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Variational approach to bifurcation from infinity for nonlinear elliptic problems

Jaeyoung Byeon and **Youngae Lee**
Department of Mathematics, Pohang University of Science and Technology, Pohang, Kyungbuk 790-784, Republic of Korea
(jbyeon@postech.ac.kr; duddofl@postech.ac.kr)

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For any $N \geqslant 1$ and sufficiently small $\varepsilon > 0$, we find a positive solution of a nonlinear elliptic equation

$$\Delta u = \varepsilon^2(V(x)u - f(u)), \quad x \in \mathbb{R}^N,$$

when $\lim_{|x|\to\infty} V(x) = m > 0$ and some optimal conditions on f are satisfied. Furthermore, we investigate the asymptotic behaviour of the solution as $\varepsilon \to 0$.

1. Introduction

Consider a nonlinear eigenvalue problem

$$-\Delta u = \lambda(u - g(x,u)) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $g \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, and $\lim_{u\to 0} g(x,u)/u = 0$ uniformly for $x \in \Omega$. For any $\lambda \in \mathbb{R}$, $u \equiv 0$ is a trivial solution of (1.1).

Let Ω be a bounded domain of $\mathbb{R}^N$ and let $\lambda_k(\Omega) > 0$ be the kth eigenvalue of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

It is a classical result that $(\lambda_k(\Omega), 0)$ is a bifurcation point of problem (1.1), that is, any neighbourhood of $(\lambda_k(\Omega), 0)$ in $\mathbb{R} \times H_0^1(\Omega)$ contains a non-trivial solution of (1.1). In particular, when $k = 1$, there exist smooth functions $\lambda\colon (-\delta,\delta) \to \mathbb{R}$ and $\varphi\colon (-\delta,\delta) \to H_0^1(\Omega)$ such that $\lim_{s\to 0} \varphi(s)/s = u_1$, a first eigenfunction of (1.2), and $(\lambda(s), \varphi(s))$ is a solution of (1.1) (see [14]).

Stuart initially studied a case $\Omega = \mathbb{R}^N$, $N \geqslant 3$, in [33], typically when $g(x,u) = h(x)|u|^{p-1}u$, $h(x) \geqslant 0$, $\lim_{|x|\to\infty} h(x) = 0$, $\liminf_{|x|\to\infty} h(x)(1+|x|)^t > 0$ for $t \in (0,2)$ and $p \in (1, (N+2-2t)/(N-2))$. In this case, the result was that a bifurcation occurs from infinity at $\lambda = 0$, that is, there exist solutions $\{(v_l, \lambda_l)\}_{l=1}^\infty$ of (1.1) such that $\lim_{l\to\infty} \|\nabla v_l\|_{L^2(\mathbb{R}^N)} = \infty$ and $\lim_{l\to\infty} \lambda_l = 0$. Thus, a bifurcation from infinity occurs at the lowest point of the essential spectrum $[0,\infty)$ of $-\Delta$ on $\mathbb{R}^N$ without eigenvalues. The proof, based on a constraint minimization, states that $\lambda_l < 0$ and $u_l > 0$ or $u_l < 0$. He obtained a similar result for the similar type of problem

$$-\Delta u = \lambda u - g(x,u) \quad \text{in } \mathbb{R}^N \tag{1.3}$$

(see also [6,24,34,35] for further studies on the bifurcation problem, and the survey paper [36]).

On the other hand, Ambrosetti and Badiale in [2] applied the Lyapunov–Schmidt reduction method to the bifurcation problem

$$\left.\begin{aligned} v'' - \varepsilon^2 v + h(x)|v|^{p-1}v = 0, \quad x \in \mathbb{R}, \\ \lim_{|x|\to\infty} v(x) = 0. \end{aligned}\right\} \tag{1.4}$$

They showed, amongst other things, that if there exists $L > 0$ such that

$$\lim_{|x|\to\infty} h(x) = L, \qquad h(x) - L \in L^1(\mathbb{R}), \qquad \int_{\mathbb{R}} (h(x) - L)\,\mathrm{d}x \neq 0$$

and $1 < p < 5$, then (1.4) has a family of positive solutions bifurcating from the trivial solutions for small $\varepsilon > 0$. The same result holds in higher dimensions if $p \in (1, (N+2)/(N-2))$ (see [4,5]). For $p \geqslant 5$, there exist non-trivial solutions of (1.4), but they do not bifurcate from the trivial one. Note that, by a transformation $u(x) = \varepsilon^{-2/(p-1)}v(x/\varepsilon)$, (1.4) is transformed to

$$\left.\begin{aligned} u'' - u + h(x/\varepsilon)|u|^{p-1}u = 0, \quad x \in \mathbb{R}, \\ \lim_{|x|\to\infty} u(x) = 0. \end{aligned}\right\} \tag{1.5}$$

For any $R > 0$, $\lim_{\varepsilon\to 0} h(x/\varepsilon) = L = \lim_{|x|\to\infty} h(x)$ uniformly on $\mathbb{R}^N \setminus B(0,R)$. Thus, we have a limiting problem

$$\left.\begin{aligned} u'' - u + L|u|^{p-1}u = 0, \quad x \in \mathbb{R}, \\ \lim_{|x|\to\infty} u(x) = 0. \end{aligned}\right\} \tag{1.6}$$

Indeed, Ambrosetti and Badiale [2] constructed a solution of (1.5) as a perturbation of a solution of (1.6) for small $\varepsilon > 0$. Here we note that via a transformation $w(x) = u(\varepsilon x)$, equation (1.5) is transformed

$$\left.\begin{aligned} \frac{1}{\varepsilon^2} w'' - w + h(x)|w|^{p-1}w = 0, \quad x \in \mathbb{R}, \\ \lim_{|x|\to\infty} w(x) = 0. \end{aligned}\right\} \tag{1.7}$$

In this paper we study a similar type of equation:

$$\Delta u = \varepsilon^2 (V(x)u - f(u)), \quad u > 0, \quad u \in H^1(\mathbb{R}^N). \tag{1.8}$$

When $\varepsilon > 0$ is very large this corresponds to an equation for semiclassical standing waves of nonlinear Schrödinger equations. In this case, following work based on the Lyapunov–Schmidt reduction [19] and that based on a variational approach [31], there have been numerous further results to problem (1.8) (see [3, 10, 11, 15, 16, 18, 21, 25] and references therein). Note that, by a transformation $v(x) = u(x/\varepsilon)$, (1.8) is transformed to

$$\Delta v - V(x/\varepsilon)v + f(v) = 0, \quad v > 0, \quad v \in H^1(\mathbb{R}^N). \tag{1.9}$$

Although two opposite cases $0 < \varepsilon \ll 1$ and $1 \ll \varepsilon$ look quite contrastive, they share the same types of limiting equations

$$\Delta U - cU + f(U) = 0, \quad U > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} U(x) = 0, \tag{1.10}$$

where c is a positive constant. Our motivation comes from a classical result of Berestycki and Lions [7] which notes the existence of a least energy solution of (1.10) under some optimal conditions ((F1)–(F3) below) on f. Thus, it is desirable to construct a solution of (1.8) for small $\varepsilon > 0$ under the optimal conditions. Such a construction, for $\varepsilon > 0$ sufficiently large, was successfully carried out using a variational method in [10–12].

In addition to showing the existence of a solution to problem (1.8), we are concerned with the asymptotic behaviour of the solution. To see a fine asymptotic behaviour of a solution as $\varepsilon \to 0$, we need to know the shape of a least energy solution of limiting problem (1.10). If f is C^1, any solution of (1.10) is radially symmetric up to a translation and strictly decreasing. When f is just continuous, the symmetry and monotonicity of a least energy solution is proven in [13].

In § 2, we further prove that the radially symmetric solution is strictly decreasing; this property is essential to see a fine asymptotic behaviour of a solution as $\varepsilon \to 0$. It seems that the strict decreasing property of a radially symmetric solution cannot be derived by the rearrangement argument or maximum principles; interestingly we could derive the property from a generalized Pohozaev identity. Furthermore, when we try to see a fine asymptotic behaviour of a solution u_ε without monotonicity of $f(t)/t$, we have particular difficulty for the cases $N = 1, 2$ in contrast with the case $N \geqslant 3$. For some singularly perturbed nonlinear problems in bounded domain (see original papers [26–28] and some recent works [8, 9, 17]), it remains to show the asymptotic behaviour of a maximum point for a least energy solution under conditions (F1)–(F3) when $N = 2$. We believe that the argument in this paper for $N = 1, 2$ can be applied to the singularly perturbed problems.

We assume the following conditions for the potential function V.

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$.

(V2) $\lim_{|x|\to\infty} V(x) = m,\ m > 0$.

(V3) $V - m \in L^1(\mathbb{R}^N)$ and
$$\int_{\mathbb{R}^N} (V(x) - m)\,\mathrm{d}x < 0.$$

We also assume that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following.

(F1) $\lim_{t\to 0^+} f(t)/t = 0$.

(F2) If $N \geqslant 3$, $\limsup_{t\to\infty} f(t)/t^p < \infty$ for some $p \in (1, (N+2)/(N-2))$ and if $N = 2$, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|f(t)| \leqslant C_\alpha \exp(\alpha t^2)$ for all $t \geqslant 0$.

(F3) There exists $T > 0$ such that if $N \geqslant 2$, $\frac{1}{2}mT^2 < F(T)$ and if $N = 1$, $\frac{1}{2}mt^2 > F(t)$ for $0 < t < T$, $\frac{1}{2}mT^2 = F(T)$ and $mT < f(T)$, where
$$F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

Now we state our main theorem, showing the existence of solutions of (1.8) for small $\varepsilon > 0$.

Theorem 1.1. *Assume that hypotheses (V1)–(V3), (F1)–(F3) hold. Then for sufficiently small $\varepsilon > 0$, there exists a positive solution w_ε of (1.8) such that, after a transformation $u_\varepsilon(x) \equiv w_\varepsilon(x/\varepsilon)$, u_ε converges (up to a subsequence) uniformly to a radially symmetric least energy solution U of*

$$\Delta u - mu + f(u) = 0, \quad u > 0, \quad \lim_{|x|\to\infty} u(x) = 0 \tag{1.11}$$

satisfying $U(0) = \max\{\tilde{U}(0) \mid \tilde{U}$ solves (1.11)$\}$. Moreover, for a maximum point x_ε of u_ε it holds that $\lim_{\varepsilon\to 0} x_\varepsilon = 0$, and that, for some $c, C > 0$,

$$u_\varepsilon(x) + |\nabla u_\varepsilon(x)| \leqslant C \exp(-c|x|), \quad x \in \mathbb{R}^N.$$

In § 2, we introduce a variational framework and prepare some necessary propositions. In § 3, we prove theorem 1.1 in earnest. In § 4, we prove the existence of a solution u_ε for some more general class of V without a study of the asymptotic behaviour of the solution u_ε.

2. Preliminaries

Throughout this section, we assume that (F1)–(F3) hold. Instead of (1.8), we proceed with a transformed equation (1.9), since it is directly related to limiting problem (1.11).

The inner product $(\cdot,\cdot)$ is defined by

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + muv)\,\mathrm{d}x.$$

Let $H^1(\mathbb{R}^N)$ be a real Hilbert space, which is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + mu^2\,\mathrm{d}x\right)^{1/2}.$$

We also define $\Gamma_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Gamma_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V_\varepsilon u^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x,$$

where $V_\varepsilon(x) = V(x/\varepsilon)$. Since we are concerned with positive solutions, we may assume without loss of generality that $f(t) = 0$ for all $t \leqslant 0$. It is trivial to show that $\Gamma_\varepsilon \in C^1(H^1(\mathbb{R}^N))$. Clearly, a critical point of Γ_ε corresponds to a solution of (1.9).

The following is an associated limiting equation of (1.9):

$$\Delta u - mu + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbb{R}^N). \tag{2.1}$$

We define an energy functional for limiting equation (2.1) by

$$\Gamma(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + mu^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

We note that each solution U of (2.1) satisfies Pohozaev's identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla U|^2\,\mathrm{d}x + N\int_{\mathbb{R}^N}\frac{mU^2}{2} - F(U)\,\mathrm{d}x = 0. \tag{2.2}$$

Let S_m be the set of least energy solutions U of (2.1) satisfying

$$U(0) = \max_{x\in\mathbb{R}^N} U(x).$$

If f is C^1, any solution of (1.10) is obviously radially symmetric up to a translation and strictly decreasing. For the case when f is just continuous, the symmetry and monotonicity of a least energy solution is proven in [13]. Then, the symmetry and monotonicity of a least energy solution imply that there exist $C, c > 0$, independent of $U \in S_m$ such that

$$U(x) + |\nabla U(x)| \leqslant C\exp(-c|x|) \quad \text{for all } x \in \mathbb{R}^N. \tag{2.3}$$

Now we can also deduce that S_m is compact (see also previous works for $N \geqslant 3$ [10], and for $N = 1, 2$ [12]). Moreover, we have the following symmetry and strict monotone property of $U \in S_m$.

