 1 \leqslant r_i \leqslant N - (n-1)m$. Consequently, the passenger sequence is sorted back-to-front on a scale larger than groups, but remains randomized within each group. Thus, m can be interpreted as the range of sequential disorder. If N/m (m) is fixed as $N \to \infty$, we can say that sequential disorder is purely globalized (localized), as illustrated in Figs. 2(a) and 2(b), respectively.

Meanwhile, some passengers would deviate from the boarding policy due to their unpunctuality. This justifies the incorporation of arrival time fluctuations as another determinant of sequential disorder. We consider Gaussian shuffling as a model of such fluctuations, which is defined as follows: we add an independent and identically distributed Gaussian random variable η_i of zero mean and of variance σ^2 to the sequential index i of each passenger, and then sort the sequence in the increasing order of $i + \eta_i$ [see Fig. 2(c)]. As long as m is finite, Gaussian shuffling makes sequential disorder purely localized (globalized) if σ (σ/N) is fixed in the asymptotic limit. Alternatively, we can also consider uniform shuffling, in which every passenger is randomly relocated in the sequence with probability p [see Fig. 2(d)]. If m is finite, uniform shuffling allows localized and globalized disorder to coexist until the sequence is completely randomized at $p = 1$.

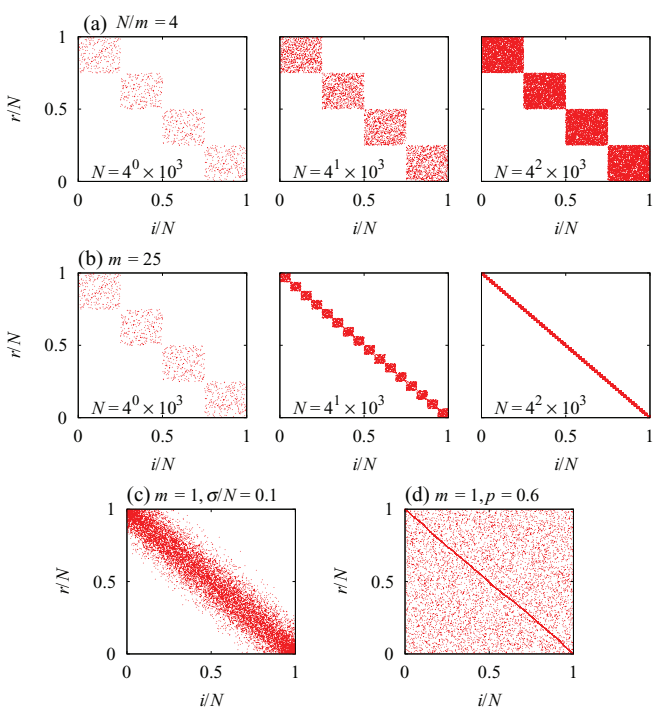

FIG. 2. (Color online) Examples of the passenger sequence $\{r_i\}$. As $N \to \infty$, (a) fixed N/m means globalized disorder; (b) fixed m means localized disorder; (c) Gaussian shuffling gives either purely globalized or purely localized disorder; (d) uniform shuffling allows coexistence of both kinds of disorder. We set $N = 10^4$ in (c) and (d).

III. THEORETICAL PREDICTIONS

A. Scaling behaviors for globalized disorder

As illustrated in Fig. 2, a passenger sequence can be rendered as a two-dimensional scatter plot, horizontal and vertical axes representing the sequential index i and the row index r, respectively. In the asymptotic limit $N \to \infty$, an ensemble of sequences becomes equivalent to a probability density function (PDF) $p(i/N, r/N)$ defined over a continuous two-dimensional area [11–14]. This representation enables us to utilize the following mathematical theorem [15,16].