PROPOSITION 2.1. *Any $U \in S_m$ is radially symmetric and strictly decreasing with respect to $r = |x|$.*

Proof. As mentioned above, it is shown in [13] that any $U \in S_m$ is radially symmetric up to a translation and non-increasing with respect to $r = |x|$. Thus, it is sufficient to show that any radially symmetric least energy solution U of (2.1) is strictly decreasing. Let $|x| = r$. For any radially symmetric function $G(x) = G(|x|) \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and $a, b \geqslant 0$ we see that

$$\begin{aligned}
0 &= \int_a^b \left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}U}{\mathrm{d}r} - mU + f(U)\right)G(r)\frac{\mathrm{d}U}{\mathrm{d}r}r^N\,\mathrm{d}r \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}r}\left\{\left(\frac{1}{2}\left|\frac{\mathrm{d}U}{\mathrm{d}r}\right|^2 - \frac{mU^2}{2} + F(U)\right)G(r)r^N\right\} \\
&\qquad + \left(\frac{N-2}{2}G(r)r^{N-1} - \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}r}r^N\right)\left|\frac{\mathrm{d}U}{\mathrm{d}r}\right|^2 \\
&\qquad\qquad + \left(NG(r)r^{N-1} + \frac{\mathrm{d}G}{\mathrm{d}r}r^N\right)\left(\frac{mU^2}{2} - F(U)\right)\mathrm{d}r.
\end{aligned} \tag{2.4}$$

From the exponential decaying property of U and $|\mathrm{d}U/\mathrm{d}r|$, we see that for any $G \in C^1(\mathbb{R}^N)$ with an algebraic growth near ∞,

$$\begin{aligned}
\int_0^\infty &\left\{\left(\frac{N-2}{2}G(r) - \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left|\frac{\mathrm{d}U}{\mathrm{d}r}\right|^2\right. \\
&\qquad \left. + \left(NG(r) + \frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left(\frac{mU^2}{2} - F(U)\right)\right\}r^{N-1}\,\mathrm{d}r = 0.
\end{aligned}$$

Suppose that $U(r)$ is a constant M on some interval $I \subset [0, \infty)$.

First, we consider a case $N \geqslant 3$. Then, we choose any C^1-function G such that $G(r) = r^{N-2}$ on $[0,\infty) \setminus I$. Then, it follows that

$$\left(\frac{N-2}{2}G(r) - \frac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left|\frac{\mathrm{d}U}{\mathrm{d}r}\right|^2 \equiv 0 \quad \text{on } [0,\infty).$$

Now we get that

$$\begin{aligned} 0 &= \int_0^\infty \left(NG(r) + \frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left(\frac{mU^2}{2} - F(U)\right)r^{N-1}\,\mathrm{d}r \\ &= \int_{r\in I} \left(NG(r) + \frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left(\frac{mU^2}{2} - F(U)\right)r^{N-1}\,\mathrm{d}r \\ &\quad + \int_{r\notin I} \left(NG(r) + \frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left(\frac{mU^2}{2} - F(U)\right)r^{N-1}\,\mathrm{d}r. \end{aligned} \tag{2.5}$$

This means that an integration

$$\int_{r\in I} \left(NG(r) + \frac{\mathrm{d}G}{\mathrm{d}r}r\right)\left(\frac{mU^2}{2} - F(U)\right)r^{N-1}\,\mathrm{d}r$$

is independent for any C^1-function G satisfying $G(r) = r^{N-2}$ on $[0,\infty) \setminus I$. This implies that $mM^2/2 - F(M) = 0$. Since U is a C^2-solution of (2.1) on $r > 0$, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^2 - \frac{m}{2}U^2 + F(U)\right) &= \left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2} - mU + f(U)\right)\frac{\mathrm{d}U}{\mathrm{d}r} \\ &= -\frac{(N-1)}{r}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^2 \\ &\leqslant 0. \end{aligned} \tag{2.6}$$

Thus, a function

$$A(r) = \frac{1}{2}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^2 - \frac{m}{2}U^2 + F(U)$$

is monotone decreasing with respect to $r = |x|$. Then, since $\lim_{r\to\infty} A(r) = 0$ and $A(r) = 0$ on I, there exists $R > 0$ such that

$$A(r) = A'(r) = -\frac{(N-1)}{r}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^2 = 0 \quad \text{for all } r \geqslant R.$$

Thus, we get that U has compact support. By the Hopf lemma (see [20, § 3]),

$$\frac{\mathrm{d}U}{\mathrm{d}r}(x_0) \neq 0$$

for $x_0 \in \partial(\operatorname{supp} U)$. This contradicts $U \in C^2(\mathbb{R}^N/\{0\})$.

For $N = 2$, we choose any C^1-function G such that G is constant on $\mathbb{R}^N \setminus I$. Then, we get a contradiction in the same way as with the case $N \geqslant 3$.

For $N = 1$, since

$$\frac{1}{2}\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^2 - \frac{m}{2}U^2 + F(U) \equiv 0,$$

we get that

$$\int_{U(t_1)}^{U(t_2)} \frac{\mathrm{d}s}{\sqrt{ms^2 - 2F(s)}} = -(t_2 - t_1).$$

This implies that U is strictly decreasing. This completes the proof. □

To get an energy estimate, we will use the following estimation.

PROPOSITION 2.2. *Assume that (V1)–(V3) hold. Let* $W \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $W > 0$. *Then,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m) W(x) \, \mathrm{d}x = W(0) \int_{\mathbb{R}^N} (V(x) - m) \, \mathrm{d}x.$$

Proof. For any $k > 0$, there exists $r_k > 0$ such that $|W(x) - W(0)| < 1/k$ for $|x| \leqslant r_k$. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (V_\varepsilon(x) - m) W(x) \, \mathrm{d}x \\
&\quad = \varepsilon^N \biggl\{ \int_{|x| \leqslant r_k/\varepsilon} (V(x) - m) W(\varepsilon x) \, \mathrm{d}x + \int_{|x| \geqslant r_k/\varepsilon} (V(x) - m) W(\varepsilon x) \, \mathrm{d}x \biggr\} \\
&\quad = \varepsilon^N \biggl\{ \int_{|x| \leqslant r_k/\varepsilon} (V(x) - m)(W(\varepsilon x) - W(0)) \, \mathrm{d}x + W(0) \int_{\mathbb{R}^N} (V(x) - m) \, \mathrm{d}x \\
&\qquad\qquad + \int_{|x| \geqslant r_k/\varepsilon} (V(x) - m)(W(\varepsilon x) - W(0)) \, \mathrm{d}x \biggr\}.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\left| \varepsilon^{-N} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m) W(x) \, \mathrm{d}x - W(0) \int_{\mathbb{R}^N} (V(x) - m) \, \mathrm{d}x \right| \\
&\qquad\qquad \leqslant \frac{\|V - m\|_{L^1}}{k} + 2\|W\|_{L^\infty} \int_{|x| \geqslant r_k/\varepsilon} |V(x) - m| \, \mathrm{d}x.
\end{aligned}$$

This implies that

$$\lim_{\varepsilon \to 0} \left| \varepsilon^{-N} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m) W(x) \, \mathrm{d}x - W(0) \int_{\mathbb{R}^N} (V(x) - m) \, \mathrm{d}x \right| \leqslant \frac{\|V - m\|_{L^1}}{k};$$

then the conclusion follows. □

3. Proof of theorem 1.1

Throughout this section, we assume that (V1)–(V3) and (F1)–(F3) hold. As stated in § 2, we define S_m as the set of least energy solutions U of (2.1) satisfying $U(0) = \max_{x \in \mathbb{R}^N} U(x)$. Now we set $E_m = \Gamma(U)$ for $U \in S_m$. We will find a solution near the set

$$X \equiv \{U(\cdot - a) \mid a \in \mathbb{R}^N,\ U \in S_m\}.$$

For $\alpha \in \mathbb{R}$, we define $\Gamma_\varepsilon^\alpha = \{u \in H^1(\mathbb{R}^N) \mid \Gamma_\varepsilon(u) \leqslant \alpha\}$, and for a set $A \subset H^1(\mathbb{R}^N)$ and $d > 0$ let $A^d \equiv \{u \in H^1(\mathbb{R}^N) \mid \inf_{v \in A} \|u - v\| \leqslant d\}$.

Proposition 3.1. *There exists some* $t_0 > 0$ *and a continuous path* $\zeta\colon [0,t_0] \to H^1(\mathbb{R}^N)$ *satisfying* $\zeta(0) = 0$ *and* $\Gamma_\varepsilon(\zeta(t_0)) < -1$ *such that, for any* $U \in S_m$,

$$\max_{t\in[0,t_0]} \Gamma_\varepsilon(\zeta(t)) \leqslant E_m + \varepsilon^N \left\{ \frac{U^2(0)}{2} \int_{\mathbb{R}^N} (V(x) - m)\,\mathrm{d}x + o(1) \right\} \quad \textit{as } \varepsilon \to 0.$$

Moreover, for any small $\alpha > 0$, *there exists a constant* $\beta > 0$ *such that, for any* $t \in (0, t_0)$,

$$\zeta(t) \in X^\alpha \cup \Gamma_\varepsilon^{E_m-\beta}.$$

Proof. First, we consider the case $N \geqslant 3$. Now defining $\zeta\colon (0,\infty) \to H^1(\mathbb{R}^N)$ by

$$\zeta(t)(x) = U(x/t) \quad \text{and} \quad \zeta(0) = 0,$$

we see that $\zeta\colon [0,\infty) \to H(\mathbb{R}^N)$ is continuous. It is easy to see from (2.2) that

$$\lim_{t\to\infty} \Gamma(\zeta(t)) = -\infty.$$

Since

$$\Gamma_\varepsilon(\zeta(t)) = \Gamma(\zeta(t)) + \frac{1}{2}\int (V_\varepsilon - m)(\zeta(t))^2\,\mathrm{d}x = \Gamma(\zeta(t)) + O(\varepsilon^N),$$

there exists some large $t_0 > 0$ such that $\Gamma_\varepsilon(\zeta(t_0)) < -1$. Moreover, we compute that

$$\begin{aligned}\Gamma_\varepsilon(\zeta(t)) = {}& \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla U|^2\,\mathrm{d}x + t^N \int_{\mathbb{R}^N} \frac{m}{2}U^2 - F(U)\,\mathrm{d}x \\ & + \frac{1}{2}\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)U^2(x/t)\,\mathrm{d}x \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t} = {}& \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N} |\nabla U|^2\,\mathrm{d}x + Nt^{N-1}\int_{\mathbb{R}^N} \frac{m}{2}U^2 - F(U)\,\mathrm{d}x \\ & + \int_{\mathbb{R}^N} (V_\varepsilon(x) - m)U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Then, from the exponential decay of U and $|\nabla U|$ in (2.3), we see that

$$\left|\frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t}\right|_{t=1} = \left|\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)U(x)\nabla U\cdot x\,\mathrm{d}x\right| = O(\varepsilon^N) \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

Setting $x/t = y = (y_1,\dots,y_N)$ and $r = |y|$, we get from the radial symmetric property of $U \in S_m$ that

$$\sum_{i,j=1}^N D_{ij}U(y)y_iy_j = r^2\frac{\mathrm{d}^2U}{\mathrm{d}r^2} = -r(N-1)\frac{\mathrm{d}U}{\mathrm{d}r} + r^2(mU - f(U)). \tag{3.4}$$

Then, we see that $\Gamma_\varepsilon(\zeta(t))$ is a C^2-function with respect to $t \in (0,\infty)$, and that

$$\begin{aligned}
&\frac{\mathrm{d}^2\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t^2}\\
&\quad= \frac{(N-2)(N-3)}{2}t^{N-4}\int_{\mathbb{R}^N}|\nabla U|^2\,\mathrm{d}x\\
&\qquad + N(N-1)t^{N-2}\int_{\mathbb{R}^N}\frac{m}{2}U^2 - F(U)\,\mathrm{d}x\\
&\qquad + t^{-4}\int_{\mathbb{R}^N}(V_\varepsilon(x)-m)\biggl(|\nabla U(x/t)\cdot x|^2 + U(x/t)\sum_{i,j=1}^{N}D_{ij}U(x/t)x_ix_j\biggr)\,\mathrm{d}x\\
&\qquad + 2t^{-3}\int_{\mathbb{R}^N}(V_\varepsilon(x)-m)U(x/t)\nabla U(x/t)\cdot x\,\mathrm{d}x. \qquad (3.5)
\end{aligned}$$

Moreover, from (2.2), (2.3) and (3.4), we see that, if $\rho > 0$ is sufficiently small,

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}^2\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t^2} \leqslant -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla U|^2\,\mathrm{d}x, \qquad (3.6)$$

uniformly on $t \in (1-\rho, 1+\rho)$. This implies that there exists $t_\varepsilon \in [0, t_0]$ satisfying

$$\max_{s\in[0,1]}\Gamma_\varepsilon(\zeta(st_0)) = \Gamma_\varepsilon(\zeta(t_\varepsilon)) \quad\text{and}\quad \lim_{\varepsilon\to 0}t_\varepsilon = 1.$$

Then, there exists a point $\hat{t}_\varepsilon > 0$ between t_ε and 1 such that

$$0 = \frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t}\bigg|_{t=t_\varepsilon} = \frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t}\bigg|_{t=1} + (t_\varepsilon - 1)\frac{\mathrm{d}^2\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t^2}\bigg|_{t=\hat{t}_\varepsilon}.$$

From (3.3) and (3.6), we get that $|t_\varepsilon - 1| = O(\varepsilon^N)$ as $\varepsilon \to 0$. There also exists a point $t'_\varepsilon > 0$ between t_ε and 1 such that

$$\Gamma_\varepsilon(\zeta(t_\varepsilon)) = \Gamma_\varepsilon(\zeta(1)) + (t_\varepsilon - 1)\frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t}\bigg|_{t=t'_\varepsilon}.$$

We note that $\zeta(1) = U$ and

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t}\bigg|_{t=t'_\varepsilon} = 0.$$

Then, it follows from proposition 2.2 that

$$\begin{aligned}
\max_{s\in[0,1]}\Gamma_\varepsilon(\zeta(st_0)) &= \Gamma_\varepsilon(\zeta(t_\varepsilon)) = \Gamma_\varepsilon(U) + o(\varepsilon^N)\\
&= \Gamma(U) + \frac{1}{2}\int_{\mathbb{R}^N}(V_\varepsilon - m)U^2\,\mathrm{d}x + o(\varepsilon^N)\\
&\leqslant E_m + \varepsilon^N\biggl\{\frac{U^2(0)}{2}\int_{\mathbb{R}^N}(V(x)-m)\,\mathrm{d}x + o(1)\biggr\} \quad \text{as } \varepsilon\to 0. \qquad (3.7)
\end{aligned}$$

Second, we consider a case $N = 2$. Here we use an idea similar to [12, 23]. We denote $h(s) \equiv -ms + f(s)$, $H(s) \equiv -\frac{1}{2}ms^2 + F(s)$. Define a function $g(\theta, t)\colon (0,\infty)\times$

$(0,\infty)\to\mathbb{R}$ by

$$g(\theta,t)\equiv\Gamma(\theta U(\cdot/t))=\frac{\theta^2}{2}\|\nabla U\|_{L^2}^2-t^2\int_{\mathbb{R}^2}H(\theta U)\,\mathrm{d}x,$$

and a function $g_\varepsilon(\theta,t)\colon(0,\infty)\times(0,\infty)\to\mathbb{R}$ by

$$g_\varepsilon(\theta,t)\equiv\Gamma_\varepsilon(\theta U(\cdot/t))=g(\theta,t)+\frac{1}{2}\int_{\mathbb{R}^2}(V_\varepsilon(x)-m)\theta^2U^2(x/t)\,\mathrm{d}x.$$

Then we see that

$$\begin{aligned}g_\theta(\theta,t)&=\theta\|\nabla U\|_{L^2}^2-t^2\int_{\mathbb{R}^2}h(\theta U)U\,\mathrm{d}x,\\ g_t(\theta,t)&=-2t\int_{\mathbb{R}^2}H(\theta U)\,\mathrm{d}x.\end{aligned}$$

Then, we can find a small $\tau_0\in(0,1)$ such that

$$g_\theta(\theta,t)=\theta\bigg(\|\nabla U\|_{L^2}^2-t^2\int_{\mathbb{R}^2}\frac{h(\theta U)}{\theta U}U^2\,\mathrm{d}x\bigg)\geqslant\frac{\theta}{2}\|\nabla U\|_{L^2}^2>0,\tag{3.8}$$

for $\theta\in(0,2]$, $t\in[0,\tau_0]$. Similarly, we see that if $\varepsilon>0$ is sufficiently small,

$$\begin{aligned}(g_\varepsilon)_\theta(\theta,t)&=g_\theta(\theta,t)+\theta\int_{\mathbb{R}^2}(V_\varepsilon(x)-m)U^2(x/t)\,\mathrm{d}x\\&>\tfrac{1}{4}\theta\|\nabla U\|_{L^2}^2>0\quad\text{for }\theta\in(0,2],\ t\in[0,\tau_0].\end{aligned}\tag{3.9}$$

Since U satisfies (2.1) and (2.2), we get

$$\int_{\mathbb{R}^2}H(U)=0,\qquad\int_{\mathbb{R}^2}h(U)U\,\mathrm{d}x=\|\nabla U\|_{L^2}^2>0.$$

Thus there exist constants $\theta_1,\theta_2>0$ satisfying $\theta_1<1<\theta_2<2$ such that

$$\frac{\partial}{\partial\theta}\int_{\mathbb{R}^2}H(\theta U)\,\mathrm{d}x=\int_{\mathbb{R}^2}h(\theta U)U\,\mathrm{d}x>\frac{1}{2}\|\nabla U\|_{L^2}^2>0\quad\text{for }\theta\in[\theta_1,\theta_2].\tag{3.10}$$

From the exponential decaying property of $|\nabla U|$ in (2.3), we see that, for $\tau_0\leqslant t$, $\theta_1\leqslant\theta\leqslant\theta_2$,

$$\begin{aligned}(g_\varepsilon)_{t\theta}(\theta,t)&=-2t\int_{\mathbb{R}^2}h(\theta U)U\,\mathrm{d}x+2\theta\int_{\mathbb{R}^2}(V_\varepsilon(x)-m)U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x\\&\leqslant-2\tau_0\int_{\mathbb{R}^2}h(\theta U)U\,\mathrm{d}x+2\theta\int_{\mathbb{R}^2}(V_\varepsilon(x)-m)U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x\\&\leqslant-\tau_0\|\nabla U\|_{L^2}^2+2\theta\int_{\mathbb{R}^2}(V_\varepsilon(x)-m)U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x\\&\leqslant-\frac{\tau_0}{2}\|\nabla U\|_{L^2}^2\\&<0,\end{aligned}\tag{3.11}$$

$$
\begin{aligned}
(g_\varepsilon)_t(\theta_1, t) &= -2t \int_{\mathbb{R}^2} H(\theta_1 U)\,\mathrm{d}x + \int_{\mathbb{R}^2} (V_\varepsilon(x) - m)\theta_1^2 U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x \\
&\geqslant -2\tau_0 \int_{\mathbb{R}^2} H(\theta_1 U)\,\mathrm{d}x + \int_{\mathbb{R}^2} (V_\varepsilon(x) - m)\theta_1^2 U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x \\
&\geqslant -\tau_0 \int_{\mathbb{R}^2} H(\theta_1 U)\,\mathrm{d}x \\
&> 0
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
(g_\varepsilon)_t(\theta_2, t) &= -2t \int_{\mathbb{R}^2} H(\theta_2 U)\,\mathrm{d}x + \int_{\mathbb{R}^2} (V_\varepsilon(x) - m)\theta_2^2 U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x \\
&\leqslant -2\tau_0 \int_{\mathbb{R}^2} H(\theta_2 U)\,\mathrm{d}x + \int_{\mathbb{R}^2} (V_\varepsilon(x) - m)\theta_2^2 U(x/t)\nabla U(x/t)\cdot(-x/t^2)\,\mathrm{d}x \\
&\leqslant -\tau_0 \int_{\mathbb{R}^2} H(\theta_2 U)\,\mathrm{d}x \\
&< 0
\end{aligned} \tag{3.13}
$$

if $\varepsilon > 0$ is sufficiently small. Applying the mean-value theorem and the implicit function theorem to (3.11)–(3.13), we see that there exists a continuous function $\theta_\varepsilon\colon [\tau_0, \infty) \to \mathbb{R}$ such that $\theta_\varepsilon(t) \in (\theta_1, \theta_2)$ satisfies

$$
(g_\varepsilon)_t(\theta, t) \begin{cases} > 0 & \text{for } \theta \in [\theta_1, \theta_\varepsilon(t)), \\ = 0 & \text{for } \theta = \theta_\varepsilon(t), \\ < 0 & \text{for } \theta \in (\theta_\varepsilon(t), \theta_2]. \end{cases} \tag{3.14}
$$

Moreover, there exists $C > 0$ such that for $t \geqslant \tau_0$

$$
|(g_\varepsilon)_t(1, t)| = \left| \int_{\mathbb{R}^2} (V_\varepsilon(x) - m) U(x/t)\nabla U(x/t)\cdot(x/t^2)\,\mathrm{d}x \right| \leqslant C\varepsilon^2 \tag{3.15}
$$

if $\varepsilon > 0$ is sufficiently small. From (3.11), there exists a constant $D > 0$ such that $|\theta_\varepsilon(t) - 1| \leqslant D\varepsilon^2$ for $t \geqslant \tau_0$ and small $\varepsilon > 0$. Now we define that

$$
\left.\begin{aligned} \inf_{t\geqslant\tau_0} \theta_\varepsilon(t) &\equiv \underline{\theta_\varepsilon}, \\ \sup_{t\geqslant\tau_0} \theta_\varepsilon(t) &\equiv \overline{\theta_\varepsilon}. \end{aligned}\right\} \tag{3.16}
$$

Then, we get that for small $\varepsilon > 0$, $|\underline{\theta_\varepsilon} - 1| \leqslant D\varepsilon^2$ and $|\overline{\theta_\varepsilon} - 1| \leqslant D\varepsilon^2$; this implies that $\underline{\theta_\varepsilon} - D\varepsilon^2 \leqslant 1 \leqslant \overline{\theta_\varepsilon} + D\varepsilon^2$. For $t \geqslant \tau_0$, we see that

$$
\left.\begin{aligned} (g_\varepsilon)_t(\underline{\theta_\varepsilon} - D\varepsilon^2, t) > 0, \\ (g_\varepsilon)_t(\overline{\theta_\varepsilon} + D\varepsilon^2, t) < 0. \end{aligned}\right\} \tag{3.17}
$$

For small $\varepsilon > 0$, let $\hat{\zeta}(s) = (\theta(s), t(s))\colon [0, \infty) \to \mathbb{R}^2$ be a piecewise linear injective curve joining

$$
(0, \tau_0) \to (\underline{\theta_\varepsilon} - D\varepsilon^2, \tau_0) \to (\underline{\theta_\varepsilon} - D\varepsilon^2, 1) \to (\overline{\theta_\varepsilon} + D\varepsilon^2, 1) \to (\overline{\theta_\varepsilon} + D\varepsilon^2, \infty), \tag{3.18}
$$

where each line segment in the image of $\tilde{\zeta}$ is parallel to axes. Let $0 \equiv \hat{s}_0 < \hat{s}_1 < \cdots < \hat{s}_4 \equiv \infty$ be such that for each $i = 0, \ldots, 4$, $\hat{\zeta}(\hat{s}_i)$ is the end point of a linear segment of the piecewise linear curve $\hat{\zeta}$. Then, we see that the function $s \mapsto \Gamma_\varepsilon(\theta(s)U(x/t(s)))$ is strictly increasing on $(\hat{s}_0, \hat{s}_1)$, $(\hat{s}_1, \hat{s}_2)$ by (3.9), (3.17) respectively. We also see that the function $s \mapsto \Gamma_\varepsilon(\theta(s)U(x/t(s)))$ is strictly decreasing on $(\hat{s}_3, \hat{s}_4)$ by (3.17). There exists $s_0 > 0$ such that $\Gamma_\varepsilon(\theta(s_0)U(\cdot/t(s_0))) < -1$, where $\theta(s_0) = \overline{\theta_\varepsilon} + D\varepsilon^2$ and $t(s_0) > 1$. Now for $N = 2$, we define $\zeta(s)(x) = \theta(s)U(x/t(s))$, which is actually dependent on $\varepsilon > 0$. From the monotone property of $\Gamma_\varepsilon(\zeta(\cdot))$ on $(\hat{s}_i, \hat{s}_{i+1})$, $i = 0, 1, 3$, we get that

$$\max_{s \in [0, s_0]} \Gamma_\varepsilon(\zeta(s)) = \Gamma_\varepsilon(\theta_\varepsilon U) \quad \text{for some } \theta_\varepsilon \in [\underline{\theta_\varepsilon} - D\varepsilon^2, \overline{\theta_\varepsilon} + D\varepsilon^2].$$

Now we note that there exists $\hat{\theta}_\varepsilon > 0$ between 1 and θ_ε satisfying

$$\Gamma(\theta_\varepsilon U) = \Gamma(U) + (\theta_\varepsilon - 1)\frac{\mathrm{d}\Gamma(\theta U)}{\mathrm{d}\theta}\bigg|_{\theta = \hat{\theta}_\varepsilon}. \tag{3.19}$$

Since $|\theta_\varepsilon - 1| \leqslant 2D\varepsilon^2$, it follows that

$$\lim_{\varepsilon \to 0} \frac{\mathrm{d}\Gamma(\theta U)}{\mathrm{d}\theta}\bigg|_{\theta = \hat{\theta}_\varepsilon} = 0.$$

Then, it follows from proposition (2.2) that

$$\begin{aligned}
\max_{s \in [0, s_0]} \Gamma_\varepsilon(\zeta(s)) &= \Gamma_\varepsilon(\theta_\varepsilon U) \\
&= \Gamma(\theta_\varepsilon U) + \frac{1}{2}\int_{\mathbb{R}^2} (V_\varepsilon(x) - m)\theta_\varepsilon^2 U^2(x)\,\mathrm{d}x \\
&= \Gamma(U) + (\theta_\varepsilon - 1)\frac{\mathrm{d}\Gamma(\theta U)}{\mathrm{d}\theta}\bigg|_{\theta = \hat{\theta}_\varepsilon} + \frac{1}{2}\int_{\mathbb{R}^2} (V_\varepsilon(x) - m)\theta_\varepsilon^2 U^2(x)\,\mathrm{d}x \\
&\leqslant E_m + \frac{\varepsilon^2}{2}\bigg\{U^2(0)\int_{\mathbb{R}^2} (V(x) - m)\,\mathrm{d}x + o(1)\bigg\} \quad \text{as } \varepsilon \to 0. \qquad (3.20)
\end{aligned}$$

Finally, we consider the case $N = 1$. We note that S_m consists of one element $U \in H^1(\mathbb{R})$ and, in addition, $U(0) = T$, where $T > 0$ is given in (F3). Let $\rho > 0$ and define $q \colon \mathbb{R} \to \mathbb{R}$ by

$$q(x) = \begin{cases} U(x), & x \in [0, \infty), \\ x^4 + U(0), & x \in [-\rho, 0], \\ \rho^4 + U(0), & x \in (-\infty, -\rho]. \end{cases} \tag{3.21}$$

From (F3) and $U(0) = T$, we can choose $\rho > 0$ so that for $x \in [-\rho, 0)$

$$\begin{aligned}
\tfrac{1}{2}(q'(x))^2 + \tfrac{1}{2}mq^2(x) - F(q(x)) &= 8x^6 + \tfrac{1}{2}m(x^4 + U(0))^2 - F(x^4 + U(0)) \\
&< 0. \qquad (3.22)
\end{aligned}$$

Now defining $\zeta \colon (0, \infty) \to H^1(\mathbb{R})$ by

$$\zeta(t)(x) = q(|x| - \ln t) \quad \text{and} \quad \zeta(0) = 0,$$

we see that $\zeta\colon [0,\infty) \to H^1(\mathbb{R})$ is continuous. Using (3.22) and (F3), it is easy to see that

$$\Gamma(\zeta(t)) = \begin{cases} E_m + \displaystyle\int_{-\ln t}^{0} |U'(x)|^2 + mU^2(x) - 2F(U(x))\,\mathrm{d}x < E_m, & 0 < t < 1, \\ E_m + \displaystyle\int_{-\ln t}^{0} |q'(x)|^2 + mq^2(x) - 2F(q(x))\,\mathrm{d}x < E_m, & t > 1. \end{cases} \tag{3.23}$$