Theorem—If (x_α, y_α), $\alpha = 1, \ldots, N$ are pairs of real numbers with $0 \leqslant x_\alpha \leqslant 1$ and $0 \leqslant y_\alpha \leqslant 1$, we say that a subsequence $\{(x_{i_1}, y_{i_1}), \ldots, (x_{i_l}, y_{i_l})\}$ is an *increasing subsequence* if

$$x_{i_j} < x_{i_{j+1}} \quad \text{and} \quad y_{i_j} < y_{i_{j+1}} \quad \text{for} \quad j = 1, \ldots, l-1,$$

where i_j is a sequence of nonrepeated indices between 1 and N. If the pairs (x_α, y_α) are generated from a finite PDF $p(x, y)$, the length of the longest increasing subsequence asymptotically scales as $N^{1/2}$.

The increasing subsequence in the theorem can be translated as the blocking subsequence of passengers, in which one passenger *blocks* the next if the latter cannot reach its seat unless the former is seated [11,14]. Since the length of the longest blocking subsequence is equal to boarding time T, the theorem indicates that T (and $\langle T \rangle$ as well) scales as $N^{1/2}$ if the PDF remains finite in the asymptotic limit $N \to \infty$. The finitude of the PDF is equivalent to the broad range of disorder in the passenger sequence, which corresponds to globalized disorder defined in the previous section. In other words, $\langle T \rangle \sim N^{1/2}$ is guaranteed for purely globalized disorder.

B. Scaling behaviors for localized disorder

Meanwhile, different scaling behaviors might arise if some part of the PDF diverges due to the presence of localized disorder in the passenger sequence. Here we present theoretical arguments on the scaling behaviors of $\langle T \rangle$ for fully reserved planes when m is kept finite without arrival time fluctuations.

1. Single-column planes ($c = 1$)

We first consider a fully reserved plane with a single column ($c = 1$) of seats. When a boarding policy fills the window seats first and the aisle seats later, there is effectively no interference between the passengers to be seated in the same row. In such

a case, even a multicolumn plane can be regarded as a single-column plane.

We start with an argument on a lower bound of T. As $N \to \infty$, the passenger sequence always contains a boarding group whose row-index configuration is exactly front-to-back, e.g., $\{1,2,\ldots,m-1,m\}$. Since every passenger blocks the next passenger, boarding time for this particular group is m. Thus, we have $\lim_{N\to\infty} T \geqslant m$.

Now we turn to an upper bound of T. We assume the worst-case scenario: a group starts to board when $m-1$ seats out of m seats reserved by its members are blocked by the previous group, and the number of blocked seats decreases by one at each time step. We choose an arbitrary passenger who has $n_< + n_> - 1$ sites to go before reaching its seat, where $n_<$ ($n_>$) denotes the number of passengers in the same group whose row indices are smaller (larger). Those $n_<$ passengers would eventually get seated before the chosen passenger, leaving empty sites in the aisle. Whenever such empty sites appear, the chosen passenger instantly advances along the aisle by as many sites in addition to the base speed of one site per unit time. Hence, the chosen passenger can reach the reserved seat within $n_> - 1$ time steps, spending one more time step to sit down. Since $n_> \leqslant m$, individual boarding time for any passenger is not greater than m, and so is the total boarding time, i.e., $T \leqslant m$. Combining both upper and lower bounds, we obtain $\lim_{N\to\infty} T = m$.

2. *Multicolumn planes ($c \geqslant 2$)*

As a next step, we consider a fully reserved plane with multiple columns ($c \geqslant 2$) of seats. This is the typical situation encountered in reality, provided that passengers belonging to different columns are allowed to mix together while entering the plane.

We again begin with a lower bound of T. The number of boarding groups whose row-index configuration is c repetitions of $\{1,2,\ldots,m\}$ grows linearly with N. When the first m members of such a group begin to take seats, the rest of the group completely occupy the rows reserved for the next group. Thus, the next group cannot reach their seats for a finite number of time steps, which implies that T increases by one or more time steps for every occurrence of this configuration. Thus, $T \geqslant aN$ as $N \to \infty$, where a is a positive constant.

The total number of passengers cN is trivially an upper bound of T, since there is always at least one passenger sitting down per unit time. Therefore, we have $\langle T\rangle \sim N$.

IV. NUMERICAL RESULTS

In this section, we numerically test the scaling behaviors of $\langle T\rangle$ discussed in the previous section and check their validity in more general cases. To clarify the true asymptotic behavior of $\langle T\rangle$, we use effective scaling exponent $\bar{z}$ defined as follows [18,19]:

$$\bar{z}(\sqrt{N_i N_{i+1}}) \equiv \frac{\ln[\langle T(N_{i+1})\rangle/\langle T(N_i)\rangle]}{\ln(N_{i+1}/N_i)}.$$

This is none other than the average slope of $\langle T\rangle$ between N_i and N_{i+1} in the log-log plot of $\langle T\rangle$ against N. We interpret the saturation value of $\bar{z}$ in the asymptotic limit as the true scaling exponent of $\langle T\rangle$.

A. Single-column planes ($c = 1$)

The scaling behaviors of $\langle T\rangle$ for $c = 1$ are shown in the upper panel of Fig. 3 and the behaviors of $\bar{z}$ in the lower panel. Without arrival time fluctuations, $\langle T\rangle$ scales as $N^{1/2}$ if N/m is fixed, while it saturates to m if m is fixed [see Fig. 3(a)]. Gaussian shuffling at finite m produces similar but slightly different scalings: $N^{1/2}$ scaling for fixed σ/N [see Fig. 3(b)] and $\ln N$ scaling for fixed σ [see Fig. 3(c)]. Since both saturation and log scaling are slower than any algebraic scaling, they can be collectively labeled as N^0 scalings. Thus, our results confirm $\langle T\rangle \sim N^{1/2}$ for purely globalized disorder, while finding $\langle T\rangle \sim N^0$ for purely localized disorder, regardless of the presence of arrival time fluctuations. When those two kinds of disorder coexist, the effect of localized disorder predominates, as implied by $N^{1/2}$ scaling observed whenever $p > 0$ [see Fig. 3(d)].

Before moving on to the multicolumn case, we remark that dividing into smaller groups always reduces boarding time if $c = 1$, as previously reported by [18]. However, we emphasize that the benefits of group division, in terms of scaling, are not

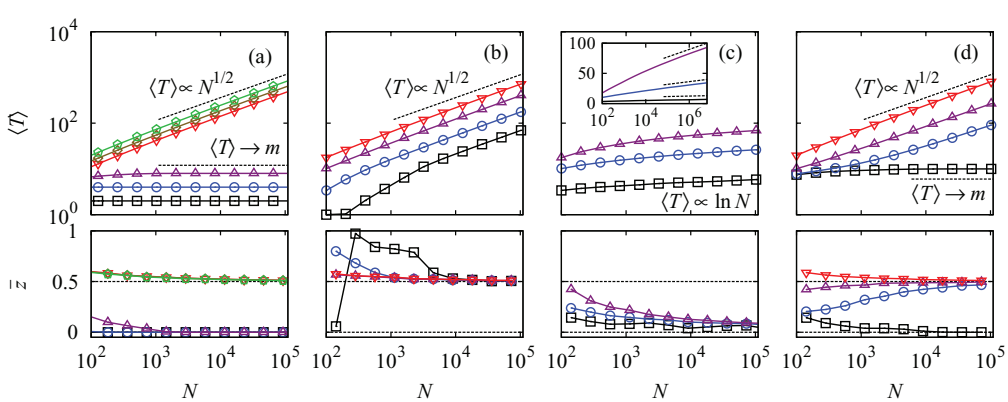

FIG. 3. (Color online) Asymptotic scaling behaviors of $\langle T\rangle$ in a fully reserved plane with $c = 1$. (Upper panel) From bottom to top, the curves represent (a) $m = 2,4,8,N/4,N/2,N$ without shuffling, and $m = 10$ with (b) $\sigma/N = 0.001,0.01,0.1,1$, (c) $\sigma = 1,10,100$ (inset: semilog plots of the main), (d) $p = 0,0.01,0.1,1$. (Lower panel) Effective scaling exponents $\bar{z}$ of corresponding curves in the upper panel. Each data point is averaged over at least 100 samples.