From (3.22), it follows that

$$\begin{aligned} \Gamma(\zeta(t)) &\leqslant E_m + \int_{-\ln t}^{-\rho} |q'(x)|^2 + mq^2(x) - 2F(q(x))\,\mathrm{d}x \\ &= E_m + (\ln t - \rho)\{m(\rho^4 + U(0))^2 - 2F(\rho^4 + U(0))\} \to -\infty \quad \text{as } t \to \infty. \end{aligned} \tag{3.24}$$

Since

$$\Gamma_\varepsilon(\zeta(t)) = \Gamma(\zeta(t)) + \frac{1}{2}\int (V_\varepsilon - m)(\zeta(t))^2\,\mathrm{d}x = \Gamma(\zeta(t)) + O(\varepsilon),$$

there exists some large $t_0 > 0$ such that $\Gamma_\varepsilon(\zeta(t_0)) < -1$. Now we define

$$Q(x) = \begin{cases} |U'(x)|^2 + mU^2(x) - 2F(U(x)) & \text{for } x \geqslant 0, \\ |q'(x)|^2 + mq^2(x) - 2F(q(x)) & \text{for } x \leqslant 0. \end{cases} \tag{3.25}$$

We see that

$$\Gamma_\varepsilon(\zeta(t)) = E_m + \int_{-\ln t}^{0} Q(x)\,\mathrm{d}x + \frac{1}{2}\int (V_\varepsilon - m)(\zeta(t))^2\,\mathrm{d}x \tag{3.26}$$

and

$$\frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t} = \frac{Q(-\ln t)}{t} + \int (V_\varepsilon - m)\zeta(t)\frac{\partial \zeta(t)}{\partial t}\,\mathrm{d}x, \tag{3.27}$$

where, for $H(t) = -mt^2/2 + F(t)$,

$$Q(-\ln t) = \begin{cases} |U'(-\ln t)|^2 - 2H(U(-\ln t)) & \text{for } 0 < t < 1, \\ 16(-\ln t)^6 - 2H((-\ln t)^4 + U(0)) & \text{for } 1 < t < \mathrm{e}^\rho, \\ -2H(\rho^4 + U(0)) & \text{for } t > \mathrm{e}^\rho. \end{cases} \tag{3.28}$$

Now we see that $\mathrm{d}\Gamma_\varepsilon(\zeta(t))/\mathrm{d}t$ is continuous on $\{t \mid 0 < t < \mathrm{e}^\rho\}$. From the exponential decaying property of $|U'(x)|$, we have that

$$\left|\frac{\partial \zeta(t)(x)}{\partial t}\right| = \begin{cases} |U'(|x| - \ln t)/t| \leqslant 1 & \text{for } 0 \leqslant t < \mathrm{e}^{|x|}, \\ |4(|x| - \ln t)^3/t| \leqslant 4\rho^3 & \text{for } \mathrm{e}^{|x|} < t < \mathrm{e}^{\rho+|x|}, \\ 0 & \text{for } \mathrm{e}^{\rho+|x|} < t. \end{cases} \tag{3.29}$$

Then we get that

$$\int (V_\varepsilon - m)\zeta(t)\frac{\partial \zeta(t)}{\partial t}\,\mathrm{d}x = O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

We note that

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}\Gamma_\varepsilon(\zeta(t))}{\mathrm{d}t}$$

$$= \begin{cases} \dfrac{|U'(-\ln t)|^2 + mU^2(-\ln t) - 2F(U(-\ln t))}{t} > 0, & 0 < t < 1, \\ \dfrac{16(-\ln t)^6 + m((-\ln t)^4 + U(0))^2 - 2F((-\ln t)^4 + U(0))}{t} < 0, & 1 < t < \mathrm{e}^\rho, \\ \dfrac{m(\rho^4 + U(0))^2 - 2F(\rho^4 + U(0))}{t} < 0, & t > \mathrm{e}^\rho. \end{cases} \tag{3.30}$$

Thus, $\Gamma_\varepsilon(\zeta(t))$ has a maximum at t_ε such that $\lim_{\varepsilon\to 0} t_\varepsilon = 1$. Also we have that

$$\zeta(t_\varepsilon)(x) - U(x) = \begin{cases} U(|x| - \ln t_\varepsilon) - U(x) & \text{for } |x| - \ln t_\varepsilon \geqslant 0, \\ (|x| - \ln t_\varepsilon)^4 + U(0) - U(x) & \text{for } -\rho < |x| - \ln t_\varepsilon \leqslant 0, \\ \rho^4 + U(0) - U(x) & \text{for } -\infty < |x| - \ln t_\varepsilon < -\rho. \end{cases} \tag{3.31}$$

Since $\lim_{\varepsilon\to 0}(\ln t_\varepsilon - \rho) < -\rho/2 < 0$, we get that

$$\zeta(t_\varepsilon)(x) - U(x) = \begin{cases} U(|x| - \ln t_\varepsilon) - U(x) & \text{for } |x| - \ln t_\varepsilon \geqslant 0, \\ (|x| - \ln t_\varepsilon)^4 + U(0) - U(x) & \text{for } -\ln t_\varepsilon \leqslant |x| - \ln t_\varepsilon \leqslant 0. \end{cases} \tag{3.32}$$

Thus, we obtain that $\max_{x\in\mathbb{R}} |\zeta(t_\varepsilon)(x) - U(x)| = o(1)$ as $\varepsilon \to 0$. Moreover, we have from (3.23) that $\max_{t\in[0,\infty)} \Gamma(\zeta(t)) = \Gamma(\zeta(1)) = \Gamma(U) = E_m$. Then, it follows from proposition 2.2 that

$$\begin{aligned} \max_{t\in[0,t_0]} \Gamma_\varepsilon(\zeta(t)) &= \Gamma_\varepsilon(\zeta(t_\varepsilon)) \\ &= \Gamma(\zeta(t_\varepsilon)) + \frac{1}{2}\int (V_\varepsilon - m)(\zeta(t_\varepsilon))^2\,\mathrm{d}x \\ &= \Gamma(\zeta(t_\varepsilon)) + \frac{1}{2}\int (V_\varepsilon - m)U^2\,\mathrm{d}x + \frac{1}{2}\int (V_\varepsilon - m)((\zeta(t_\varepsilon))^2 - U^2)\,\mathrm{d}x \\ &\leqslant E_m + \frac{\varepsilon}{2}\left\{U^2(0)\int (V(x) - m)\,\mathrm{d}x + o(1)\right\} \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.33}$$

Lastly, the property $\zeta(t) \in X^\alpha \cup \Gamma_\varepsilon^{E_m - \beta}$ comes directly from the construction of ζ. □

For a path ζ in proposition 3.1, we take a sufficiently large $G > 0$ satisfying

$$G > 2\max_{0\leqslant s\leqslant 1}\{\mathrm{dist}(\zeta(st_0), X)\}.$$

We define

$$\Phi \equiv \{\gamma \in C([0,1], X^G) \mid \gamma(0) = 0 \text{ and } \gamma(1) = \zeta(t_0)\},$$
$$D_\varepsilon = \max_{s\in[0,1]} \Gamma_\varepsilon(\zeta(st_0)),$$

and

$$C_\varepsilon = \inf_{\gamma\in\Phi} \max_{s\in[0,1]} \Gamma_\varepsilon(\gamma(s)).$$

Here we note that the min–max value C_ε is a local, not global, mountain-pass level since $\Phi \subset C([0,1], X^G)$.

From proposition 3.1, it follows that

$$C_\varepsilon \leqslant D_\varepsilon \leqslant E_m + \frac{\varepsilon^N}{2}\left\{U^2(0)\int_{\mathbb{R}^N}(V(x)-m)\,\mathrm{d}x + o(1)\right\} < E_m \quad \text{as } \varepsilon \to 0.$$

Now we get the following lower estimation of C_ε.

PROPOSITION 3.2. *$E_m \leqslant \liminf_{\varepsilon\to 0} C_\varepsilon$.*

Proof. On the contrary, we assume that $\liminf_{\varepsilon\to 0} C_\varepsilon < E_m$. Then, there exists $\alpha > 0, \varepsilon_n \to 0$ and $\gamma_n \in \Phi$ satisfying $\Gamma_{\varepsilon_n}(\gamma_n(s)) < E_m - \alpha$ for $s \in [0,1]$.

We see from (V2) that for any $k > 0$, there exists $R_k > 0$ such that $|V(x) - m| < 1/k$ for $|x| \geqslant R_k$. There is a constant $M > 0$ such that

$$\max_{s\in[0,1]} \|\gamma(s)\| \leqslant M$$

for all $\gamma \in \Phi$, since X is bounded in $H^1(\mathbb{R}^N)$ and $\Phi \subset C([0,1], X^G)$. From the facts $\Gamma(\gamma_n(0)) = 0$, $\Gamma(\gamma_n(1)) < 0$ and the results in [22] and [23] which state that

$$E_m \leqslant \max_{s\in[0,1]} \Gamma(\eta(s))$$

for any $\eta \in C([0,1], H^1(\mathbb{R}^N))$ satisfying $\eta(0) = 0$ and $\Gamma(\eta(1)) < 0$, we see that

$$\max_{s\in[0,1]} \Gamma(\gamma_n(s)) \geqslant E_m. \tag{3.34}$$

From the Sobolev inequality in [1] and the Hölder inequality, there exist some constants $c, C > 0$ such that for any $k, n > 0$,

$$\begin{aligned}
&E_m - \alpha \\
&\quad\geqslant \max_{s\in[0,1]} \Gamma_{\varepsilon_n}(\gamma_n(s)) \\
&\quad\geqslant \max_{s\in[0,1]} \left\{\Gamma(\gamma_n(s)) - \frac{1}{2}\left|\int_{|x|\geqslant\varepsilon_n R_k}(V_{\varepsilon_n} - m)\gamma_n^2(s)\,\mathrm{d}x\right|\right. \\
&\qquad\qquad\qquad\left. - \frac{1}{2}\left|\int_{|x|\leqslant\varepsilon_n R_k}(V_{\varepsilon_n} - m)\gamma_n^2(s)\,\mathrm{d}x\right|\right\} \\
&\quad\geqslant E_m - \frac{1}{2}\max_{s\in[0,1]}\left\{\left|\int_{|x|\geqslant\varepsilon_n R_k}(V_{\varepsilon_n} - m)\gamma_n^2(s)\,\mathrm{d}x\right| - \left|\int_{|x|\leqslant\varepsilon_n R_k}(V_{\varepsilon_n} - m)\gamma_n^2(s)\,\mathrm{d}x\right|\right\}
\end{aligned}$$

$$\geqslant \begin{cases} E_m - \dfrac{1}{2k} \max\limits_{s\in[0,1]} \displaystyle\int_{\mathbb{R}^N} \gamma_n^2(s)\,\mathrm{d}x - c \max\limits_{s\in[0,1]} (\varepsilon_n R_k)^2 \|\gamma_n\|^2_{L^2 N(N-2)} & \\ & \text{for } N \geqslant 3, \\ E_m - \dfrac{1}{2k} \max\limits_{s\in[0,1]} \displaystyle\int_{\mathbb{R}^N} \gamma_n^2(s)\,\mathrm{d}x - c \max\limits_{s\in[0,1]} (\varepsilon_n R_k) \|\gamma_n\|^2_{L^4} & \text{for } N = 2, \\ E_m - \dfrac{1}{2k} \max\limits_{s\in[0,1]} \displaystyle\int_{\mathbb{R}^N} \gamma_n^2(s)\,\mathrm{d}x - c \max\limits_{s\in[0,1]} (\varepsilon_n R_k) \|\gamma_n\|^2_{L^\infty(\mathbb{R}^N)} & \text{for } N = 1, \end{cases}$$

$$\geqslant \begin{cases} E_m - \dfrac{M^2}{2k} - C(\varepsilon_n R_k)^2 M^2 & \text{for } N \geqslant 3, \\ E_m - \dfrac{M^2}{2k} - C(\varepsilon_n R_k) M^2 & \text{for } N = 1, 2. \end{cases} \tag{3.35}$$

Taking $k > 0$ such that $M^2/k \leqslant \alpha$ and sufficiently large $n > 0$, we get a contradiction. □

Proposition 3.3. *Let $d_1 > d_2 > 0$ be sufficiently small. There exist constants $w > 0$ and $\varepsilon_0 > 0$ such that $\|\Gamma'_\varepsilon(u)\| \geqslant w$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X^{d_1} \setminus X^{d_2})$ and $0 < \varepsilon \leqslant \varepsilon_0$.*

Proof. On the contrary, we suppose that, for small $d_1 > d_2 > 0$, there exists $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} \varepsilon_i = 0$ and $u_{\varepsilon_i} \in X^{d_1} \setminus X^{d_2}$ satisfying $\lim_{i\to\infty} \|\Gamma'_{\varepsilon_i}(u_{\varepsilon_i})\| = 0$ and $\Gamma_{\varepsilon_i}(u_{\varepsilon_i}) \leqslant D_{\varepsilon_i}$. For the sake of convenience we write ε for ε_i. Now we set $u_\varepsilon = z_\varepsilon(\cdot - a_\varepsilon) + w_\varepsilon$ where $z_\varepsilon \in S_m$, $a_\varepsilon \in \mathbb{R}^N$, and $d_2 \leqslant \|w_\varepsilon\| \leqslant d_1$. Then,

$$\eta_\varepsilon = u_\varepsilon(\cdot + a_\varepsilon) \in X^{d_1} \setminus X^{d_2}.$$

We see from (V2) that, for any $k > 0$, there exists $R_k > 0$ such that $|V(x) - m| < 1/k$ for $|x| \geqslant R_k$. By the Sobolev inequalities in [1] and Hölder's inequality, it follows that, for some constant $C > 0$,

$$\begin{aligned} \Gamma_\varepsilon(\eta_\varepsilon) &= \Gamma_\varepsilon(u_\varepsilon) + \frac{1}{2} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m)(\eta_\varepsilon^2(x) - u_\varepsilon^2(x))\,\mathrm{d}x \\ &= \Gamma_\varepsilon(u_\varepsilon) + \frac{1}{2} \int_{|x|\geqslant \varepsilon R_k} (V_\varepsilon(x) - m)(\eta_\varepsilon^2(x) - u_\varepsilon^2(x))\,\mathrm{d}x \\ &\quad + \frac{1}{2} \int_{|x|\leqslant \varepsilon R_k} (V_\varepsilon(x) - m)(\eta_\varepsilon^2(x) - u_\varepsilon^2(x))\,\mathrm{d}x \\ &\leqslant \begin{cases} D_\varepsilon + \dfrac{\|u_\varepsilon\|^2}{k} + C(\varepsilon R_k)^2 \|u_\varepsilon\|^2_{L^{2N/(N-2)}(\mathbb{R}^N)} & \text{for } N \geqslant 3, \\ D_\varepsilon + \dfrac{\|u_\varepsilon\|^2}{k} + C\varepsilon R_k \|u_\varepsilon\|^2_{L^4(\mathbb{R}^N)} & \text{for } N = 2, \\ D_\varepsilon + \dfrac{\|u_\varepsilon\|^2}{k} + C\varepsilon R_k \|u_\varepsilon\|^2_{L^\infty(\mathbb{R}^N)} & \text{for } N = 1, \end{cases} \\ &\leqslant \begin{cases} D_\varepsilon + \|u_\varepsilon\|^2 \left(\dfrac{1}{k} + C(\varepsilon R_k)^2\right) & \text{for } N \geqslant 3, \\ D_\varepsilon + \|u_\varepsilon\|^2 \left(\dfrac{1}{k} + C\varepsilon R_k\right) & \text{for } N = 1, 2. \end{cases} \end{aligned} \tag{3.36}$$

Since X is norm bounded and $k>0$ is arbitrary, it follows that

$$\limsup_{\varepsilon\to 0} \Gamma_\varepsilon(\eta_\varepsilon) \leqslant E_m. \tag{3.37}$$

Given any $v\in C_0^\infty(\mathbb{R}^N)$, $\|v\|\leqslant 1$, and $k>0$, we see, as in the above estimate of $\Gamma_\varepsilon(\eta_\varepsilon)$, that for some $C>0$,

$$\begin{aligned}
|\Gamma'_\varepsilon(\eta_\varepsilon)(v)| &= \left|\Gamma'_\varepsilon(u_\varepsilon)(v(\cdot - a_\varepsilon)) + \int_{\mathbb{R}^N} V_\varepsilon(\eta_\varepsilon v - u_\varepsilon v(\cdot - a_\varepsilon))\,\mathrm{d}x\right| \\
&\leqslant \|\Gamma'_\varepsilon(u_\varepsilon)\| + \left|\int_{\mathbb{R}^N}(V_\varepsilon - m)(\eta_\varepsilon v - u_\varepsilon v(\cdot - a_\varepsilon))\,\mathrm{d}x\right| \\
&\leqslant \begin{cases} \|\Gamma'_\varepsilon(u_\varepsilon)\| + \|u_\varepsilon\|\left(\dfrac{1}{k} + C(\varepsilon R_k)\right) & \text{for } N\geqslant 3, \\ \|\Gamma'_\varepsilon(u_\varepsilon)\| + \|u_\varepsilon\|\left(\dfrac{1}{k} + C(\varepsilon R_k)^{1/2}\right) & \text{for } N = 1,2. \end{cases}
\end{aligned}$$

Thus, it follows that

$$\lim_{\varepsilon\to 0}\|\Gamma'_\varepsilon(\eta_\varepsilon)\| = 0. \tag{3.38}$$

By the compactness of S_m in $H^1(\mathbb{R}^N)$, there exists $z\in S_m$ such that $z_\varepsilon\to z$ in $H^1(\mathbb{R}^N)$. Then, for sufficiently small $\varepsilon>0$, it follows that

$$\|\eta_\varepsilon - z\| = \|(z_\varepsilon - z) + w_\varepsilon(\cdot + a_\varepsilon)\| \leqslant 2d_1.$$

Moreover, there exists $\eta\in H^1(\mathbb{R}^N)$ such that $\eta_\varepsilon \rightharpoonup \eta$ weakly, up to a subsequence, in $H^1(\mathbb{R}^N)$ as $\varepsilon\to 0$.

Now we claim that $\eta_\varepsilon\to\eta$ strongly in $H^1(\mathbb{R}^N)$. In fact, suppose that there exists $x_\varepsilon\in\mathbb{R}^N$ with $\lim_{\varepsilon\to 0}|x_\varepsilon| = \infty$ such that, for some $R>0$,

$$\limsup_{\varepsilon\to 0}\int_{B(x_\varepsilon,R)}(\eta_\varepsilon)^2\,\mathrm{d}x > 0.$$

We may assume that $\eta_\varepsilon(\cdot + x_\varepsilon)$ converges weakly to $\eta'\in H^1(\mathbb{R}^N)\setminus\{0\}$. Then, it is easy to see that

$$\Delta\eta' - m\eta' + f(\eta') = 0, \quad \eta' > 0 \quad \text{in } \mathbb{R}^N.$$

Then, from the Pohozaev identity we see that

$$\Gamma(\eta') = \frac{1}{N}\|\nabla\eta'\|^2_{L^2} \geqslant E_m.$$

For large $R>0$, it holds that

$$\limsup_{\varepsilon\to 0}\int_{B(x_\varepsilon,R)}|\nabla\eta_\varepsilon|^2\,\mathrm{d}y \geqslant \frac{1}{2}\int_{\mathbb{R}^N}|\nabla\eta'|^2\,\mathrm{d}y = \tfrac{1}{2}N\Gamma(\eta') \geqslant \tfrac{1}{2}NE_m. \tag{3.39}$$

We take $d_1>0$ satisfying $d_1 < \frac{1}{4}\sqrt{NE_m/2}$. Then, we get a contradiction since $\lim_{\varepsilon\to 0}|x_\varepsilon| = \infty$ and $\|\eta_\varepsilon - z\|\leqslant 2d_1$. Thus, we get that

$$\limsup_{|y|\to\infty}\int_{B(y,R)}(\eta_\varepsilon)^{p+1}\,\mathrm{d}x = \limsup_{|y|\to\infty}\int_{B(y,R)}(\eta_\varepsilon)^2\,\mathrm{d}x = 0$$

uniformly for small $\varepsilon > 0$. Applying [29, lemma 1.1] for $N \geqslant 3$, [12, lemma 1] for $N = 2$ and [12, remark 1(i)] for $N = 1$, we see that

$$\lim_{R\to\infty}\left(\int_{|x|\geqslant R} F(\eta_\varepsilon)\,\mathrm{d}x\right) = 0$$

uniformly for small $\varepsilon > 0$. Then, since

$$\lim_{\varepsilon\to 0}\int_{B(0,R)} F(\eta_\varepsilon)\,\mathrm{d}x = \int_{B(0,R)} F(\eta)\,\mathrm{d}x,$$

we get that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} F(\eta_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}^N} F(\eta)\,\mathrm{d}x. \tag{3.40}$$

From the weak convergence of η_ε to η in $H^1(\mathbb{R}^N)$, (3.37), (3.40) and (3.38) it follows that $E_m \geqslant \Gamma(\eta)$, $\Gamma'(\eta) = 0$. From the maximum principle, it also follows that $\eta(x) > 0$ for any $x \in \mathbb{R}^N$. Thus, we conclude that $\Gamma(\eta) = E_m$ and $\eta \in X$. Then, from (3.37), we get that

$$\begin{aligned} E_m &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\eta|^2 + m\eta^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(\eta)\,\mathrm{d}x \\ &\geqslant \limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla\eta_\varepsilon|^2 + V_\varepsilon\eta_\varepsilon^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(\eta_\varepsilon)\,\mathrm{d}x\right) \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\eta|^2 + m\eta^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(\eta)\,\mathrm{d}x \\ &= E_m. \end{aligned} \tag{3.41}$$

From (3.40) and (3.41), we get that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} |\nabla\eta_\varepsilon|^2 + m\eta_\varepsilon^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla\eta|^2 + m\eta^2\,\mathrm{d}x.$$

This proves the strong convergence of η_ε to $\eta \in X$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$. This contradicts that $\eta_\varepsilon \in X^{d_1} \setminus X^{d_2}$ and completes the proof. □

Now we can take a sufficiently small $d \in (0, G)$ such that, for $0 < \|u\| \leqslant 3d$,

$$\Gamma(u) > 0, \qquad \Gamma'(u)(u) > 0, \qquad \Gamma_\varepsilon(u) > 0, \qquad \Gamma_\varepsilon'(u)(u) > 0, \tag{3.42}$$

and that, for some $\omega > 0$ and $\varepsilon_0 > 0$,

$$\|\Gamma_\varepsilon'(u)\| \geqslant w \quad \text{if } u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X^d \setminus X^{d/2}) \text{ and } 0 < \varepsilon \leqslant \varepsilon_0. \tag{3.43}$$

Then, proposition 3.1 implies the following proposition.

PROPOSITION 3.4. *There exists $\alpha > 0$ such that, for sufficiently small $\varepsilon > 0$, $\Gamma_\varepsilon(\zeta(t)) \geqslant C_\varepsilon - \alpha$ with $t \in (0, t_0)$ implies that $\zeta(t) \in X^{d/2}$.*

Now for small $\varepsilon > 0$, we get a sequence $\{u_n\}_n \subset X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ with

$$\lim_{n\to\infty} \Gamma_\varepsilon'(u_n) = 0.$$

PROPOSITION 3.5. *For sufficiently small, fixed $\varepsilon > 0$ there exists a sequence*

$$\{u_n\}_{n=1}^{\infty} \subset X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$$

such that $\Gamma_\varepsilon'(u_n) \to 0$ as $n \to \infty$.

Proof. Since we have (3.43) and proposition 3.4, we can prove the above proposition following the same procedure as with the proof of [10, proposition 7], which we sketch for the reader's convenience. Suppose that proposition 3.5 does not hold for sufficiently small $\varepsilon > 0$. Then, there exists $a(\varepsilon) > 0$ such that $\|\Gamma_\varepsilon'\| \geqslant a(\varepsilon)$ on $X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$. Now there exists a pseudo-gradient vector field Q_ε on a neighbourhood Z_ε of $X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ for Γ_ε (see [32]). Let χ_ε be a Lipschitz continuous function on $H^1(\mathbb{R}^N)$ such that $0 \leqslant \chi_\varepsilon \leqslant 1$, $\chi_\varepsilon \equiv 1$ on $X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ and $\chi_\varepsilon \equiv 0$ on $H^1(\mathbb{R}^N) \setminus Z_\varepsilon$. Also, let ξ_ε be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \leqslant \xi_\varepsilon \leqslant 1$, $\xi_\varepsilon(a) \equiv 1$ if $|C_\varepsilon - a| \leqslant \alpha/2$, and $\xi_\varepsilon(a) \equiv 0$ if $|C_\varepsilon - a| \geqslant \alpha$. Then, there exists a global solution $\Lambda_\varepsilon \colon H^1(\mathbb{R}^N) \times \mathbb{R} \to H^1(\mathbb{R}^N)$ of the initial-value problem

$$\frac{\partial \Lambda_\varepsilon(u,\tau)}{\partial \tau} = -\chi_\varepsilon(\Lambda_\varepsilon(u,\tau))\xi_\varepsilon(\Gamma_\varepsilon(\Lambda_\varepsilon(u,\tau)))Q_\varepsilon(\Lambda_\varepsilon(u,\tau)),$$
$$\Lambda_\varepsilon(u,0) = u.$$

Recall that $\lim_{\varepsilon\to 0} C_\varepsilon = \lim_{\varepsilon \to 0} D_\varepsilon = E_m$. By a deformation argument using propositions 3.3 and 3.4, we get some large $\tau_\varepsilon > 0$ such that

$$\Gamma_\varepsilon(\Lambda_\varepsilon(\zeta(st_0),\tau_\varepsilon)) < E_m - \alpha/4, \quad s \in [0,1].$$

Note that $\tilde{\gamma}_\varepsilon(s) = \Lambda_\varepsilon(\zeta(st_0),\tau_\varepsilon) \in \Phi$ and $\Gamma_\varepsilon(\tilde{\gamma}_\varepsilon(s)) < E_m - \alpha/4$ for all $s \in [0,1]$. This contradicts proposition 3.2. □

The existence of a sequence $\{u_n\}_n$ in $X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ with $\lim_{n\to\infty} \Gamma_\varepsilon'(u_n) = 0$ implies the following existence result of a solution of (1.9).

PROPOSITION 3.6. *For sufficiently small $\varepsilon > 0$, Γ_ε has a critical point*

$$u_\varepsilon \in X^d \cap \Gamma_\varepsilon^{D_\varepsilon}.$$

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be the sequence as given by proposition 3.5 for sufficiently small $\varepsilon > 0$. Now we write $u_n = v_n(\cdot - a_n) + w_n$ with $v_n \in S_m$, $a_n \in \mathbb{R}^N$, $\|w_n\| \leqslant d$ and denote $\tau_n = u_n(\cdot + a_n)$. If $\{a_n\}_n$ is bounded, we can prove the claim by the proof of [10, proposition 8]. Now, we show the boundedness of $\{a_n\}_n$.