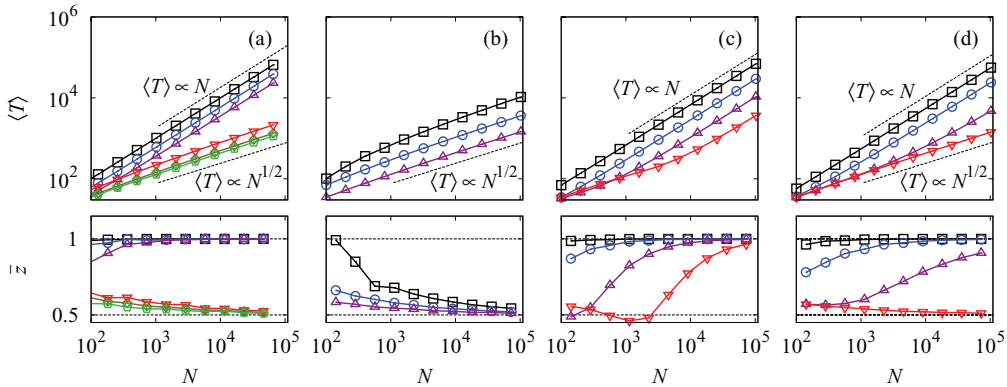

FIG. 4. (Color online) Asymptotic scaling behaviors of $\langle T\rangle$ in a fully reserved plane with $c = 2$. (Upper panel) From top to bottom, (a) $m = 1,4,16,N/16,N/4,N$ without shuffling, and $m = 5$ with (b) $\sigma/N = 0.001,0.01,0.1$, (c) $\sigma = 1,10,100,1000$, (d) $p = 0,0.5,0.9,1$. (Lower panel) Effective scaling exponents $\bar{z}$ of corresponding curves in the upper panel. Each data point is averaged over at least 100 samples.

very robust against disorder caused by arrival time fluctuations. A slightest hint of globalized disorder can revert the scaling behavior to that of the random boarding policy.

B. Multicolumn planes ($c \geqslant 2$)

Without loss of generality, we focus on the boarding process of a two-column ($c = 2$) plane since the scaling behaviors does not change in the other cases.

Figure 4 shows the scaling behaviors of $\langle T\rangle$ at $c = 2$. If no shuffling is applied, $\langle T\rangle$ scales as $N^{1/2}$ (N) if N/m (m) is fixed [see Fig. 4(a)]. Gaussian shuffling at finite m results in exactly the same scalings [see Figs. 4(b) and 4(c)], implying $\langle T\rangle \sim N^{1/2}$ ($\langle T\rangle \sim N$) for purely globalized (localized) disorder, even in the presence of arrival time fluctuations. If both kinds of disorder are mixed together, the effect of globalized disorder predominates, as linear scaling for $p < 1$ indicates [see Fig. 4(d)].

We note that division into smaller groups always increases boarding time if $c \geqslant 2$. In this case, reducing the range of disorder always results in stronger interference between passengers to be seated near each other, as reflected in the strong sensitivity of the scaling exponent to localized disorder. Thus, lack of control is better than row-wise division of groups for $c \geqslant 2$ planes (provided that we do not divide passengers column-wise, which was covered by the $c = 1$ case), which is a lesson shared by most of previous studies [5–14].

C. Effect of vacancy

We also consider the asymptotic scaling behaviors of $\langle T\rangle$ for $c = 2$ (see Fig. 5) when each seat can be vacant with probability ϕ. A similar case was also studied by [7], but only at a fixed plane size without any consideration of scaling.

Figure 5(a) shows the effect of ϕ when there are no arrival time fluctuations. $\langle T\rangle$ scales as N for $\phi < 1/2$ and as $\ln N$ for $\phi > 1/2$, implying that ϕ can interpolate between the scalings for localized disorder observed at $c = 1$ and $c \geqslant 2$. The same observation can be made even in the presence of globalized disorder, as shown in Fig. 5(b), where the scaling changes from $\langle T\rangle \sim N$ to $\langle T\rangle \sim N^{1/2}$. The transition point of the scaling exponent is $1/2$ for $c = 2$, which is confirmed to be $\phi_c = 1 - 1/c$ for the general value of c (even including $c = 1$, in which case the scaling is not affected by vacancy). Interestingly, $\langle T\rangle \sim N^{1/2}$ seems to hold exactly at $\phi = \phi_c$, regardless of the nature of sequential disorder.

D. Generalization of localized disorder

Finally, we generalize the range of sequential disorder even further by applying Gaussian shuffling with $\sigma \sim N^s$, where s is freely varied between 0 and 1. This allows us to interpolate between purely localized and purely globalized disorders without mixing different kinds of disorders as in the case of uniform shuffling.

N dependence of $\langle T\rangle$ and $\bar{z}$ for $c = 1$ and $c = 2$ are plotted in Figs. 6(a) and 6(b), respectively. It is clear that the scaling exponent of $\langle T\rangle$ changes continuously as s is varied, monotonically increasing (decreasing) with s if $c = 1$ ($c = 2$), as indicated by the s dependence of $\bar{z}$ in Fig. 6(c). The results

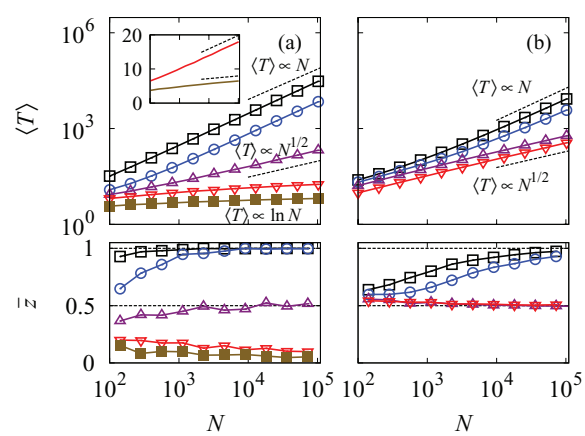

FIG. 5. (Color online) Asymptotic scaling behaviors of $\langle T\rangle$ in a partially reserved two-column ($c = 2$) airplane with group size $m = 5$. (Upper panel) For the curves from top to bottom, (a) $\phi = 0.25,0.45,0.5,0.55,0.75$ without shuffling (inset: semilog plots of the lowest two curves in the main) and (b) $\phi = 0.25,0.35,0.5,0.55$ with uniform shuffling $p = 0.5$. (Lower panel) Effective scaling exponents $\bar{z}$ of corresponding curves in the upper panel. Each data point is averaged over at least 100 samples.