On the contrary, suppose that $\liminf_{n\to\infty} |a_n| = \infty$. Since S_m is compact, we may assume that v_n converges to some v in $H^1(\mathbb{R}^N)$. Then, the function v satisfies $\Delta v - mv + f(v) = 0$ and $v > 0$. We may assume that w_n converges weakly to some w in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Then, we see that $\Delta w - V_\varepsilon w + f(w) = 0$ in $\mathbb{R}^N$. From (3.42), we see that $w = 0$. This implies that, for each $R > 0$,

$$\lim_{n\to\infty} \int_{B(0,R)} (w_n)^2 \, dx = 0.$$

Note that, for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\Gamma_\varepsilon'(u_n)(\phi) = \Gamma'(u_n)(\phi) + \int_{\mathbb{R}^N} (V_\varepsilon - m)(v_n(\cdot - a_n) + w_n)\phi \, dx,$$

and that, for each $R > 0$,

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(V_\varepsilon - m)(v_n(\cdot - a_n) + w_n)\phi\,\mathrm{d}x\right| &\leqslant \|V - m\|_{L^\infty(\mathbb{R}^N)}\left(\int_{B(0,R)}(v_n(\cdot - a_n) + w_n)^2\,\mathrm{d}x\right)^{1/2}\|\phi\| \\ &\quad + \|V_\varepsilon - m\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))}\|v_n(\cdot - a_n) + w_n\|\|\phi\|\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(V_\varepsilon - m)(v_n(\cdot - a_n) + w_n)^2\,\mathrm{d}x\right| &\leqslant \|V - m\|_{L^\infty(\mathbb{R}^N)}\int_{B(0,R)}(v_n(\cdot - a_n) + w_n)^2\,\mathrm{d}x \\ &\quad + \|V_\varepsilon - m\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))}\|v_n(\cdot - a_n) + w_n\|^2.\end{aligned}$$

This implies that $\lim_{n\to\infty}\Gamma'(u_n) = 0$ and $\lim_{n\to\infty}\Gamma(u_n) \leqslant D_\varepsilon$. Then, by the same argument as that in the proof of the strong convergence of τ_ε to τ in proposition 3.3, it follows that τ_n converges to some $\tau \in H^1(\mathbb{R}^N)\setminus\{0\}$, satisfying $\Gamma'(\tau) = 0$ and $\Gamma(\tau) \leqslant D_\varepsilon$. Since $D_\varepsilon < E_m$ for small $\varepsilon > 0$, this contradicts that E_m is the least energy level for all non-trivial critical points of Γ.

Thus, we get the boundedness of the sequence $\{a_n\}_n$. This completes the proof. □

We see from proposition 3.6 that there exist $d > 0$ and $\varepsilon_0 > 0$ such that Γ_ε has a critical point $u_\varepsilon \in X^d \cap \Gamma_\varepsilon^{D_\varepsilon}$, $0 < \varepsilon \leqslant \varepsilon_0$. Let $x_\varepsilon \in \mathbb{R}^N$ be a maximum point of u_ε. Then we get the following proposition.

Proposition 3.7. *For sufficiently small $\varepsilon > 0$, $u_\varepsilon > 0$ in $\mathbb{R}^N$, and there exist some constants $c, C > 0$, independent of small $\varepsilon > 0$, such that $u_\varepsilon(x) + |\nabla u_\varepsilon(x)| \leqslant C\exp(-c|x - x_\varepsilon|)$ for $x \in \mathbb{R}^N$.*

Proof. Since $\lim_{|x|\to\infty} V(x) = m > 0$, there exists $R > 0$ such that $V(x) \geqslant \frac{1}{2}m$ for $|x| \geqslant R$. Denote $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$. Since u_ε satisfies $\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0$ and $f(s) = 0$ for $s \leqslant 0$, we see that

$$\int_{\mathbb{R}^N}|\nabla u_\varepsilon^-|^2 + V_\varepsilon(u_\varepsilon^-)^2\,\mathrm{d}x = 0.$$

By Sobolev's inequality and Hölder's inequality, there exists some $C > 0$ such that

$$\begin{aligned}0 &= \int_{\mathbb{R}^N}|\nabla u_\varepsilon^-|^2 + V_\varepsilon|u_\varepsilon^-|^2\,\mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^N}|\nabla u_\varepsilon^-|^2\,\mathrm{d}x + \int_{|x|\leqslant\varepsilon R}V_\varepsilon|u_\varepsilon^-|^2\,\mathrm{d}x + \frac{m}{2}\int_{|x|\geqslant\varepsilon R}|u_\varepsilon^-|^2\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u_\varepsilon^-\|^2 - \left(\max_{x\in\mathbb{R}^N}|V| + \frac{m}{2}\right)\int_{|x|\leqslant\varepsilon R}|u_\varepsilon^-|^2\,\mathrm{d}x\end{aligned}$$

$$\geqslant \begin{cases} \frac{1}{2}\|u_\varepsilon^-\|^2 - \varepsilon^2 CR^2 \|u_\varepsilon^-\|_{L^{2^*}}^2 & \text{for } N \geqslant 3, \\ \frac{1}{2}\|u_\varepsilon^-\|^2 - \varepsilon CR \|u_\varepsilon^-\|_{L^4}^2 & \text{for } N = 2, \\ \frac{1}{2}\|u_\varepsilon^-\|^2 - \varepsilon CR \|u_\varepsilon^-\|_{L^\infty}^2 & \text{for } N = 1, \end{cases}$$

$$\geqslant \begin{cases} (\frac{1}{2} - \varepsilon^2 CR^2)\|u_\varepsilon^-\|^2 & \text{for } N \geqslant 3, \\ (\frac{1}{2} - \varepsilon CR)\|u_\varepsilon^-\|^2 & \text{for } N = 1, 2. \end{cases} \tag{3.44}$$

Now we get $u_\varepsilon^- \equiv 0$ in $\mathbb{R}^N$, $u_\varepsilon \geqslant 0$ for sufficiently small $\varepsilon > 0$. Applying the strong maximum principle (see [30]) to the following equation:

$$\Delta u_\varepsilon - \left(V_\varepsilon u_\varepsilon - \frac{f(u_\varepsilon)}{u_\varepsilon}\right)^+ u_\varepsilon = \left(V_\varepsilon u_\varepsilon - \frac{f(u_\varepsilon)}{u_\varepsilon}\right)^- u_\varepsilon \leqslant 0,$$

we get $u_\varepsilon > 0$ in $\mathbb{R}^N$.

Moreover, from elliptic estimates through the Moser iteration scheme [20], we deduce that $\{\|u_\varepsilon\|_{L^\infty}\}_\varepsilon$ is bounded. Since $\Gamma_\varepsilon(u_\varepsilon) \leqslant D_\varepsilon \to E_m$, we deduce from comparison principles that for some $C, c > 0$, independent of small $\varepsilon > 0$, $u_\varepsilon(x) + |\nabla u_\varepsilon(x)| \leqslant C \exp(-c|x - x_\varepsilon|)$ for all $x \in \mathbb{R}^N$. This completes the proof. □

Let x_ε be a maximum point of u_ε. Then, it follows from proposition 3.7 and the fact that $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon) \leqslant E_m$, that $u_\varepsilon(\cdot + x_\varepsilon)$ converges uniformly, up to a subsequence, in the C^1-sense to a function $\tilde{U} \in S_m$ as $\varepsilon \to 0$. To see the asymptotic behaviour of x_ε, we need to obtain the following lower energy estimation of u_ε.

PROPOSITION 3.8. *For $N \geqslant 2$,*

$$\Gamma_\varepsilon(u_\varepsilon) \geqslant E_m + \varepsilon^N \left(\frac{(\tilde{U}(x_\varepsilon))^2}{2} \int_{\mathbb{R}^N} (V(x) - m \right) \mathrm{d}x + o(1))$$

as $\varepsilon \to 0$. Moreover, for any $N \geqslant 1$, a maximum point x_ε of u_ε converges to 0 as ε goes to 0.

Proof. Taking a subsequence, if it is necessary, we may also assume that $u_\varepsilon(\cdot + x_\varepsilon)$ converges weakly to $\tilde{U} \in S_m$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$. Then, we see from the exponential decay in proposition 3.7 that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} F(u_\varepsilon(\cdot + x_\varepsilon))\, \mathrm{d}x = \int_{\mathbb{R}^N} F(\tilde{U})\, \mathrm{d}x. \tag{3.45}$$

Then, it follows that $\limsup_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon) \leqslant E_m$ and

$$\Gamma_\varepsilon(u_\varepsilon) = \Gamma(u_\varepsilon) + \frac{1}{2} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m)(u_\varepsilon(x))^2\, \mathrm{d}x = \Gamma(u_\varepsilon) + o(1).$$

Thus, it follows from the weak convergence of $u_\varepsilon(\cdot + x_\varepsilon)$ to $\tilde{U}$ in $H^1(\mathbb{R}^N)$ that

$$E_m \geqslant \liminf_{\varepsilon\to 0} \Gamma(u_\varepsilon(\cdot + x_\varepsilon)) \geqslant \Gamma(\tilde{U}) \geqslant E_m. \tag{3.46}$$

This implies that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\nabla u_\varepsilon(\cdot + x_\varepsilon)|^2 + m u_\varepsilon(\cdot + x_\varepsilon)^2\, \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla \tilde{U}|^2 + m\tilde{U}^2\, \mathrm{d}x.$$

This proves the strong convergence of $u_\varepsilon(\cdot + x_\varepsilon)$ to $\tilde U$ in $H^1(\mathbb{R}^N)$. Since we can write

$$\Gamma_\varepsilon(u_\varepsilon) = \Gamma(u_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)(u_\varepsilon)^2 \,\mathrm{d}x,$$

we estimate the two right-hand terms separately.

First, we estimate

$$\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)u_\varepsilon^2(x)\,\mathrm{d}x.$$

By the elliptic estimates for $\{u_\varepsilon\}$ (see [20]) and an imbedding $W^{2,q}_{\mathrm{loc}} \hookrightarrow C^1_{\mathrm{loc}}$ for large $q > 0$, we see that, for a given $k > 0$, there exists $r_k > 0$ such that if $|x| \leqslant r_k$, then $|u_\varepsilon^2(x) - u_\varepsilon^2(0)| < 1/k$ for uniformly small $\varepsilon > 0$. Then, we have the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)u_\varepsilon^2(x)\,\mathrm{d}x \\
&\quad = \varepsilon^N\biggl\{u_\varepsilon^2(0)\int_{\mathbb{R}^N}(V(x)-m)\,\mathrm{d}x + \int_{|x|\leqslant r_k/\varepsilon}(V(x)-m)(u_\varepsilon^2(\varepsilon x) - u_\varepsilon^2(0))\,\mathrm{d}x \\
&\qquad\qquad + \int_{|x|\geqslant r_k/\varepsilon}(V(x)-m)(u_\varepsilon^2(\varepsilon x) - u_\varepsilon^2(0))\,\mathrm{d}x\biggr\} \\
&\quad \geqslant \varepsilon^N\biggl\{u_\varepsilon^2(0)\int_{\mathbb{R}^N}(V(x)-m)\,\mathrm{d}x - \frac{\|V-m\|_{L^1}}{k} \\
&\qquad\qquad - 2\|u_\varepsilon\|^2_{L^\infty}\int_{|x|\geqslant r_k/\varepsilon}|V(x)-m|\,\mathrm{d}x\biggr\}.
\end{aligned}$$

Then, we get that, for small $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)u_\varepsilon^2(x)\,\mathrm{d}x \geqslant \varepsilon^N\biggl\{u_\varepsilon^2(0)\int_{\mathbb{R}^N}(V(x)-m)\,\mathrm{d}x + o(1)\biggr\}. \tag{3.47}$$

Since $u_\varepsilon(\cdot + x_\varepsilon)$ converges uniformly to $\tilde U \in S_m$, it follows from the radial symmetry of $\tilde U \in S_m$ that

$$\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)u_\varepsilon^2(x)\,\mathrm{d}x \geqslant \varepsilon^N\biggl\{\tilde U^2(x_\varepsilon)\int_{\mathbb{R}^N}(V(x)-m)\,\mathrm{d}x + o(1)\biggr\} \tag{3.48}$$

as $\varepsilon \to 0$.

Now we estimate $\Gamma(u_\varepsilon)$.

First, we consider a case $N \geqslant 3$. (Here we modify the argument in the proof of [8, proposition 3.5] for this problem.) Defining $u_\varepsilon^t(x) = u_\varepsilon(x/t)$, we get from the Pohozaev identity (2.2) that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\Gamma(u_\varepsilon^t) &= \lim_{\varepsilon\to 0}\biggl\{\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,\mathrm{d}x + t^N\int_{\mathbb{R}^N}\frac{mu_\varepsilon^2}{2} - F(u_\varepsilon)\,\mathrm{d}x\biggr\} \\
&= \biggl(\frac{t^{N-2}}{2} - \frac{(N-2)t^N}{2N}\biggr)\int_{\mathbb{R}^N}|\nabla\tilde U|^2\,\mathrm{d}x,
\end{aligned} \tag{3.49}$$

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}\Gamma(u_\varepsilon^t)}{\mathrm{d}t} = \lim_{\varepsilon\to 0}\left\{\frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,\mathrm{d}x + Nt^{N-1}\int_{\mathbb{R}^N}\frac{mu_\varepsilon^2}{2} - F(u_\varepsilon)\,\mathrm{d}x\right\}$$
$$= \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N}|\nabla\tilde{U}|^2\,\mathrm{d}x + Nt^{N-1}\int_{\mathbb{R}^N}\frac{m\tilde{U}^2}{2} - F(\tilde{U})\,\mathrm{d}x \quad (3.50)$$

and

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}^2\Gamma(u_\varepsilon^t)}{\mathrm{d}t^2} = \lim_{\varepsilon\to 0}\left\{\frac{(N-2)(N-3)}{2}t^{N-4}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,\mathrm{d}x + N(N-1)t^{N-2} \times \int_{\mathbb{R}^N}\frac{mu_\varepsilon^2}{2} - F(u_\varepsilon)\,\mathrm{d}x\right\}$$
$$= \left(\frac{(N-2)(N-3)}{2}t^{N-4} - \frac{(N-1)(N-2)}{2}t^{N-2}\right)\int_{\mathbb{R}^N}|\nabla\tilde{U}|^2\,\mathrm{d}x, \quad (3.51)$$

uniformly for $t \in (0, t_0)$. Note that

$$\left(\frac{(N-2)(N-3)}{2}t^{N-4} - \frac{(N-1)(N-2)}{2}t^{N-2}\right)_{t=1} < 0. \quad (3.52)$$

This implies that a function $Y_\varepsilon(t) = \Gamma(u_\varepsilon^t)$ has a maximum at $t_\varepsilon \in (0, t_0)$ such that $\lim_{\varepsilon\to 0} t_\varepsilon = 1$. Now we estimate $|t_\varepsilon - 1|$ for small $\varepsilon > 0$. Note that

$$\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0. \quad (3.53)$$

Multiplying both sides of (3.53) by $(x - x_\varepsilon)\cdot\nabla u_\varepsilon$, we get

$$(V_\varepsilon u_\varepsilon - mu_\varepsilon)(x - x_\varepsilon)\cdot\nabla u_\varepsilon$$
$$= (\Delta u_\varepsilon - mu_\varepsilon + f(u_\varepsilon))(x - x_\varepsilon)\cdot\nabla u_\varepsilon$$
$$= \operatorname{div}\left(\nabla u_\varepsilon((x - x_\varepsilon)\cdot\nabla u_\varepsilon) - (x - x_\varepsilon)\frac{|\nabla u_\varepsilon|^2}{2} + (x - x_\varepsilon)\left(-\frac{mu_\varepsilon^2}{2} + F(u_\varepsilon)\right)\right)$$
$$+ \frac{N-2}{2}|\nabla u_\varepsilon|^2 + N\left(\frac{mu_\varepsilon^2}{2} - F(u_\varepsilon)\right). \quad (3.54)$$

Integrating (3.54) over $\mathbb{R}^N$, we get from the exponential decay in proposition 3.7 that

$$O(\varepsilon^N) = \int_{\mathbb{R}^N}(V_\varepsilon u_\varepsilon - mu_\varepsilon)((x - x_\varepsilon)\cdot\nabla u_\varepsilon)\,\mathrm{d}x$$
$$= \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,\mathrm{d}x + N\int_{\mathbb{R}^N}\frac{mu_\varepsilon^2}{2} - F(u_\varepsilon)\,\mathrm{d}x \quad (3.55)$$

as $\varepsilon \to 0$. Then, we see that

$$\left.\frac{\mathrm{d}\Gamma(u_\varepsilon^t)}{\mathrm{d}t}\right|_{t=1} = \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,\mathrm{d}x + N\int_{\mathbb{R}^N}\frac{mu_\varepsilon^2}{2} - F(u_\varepsilon)\,\mathrm{d}x = O(\varepsilon^N) \quad (3.56)$$

as $\varepsilon \to 0$. By the mean-value theorem, there exists $\hat{t_\varepsilon} > 0$ between 1 and t_ε satisfying

$$0 = \left.\frac{\mathrm{d}\Gamma(u_\varepsilon^t)}{\mathrm{d}t}\right|_{t=t_\varepsilon} = \left.\frac{\mathrm{d}\Gamma(u_\varepsilon^t)}{\mathrm{d}t}\right|_{t=1} + (t_\varepsilon - 1)\left.\frac{\mathrm{d}^2\Gamma(u_\varepsilon^t)}{\mathrm{d}t^2}\right|_{t=\hat{t_\varepsilon}}. \quad (3.57)$$

Then, it follows from (3.51), (3.52) and (3.56) that $|t_\varepsilon - 1| = O(\varepsilon^N)$ as $\varepsilon \to 0$. Note that there exists $t'_\varepsilon > 0$ between 1 and t_ε satisfying

$$\Gamma(u_\varepsilon^{t_\varepsilon}) = \Gamma(u_\varepsilon) + (t_\varepsilon - 1)\frac{\mathrm{d}\Gamma(u_\varepsilon^t)}{\mathrm{d}t}\bigg|_{t=t'_\varepsilon}. \tag{3.58}$$

From

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}\Gamma(u_\varepsilon^t)}{\mathrm{d}t}\bigg|_{t=t'_\varepsilon} = 0,$$

it follows that

$$\Gamma(u_\varepsilon^{t_\varepsilon}) = \Gamma(u_\varepsilon) + o(\varepsilon^N) \quad \text{as } \varepsilon \to 0. \tag{3.59}$$

Note that $\Gamma(u_\varepsilon^0) = 0$ and $\Gamma(u_\varepsilon^{t_0}) < 0$ for small $\varepsilon > 0$. A result of [22] implies that $\Gamma(u_\varepsilon^{t_\varepsilon}) \geqslant E_m$. Thus, we get that for small $\varepsilon > 0$,

$$\Gamma(u_\varepsilon) \geqslant E_m + o(\varepsilon^N). \tag{3.60}$$

Then, combining (3.60) with (3.48), we get the required lower estimation for $N \geqslant 3$.

Second, we consider a case $N = 2$. We need to recall some notation and contents stated in the proof of proposition 3.1. Now we define $\tilde{g}_\varepsilon(\theta, s)\colon (0,\infty)\times(0,\infty) \to \mathbb{R}$ by

$$\tilde{g}_\varepsilon(\theta, s) = \Gamma(\theta u_\varepsilon(\cdot/s)) = \frac{\theta^2}{2}\|\nabla u_\varepsilon\|_{L^2}^2 - s^2\int_{\mathbb{R}^2} H(\theta u_\varepsilon)\,\mathrm{d}x, \tag{3.61}$$

where

$$H(t) \equiv \int_0^t h(s)\,\mathrm{d}s$$

and $h(s) \equiv -ms + f(s)$. Note that

$$\left.\begin{aligned} (\tilde{g}_\varepsilon)_\theta(\theta, s) &= \theta\|\nabla u_\varepsilon\|_{L^2}^2 - s^2\int_{\mathbb{R}^2} h(\theta u_\varepsilon)u_\varepsilon\,\mathrm{d}x,\\ (\tilde{g}_\varepsilon)_s(\theta, s) &= -2s\int_{\mathbb{R}^2} H(\theta u_\varepsilon)\,\mathrm{d}x,\\ \frac{\partial}{\partial\theta}\int_{\mathbb{R}^2} H(\theta u_\varepsilon)\,\mathrm{d}x &= \int_{\mathbb{R}^2} h(\theta u_\varepsilon)u_\varepsilon\,\mathrm{d}x. \end{aligned}\right\} \tag{3.62}$$

Using (3.10) and the strong convergence of $u_\varepsilon(\cdot + x_\varepsilon)$ to $\tilde{U}$ in $H^1(\mathbb{R}^N)$, there exist $\theta_1 \in (0,1)$ and $\theta_2 \in (1,2)$ such that

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\theta}\int_{\mathbb{R}^2} H(\theta u_\varepsilon)\,\mathrm{d}x = \frac{\partial}{\partial\theta}\int_{\mathbb{R}^2} H(\theta\tilde{U})\,\mathrm{d}x \geqslant \tfrac{1}{2}\|\nabla\tilde{U}\|_{L^2}^2 > 0 \quad \text{for } \theta \in [\theta_1, \theta_2]. \tag{3.63}$$

We also note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} H(\theta_1 u_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}^2} H(\theta_1\tilde{U})\,\mathrm{d}x < 0 \tag{3.64}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} H(\theta_2 u_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}^2} H(\theta_2\tilde{U})\,\mathrm{d}x > 0. \tag{3.65}$$

Then, there exists $\theta_\varepsilon \in (\theta_1, \theta_2)$ such that

$$\int_{\mathbb{R}^2} H(\theta u_\varepsilon)\,\mathrm{d}x \begin{cases} < 0 & \text{for } \theta \in [\theta_1, \theta_\varepsilon), \\ = 0 & \text{for } \theta = \theta_\varepsilon, \\ > 0 & \text{for } \theta \in (\theta_\varepsilon, \theta_2]. \end{cases} \tag{3.66}$$

We also have, from (3.62), that

$$(\tilde{g}_\varepsilon)_s(\theta, s) \quad \begin{cases} > 0 & \text{for } \theta \in [\theta_1, \theta_\varepsilon),\ s \in (0, \infty), \\ = 0 & \text{for } \theta = \theta_\varepsilon,\ s \in (0, \infty), \\ < 0 & \text{for } \theta \in (\theta_\varepsilon, \theta_2],\ s \in (0, \infty). \end{cases} \tag{3.67}$$

Note that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} H(u_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}^2} H(\tilde{U})\,\mathrm{d}x = 0.$$

Then from (3.66), we see that $\lim_{\varepsilon \to 0} \theta_\varepsilon = 1$. Note that

$$\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0. \tag{3.68}$$

Multiplying both sides of (3.68) by $(x - x_\varepsilon) \cdot \nabla u_\varepsilon$, we see that

$$\begin{aligned} &(V_\varepsilon u_\varepsilon - m u_\varepsilon)((x - x_\varepsilon) \cdot \nabla u_\varepsilon) \\ &\quad = (\Delta u_\varepsilon - m u_\varepsilon + f(u_\varepsilon))((x - x_\varepsilon) \cdot \nabla u_\varepsilon) \\ &\quad = \operatorname{div}\left(\nabla u_\varepsilon((x - x_\varepsilon) \cdot \nabla u_\varepsilon) - (x - x_\varepsilon)\frac{|\nabla u_\varepsilon|^2}{2} + (x - x_\varepsilon) H(u_\varepsilon)\right) - 2H(u_\varepsilon). \end{aligned}$$

Then, from the exponential decay of $u_\varepsilon(\cdot + x_\varepsilon)$ in proposition 3.7, we get that

$$\int_{\mathbb{R}^2} H(u_\varepsilon)\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^2} (V_\varepsilon u_\varepsilon - m u_\varepsilon)((x - x_\varepsilon) \cdot \nabla u_\varepsilon)\,\mathrm{d}x = O(\varepsilon^2) \tag{3.69}$$

as $\varepsilon \to 0$. By the mean-value theorem, there exists $\hat{\theta}_\varepsilon$ between 1 and θ_ε satisfying

$$0 = \int_{\mathbb{R}^2} H(\theta_\varepsilon u_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}^2} H(u_\varepsilon)\,\mathrm{d}x + (\theta_\varepsilon - 1)\frac{\partial}{\partial \theta}\int_{\mathbb{R}^2} H(\theta u_\varepsilon)\,\mathrm{d}x\bigg|_{\theta = \hat{\theta}_\varepsilon}.$$

Then from (3.63) and (3.69), we get that

$$|1 - \theta_\varepsilon| = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{3.70}$$

As in (3.8), there exists a small $s_0 > 0$ such that, for sufficiently small $\varepsilon > 0$, we see that

$$(\tilde{g}_\varepsilon)_\theta(\theta, s) = \theta\left(\|\nabla u_\varepsilon\|_{L^2}^2 - s^2 \int_{\mathbb{R}^2} \frac{h(\theta u_\varepsilon)}{\theta u_\varepsilon} u_\varepsilon^2\,\mathrm{d}x\right) > 0 \quad \text{for } s \in [0, s_0],\ \theta \in (0, 2]. \tag{3.71}$$

Let $\gamma_\varepsilon(t) = (\theta(t), s(t))\colon [0, \infty) \to \mathbb{R}^2$ be a piecewise linear curve joining

$$\left.\begin{aligned} &(0, s_0) \to (\theta_\varepsilon - \varepsilon^4, s_0) \to (\theta_\varepsilon - \varepsilon^4, 1) \to (1 + \varepsilon^4, 1) \to (1 + \varepsilon^4, \infty) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \theta_1 < \theta_\varepsilon \leqslant 1 < \theta_2, \\ &(0, s_0) \to (1 - \varepsilon^4, s_0) \to (1 - \varepsilon^4, 1) \to (\theta_\varepsilon + \varepsilon^4, 1) \to (\theta_\varepsilon + \varepsilon^4, \infty) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \theta_1 < 1 \leqslant \theta_\varepsilon < \theta_2, \end{aligned}\right\} \tag{3.72}$$

where each line segment in the image of γ_ε is parallel to one of the axes. We take $0 \equiv t_0 < t_1 < \cdots < t_4 \equiv \infty$ such that, for each $i = 0, \ldots, 4$, $\gamma_\varepsilon(t_i)$ is the end point of a linear segment of the piecewise linear curve γ_ε. Moreover, we see that the function $t \mapsto \Gamma(\theta(t)u_\varepsilon(x/s(t)))$ is strictly increasing on (t_0, t_1), (t_1, t_2) by (3.71), (3.67), respectively. We also see that the function is strictly decreasing on (t_3, t_4) by (3.67). Then, we get that $\tilde{g}_\varepsilon(\gamma_\varepsilon(0)) = 0$, $\lim_{t\to\infty} \tilde{g}_\varepsilon(\gamma_\varepsilon(t)) = -\infty$. From [22], we see that

$$\max_{t\in(0,\infty)} \tilde{g}_\varepsilon(\gamma_\varepsilon(t)) \geqslant E_m.$$

Moreover, there exists $t_\varepsilon > 0$ such that $\max_{t\in(0,\infty)} \tilde{g}_\varepsilon(\gamma_\varepsilon(t))$ is attained at $\gamma_\varepsilon(t_\varepsilon) = (\theta(t_\varepsilon), 1)$ satisfying $\theta(t_\varepsilon) \in [\theta_\varepsilon - \varepsilon^4, 1 + \varepsilon^4]$ if $\theta_1 < \theta_\varepsilon \leqslant 1 < \theta_2$, or $\theta(t_\varepsilon) \in [1 - \varepsilon^4, \theta_\varepsilon + \varepsilon^4]$ if $\theta_1 < 1 \leqslant \theta_\varepsilon < \theta_2$, respectively. By the mean-value theorem, there exists θ^*_ε between $\theta(t_\varepsilon)$ and 1 such that

$$\tilde{g}_\varepsilon(\theta(t_\varepsilon), 1) = \tilde{g}_\varepsilon(1, 1) + (\tilde{g}_\varepsilon)_\theta(\theta^*_\varepsilon, 1)(\theta(t_\varepsilon) - 1).$$

Now, using (3.70) and $\lim_{\varepsilon\to 0}(\tilde{g}_\varepsilon)_\theta(\theta^*_\varepsilon, 1) = 0$, we get that

$$\tilde{g}_\varepsilon(\theta(t_\varepsilon), 1) = \Gamma(u_\varepsilon) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Then, combining this with (3.48), we get the required lower estimation for $N = 2$. In proposition 3.1, we take $U \in S_m$ so that $U(0) = \max_{W\in S_m} W(0)$. Then, we see that $\tilde{U}(0) = U(0)$ and from the strict decreasing property of $\tilde{U}, U \in S_m$ that $\lim_{\varepsilon\to 0} x_\varepsilon = 0$.