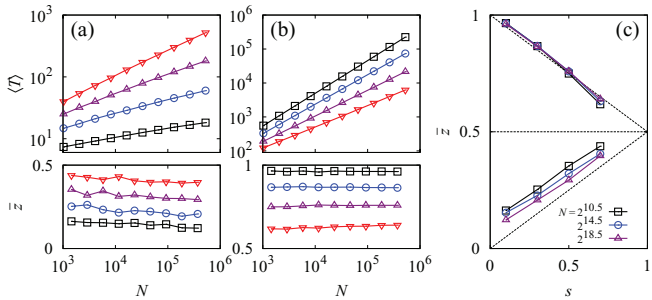

FIG. 6. (Color online) Asymptotic scaling behaviors of $\langle T \rangle$ when the range of disorder is given by Gaussian shuffling with $\sigma = N^s$. In the upper panel, (a) from bottom to top, $c = 1$ with $s = 0.1, 0.3, 0.5, 0.7$; (b) from top to bottom, $c = 2$ with $s = 0.1, 0.3, 0.5, 0.7$. Behaviors of $\bar{z}$ for corresponding curves in the upper panels are plotted in the lower panels of (a) and (b). (c) $\bar{z}$ as a function of s for different values of N. Each data point is averaged over at least 100 samples.

hint at a possible linear dependence of the scaling exponent on s [represented by dashed lines in Fig. 6(c)], which remains a speculation at the moment.

V. SUMMARY AND DISCUSSIONS

Scaling behaviors of average boarding time are summarized in Fig. 7. In a single-column ($c = 1$) plane, purely localized disorder produces N^0 scaling including both saturation and logarithmic divergence, and purely globalized disorder leads to $N^{1/2}$ scaling. The boarding process is more sensitive to globalized disorder, so $N^{1/2}$ scaling is observed when both kinds of disorder are present. On the other hand, in a multicolumn ($c \geqslant 2$) plane, purely localized disorder produces linear scaling, while purely globalized one again yields $N^{1/2}$ scaling. Since localized disorder is dominant in this case, linear scaling is observed when both kinds of disorder are present. Increasing the probability of vacant seats ϕ beyond $\phi_c = 1 - 1/c$ changes scaling behaviors from the multicolumn ones to the corresponding single-column ones while keeping sequential disorder type. However, the borderline scaling at ϕ_c seems to be $N^{1/2}$, regardless of disorder type. Expanding Fig. 7 to encompass general localized disorder with noninteger s remains a challenge for future works.

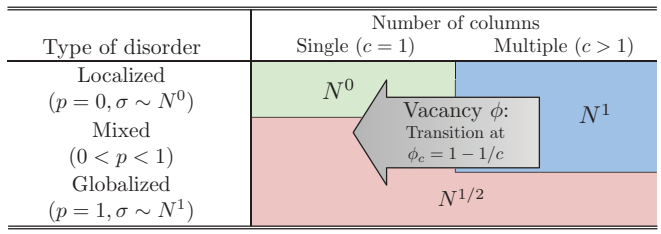

FIG. 7. (Color online) Summary of asymptotic scalings of $\langle T \rangle$. Note that N^0 stands for both saturation to a finite value and logarithmic scaling. The gray arrow indicates that vacancy effect changes the multicolumn scaling to the single-column one at ϕ_c.

To sum up, we have systematically investigated the relationship between the range of disorder in the passenger sequence and the asymptotic scaling behavior of boarding time using a simple boarding model. Our results clarify the origins of different scalings and indicate which type of disorder plays a dominant role. This offers a natural way to incorporate the boarding problem, previously regarded as an engineer's optimization problem, into the domain of physics.

Finally, we add a few remarks on other possible generalizations of our study. For example, we can allow fluctuations in the time required for each passenger to sit down. Since such fluctuations do not affect the scaling behavior of the longest blocking subsequence, they would not affect the scaling behaviors as long as they are finite and uncorrelated, whereas diverging or correlated fluctuations may produce interesting changes. We also note that the model considered in our study is similar to asymmetric simple exclusion process (ASEP) [20]. It would be interesting to check how concepts of ASEP apply to the case of boarding as differences like hopping rates of particles, diversity of destinations, and synchronous update are removed to varying extents. We hope these points to be satisfactorily addressed in future studies.

ACKNOWLEDGMENTS

The work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean Government (MEST) (No. 2012-0004216) (M.H.); (No. 2011-0028908) (Y.B., H.J.). M.H. would also acknowledge the generous hospitality of KIAS for Associate Member Program, funded by the MEST.

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