Lastly, we consider a case $N = 1$. Since S_m consists of one element $U \in H^1(\mathbb{R})$ and, in addition, $U(0) = T$, where $T > 0$ is given in (F3), it follows that $\tilde{U} = U$. Now we denote $u'_\varepsilon = \mathrm{d}u_\varepsilon/\mathrm{d}x$. Multiplying both sides of $u''_\varepsilon - V_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0$ by u'_ε, we get

$$\begin{aligned}(V_\varepsilon u_\varepsilon - m u_\varepsilon)(u'_\varepsilon) &= (u''_\varepsilon - m u_\varepsilon + f(u_\varepsilon))(u'_\varepsilon)\\ &= (\tfrac{1}{2}|u'_\varepsilon|^2 - \tfrac{1}{2}m u_\varepsilon^2 + F(u_\varepsilon))'.\end{aligned}$$

Integrating both sides from $-\infty$ to $x \in \mathbb{R}$, we get

$$\int_{-\infty}^{x} (V_\varepsilon(y) - m)u_\varepsilon(y)u'_\varepsilon(y)\,\mathrm{d}y = \tfrac{1}{2}|u'_\varepsilon(x)|^2 - \tfrac{1}{2}m u_\varepsilon^2(x) + F(u_\varepsilon(x)). \tag{3.73}$$

Then, from the exponential decay property of $u_\varepsilon(\cdot + x_\varepsilon)$ and $|\nabla u_\varepsilon(\cdot + x_\varepsilon)|$ in proposition 3.7 and that $u'_\varepsilon(x_\varepsilon) = 0$, we deduce that

$$|\tfrac{1}{2}m u_\varepsilon^2(x_\varepsilon) - F(u_\varepsilon(x_\varepsilon))| = O(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{3.74}$$

Then, since $\lim_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) = T$ and $mT - f(T) < 0$, it follows that $|u_\varepsilon(x_\varepsilon) - T| = O(\varepsilon)$ as $\varepsilon \to 0$.

Now we define

$$\mu_\varepsilon = \min\biggl\{\int_{x_\varepsilon}^{\infty} \tfrac{1}{2}|\nabla u_\varepsilon|^2 + \tfrac{1}{2}m u_\varepsilon^2 - F(u_\varepsilon)\,\mathrm{d}x, \int_{-\infty}^{x_\varepsilon} \tfrac{1}{2}|\nabla u_\varepsilon|^2 + \tfrac{1}{2}m u_\varepsilon^2 - F(u_\varepsilon)\,\mathrm{d}x\biggr\}.$$

Then, it follows that $2\mu_\varepsilon \leqslant \Gamma(u_\varepsilon)$, and we may assume that

$$\mu_\varepsilon = \int_{x_\varepsilon}^{\infty} \tfrac{1}{2}|\nabla u_\varepsilon|^2 + \tfrac{1}{2}m u_\varepsilon^2 - F(u_\varepsilon)\,\mathrm{d}x.$$

As in the proof of proposition 3.1, we take $\rho > 0$ such that $x \in [-\rho, 0)$,

$$8x^6 + \tfrac{1}{2}m(x^4 + T)^2 - F(x^4 + T) < 0. \tag{3.75}$$

Then, we define $q_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ by

$$q_\varepsilon(x) = \begin{cases} u_\varepsilon(x + x_\varepsilon), & x \in [0, \infty), \\ x^4 + u_\varepsilon(x_\varepsilon), & x \in [-\rho, 0], \\ \rho^4 + u_\varepsilon(x_\varepsilon), & x \in (-\infty, -\rho] \end{cases} \tag{3.76}$$

and $\gamma_\varepsilon \colon (0, \infty) \to H^1(\mathbb{R})$ by

$$\gamma_\varepsilon(t)(x) = q_\varepsilon(|x| - \ln t) \quad \text{and} \quad \gamma_\varepsilon(0) = 0.$$

We see that $\gamma_\varepsilon \colon [0, \infty) \to H^1(\mathbb{R})$ is continuous. Define

$$Q_\varepsilon(y) = \begin{cases} |u'_\varepsilon(y + x_\varepsilon)|^2 + mu_\varepsilon^2(y + x_\varepsilon) - 2F(u_\varepsilon(y + x_\varepsilon)) & \text{for } y \geqslant 0, \\ |q'_\varepsilon(y)|^2 + mq_\varepsilon^2(y) - 2F(q_\varepsilon(y)) & \text{for } y \leqslant 0. \end{cases} \tag{3.77}$$

Then, we obtain that

$$\Gamma(\gamma_\varepsilon(t)) = 2\mu_\varepsilon + \int_{-\ln t}^{0} Q_\varepsilon(x)\,\mathrm{d}x \tag{3.78}$$

and that for $t \in (0, \infty) \setminus \{\mathrm{e}^\rho\}$,

$$\frac{\mathrm{d}\Gamma(\gamma_\varepsilon(t))}{\mathrm{d}t} = \frac{Q_\varepsilon(-\ln t)}{t}, \tag{3.79}$$

where

$$Q_\varepsilon(-\ln t) = \begin{cases} |u'_\varepsilon(-\ln t + x_\varepsilon)|^2 + mu_\varepsilon^2(-\ln t + x_\varepsilon) - 2F(u_\varepsilon(-\ln t + x_\varepsilon)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 < t \leqslant 1, \\ 16(-\ln t)^6 + m((-\ln t)^4 + u_\varepsilon(x_\varepsilon))^2 - 2F((-\ln t)^4 + u_\varepsilon(x_\varepsilon)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leqslant t < \mathrm{e}^\rho, \\ m(\rho^4 + u_\varepsilon(x_\varepsilon))^2 - 2F(\rho^4 + u_\varepsilon(x_\varepsilon)) \qquad\qquad \text{for } t > \mathrm{e}^\rho. \end{cases} \tag{3.80}$$

Thus, $\Gamma(\gamma_\varepsilon(t))$ is a C^1-function for $t \in (0, \mathrm{e}^\rho)$. From (3.75) and (F3), we get that

$$\lim_{\varepsilon \to 0} \frac{\mathrm{d}\Gamma(\gamma_\varepsilon(t))}{\mathrm{d}t} \begin{cases} > 0 & \text{for } 0 < t < 1, \\ < 0 & \text{for } 1 < t < \mathrm{e}^\rho. \end{cases} \tag{3.81}$$

Therefore, $\Gamma(\gamma_\varepsilon(t))$ has a maximum at t_ε such that $\lim_{\varepsilon \to 0} t_\varepsilon = 1$.

Suppose that there exists $\varepsilon_n \to 0$ such that $\lim_{n \to \infty} |x_{\varepsilon_n}| > 0$. For the sake of convenience, we write ε for ε_n. Then, $\lim_{\varepsilon \to 0} V_\varepsilon(y + x_\varepsilon) = m$ whenever $|y| \leqslant |\ln t_\varepsilon|$. Since $u''_\varepsilon = V_\varepsilon u_\varepsilon - f(u_\varepsilon)$ and $\lim_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) = T$, we see from (F3) that if $\varepsilon > 0$ is small, $u''(x_\varepsilon + x) < 0$ for $|x| \leqslant |\ln t_\varepsilon|$. Then, we see that if $|y| \leqslant |\ln t_\varepsilon|$ and $\varepsilon > 0$ is sufficiently small,

$$\frac{\mathrm{d}Q_\varepsilon(y)}{\mathrm{d}y} = \begin{cases} 2u'_\varepsilon(y+x_\varepsilon)\{2mu_\varepsilon(y+x_\varepsilon) - 2f(u_\varepsilon(y+x_\varepsilon)) \\ \qquad\qquad +(V_\varepsilon(y+x_\varepsilon)-m)u_\varepsilon(y+x_\varepsilon)\} \geqslant 0 & \text{for } y \geqslant 0, \\ 8y^3\{12y^2 + m(y^4+u_\varepsilon(x_\varepsilon)) - f(y^4+u_\varepsilon(x_\varepsilon))\} \geqslant 0 & \text{for } y \leqslant 0. \end{cases} \tag{3.82}$$

This implies that, for small $\varepsilon > 0$, $Q_\varepsilon(y)$ is C^1 and increasing on the set $|y| \leqslant |\ln t_\varepsilon|$. Since $Q_\varepsilon(-\ln t_\varepsilon) = 0$, it follows that $|Q_\varepsilon(0)| \geqslant |Q_\varepsilon(y)|$ for any y between 0 and $-\ln t_\varepsilon$. Then, since $Q_\varepsilon(0) = mu_\varepsilon^2(x_\varepsilon) - 2F(u_\varepsilon(x_\varepsilon))$, we see from (3.74) that

$$\begin{aligned} |\Gamma(\gamma_\varepsilon(t_\varepsilon)) - 2\mu_\varepsilon| &= \left|\int_{-\ln t_\varepsilon}^0 Q_\varepsilon(y)\,\mathrm{d}y\right| \\ &\leqslant |Q_\varepsilon(0)||\ln t_\varepsilon| \\ &= |mu_\varepsilon^2(x_\varepsilon) - 2F(u_\varepsilon(x_\varepsilon))||\ln t_\varepsilon| \\ &\leqslant c\varepsilon|\ln t_\varepsilon|, \end{aligned} \tag{3.83}$$

for some constant $c > 0$. Since $\Gamma(\gamma_\varepsilon(0)) = 0$ and $\lim_{t\to\infty}\Gamma(\gamma_\varepsilon(t)) = -\infty$, we see from the results in [23] that $\Gamma(\gamma_\varepsilon(t_\varepsilon)) \geqslant E_m$. Now we see from proposition 3.1, (3.83) and (3.48) that

$$\begin{aligned} E_m + \frac{\varepsilon}{2}\left(\tilde{U}^2(0)\int_{\mathbb{R}}(V(x)-m)\,\mathrm{d}x + o(1)\right) \\ &\geqslant D_\varepsilon \geqslant \Gamma_\varepsilon(u_\varepsilon) \\ &= \Gamma(u_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}}(V_\varepsilon(x)-m)u_\varepsilon^2(x)\,\mathrm{d}x \\ &\geqslant 2\mu_\varepsilon + \frac{1}{2}\int_{\mathbb{R}}(V_\varepsilon(x)-m)u_\varepsilon^2(x)\,\mathrm{d}x \\ &= \Gamma(\gamma_\varepsilon(t_\varepsilon)) + \frac{1}{2}\int_{\mathbb{R}}(V_\varepsilon(x)-m)u_\varepsilon^2(x)\,\mathrm{d}x + o(\varepsilon) \\ &\geqslant E_m + \frac{\varepsilon}{2}\left(\tilde{U}^2(x_\varepsilon)\int_{\mathbb{R}}(V(x)-m\right)\mathrm{d}x + o(1)) \end{aligned} \tag{3.84}$$

as $\varepsilon \to 0$. Since

$$\int_{\mathbb{R}}(V(x)-m)\,\mathrm{d}x < 0$$

and $\tilde{U}(0) = \sup_{x\in\mathbb{R}}\tilde{U}(x) > \tilde{U}(y)$ for any $|y| > 0$, we get that a maximum point x_ε of u_ε converges to 0 as ε goes to 0. □

We note that S_m is compact. In particular, for $N = 1$, S_m consists of one element. Thus, there exists a solution $U \in S_m$ satisfying $U(0) = \sup_{W\in S_m} W(0)$. Now, combining propositions 3.6, 3.7 and 3.8, we complete the proof of theorem 1.1.

4. An extension of the existence result in theorem 1.1

Recall the definition of ζ given in the proof of proposition 3.1. Then, we introduce the following condition.

(V3') There exist $\varepsilon_0 > 0$ and $\tilde{x_\varepsilon} \in \mathbb{R}^N$ for $\varepsilon \in (0, \varepsilon_0)$ such that

$$\max_{t\in[0,t_0]} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m)(\zeta(t)(x - \tilde{x_\varepsilon}))^2 \, \mathrm{d}x \leqslant 0 \quad \text{for all } 0 < \varepsilon \leqslant \varepsilon_0.$$

Proposition 2.2 states that (V3) implies (V3′). Now we have the following, more general, existence result.

THEOREM 4.1. *Assume that conditions (V1), (V2), (V3′), and (F1)–(F3) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution w_ε of (1.8) such that, for a maximum point x_ε of w_ε, a transformation $u_\varepsilon(x) \equiv w_\varepsilon((x + x_\varepsilon)/\varepsilon)$ converges (up to a subsequence) uniformly to a radially symmetric least energy solution of*

$$\Delta u - mu + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbb{R}^N). \tag{4.1}$$

Moreover, there exist constants $c, C > 0$, independent of small $\varepsilon > 0$, such that

$$u_\varepsilon(x) + |\nabla u_\varepsilon(x)| \leqslant C \exp(-c|x|), \quad x \in \mathbb{R}^N.$$

Before proving theorem 4.1, we explore some typical V satisfying condition (V3′).

PROPOSITION 4.2. *Suppose that the potential V satisfies conditions (V1) and (V2). Then, condition (V3′) holds when one of the following is satisfied.*

(i) *$V(x) \leqslant m$ for any $x \in \mathbb{R}^N$.*

(ii) *There exists $x_0 \in \mathbb{R}^N$ such that, for any $r \in (0, \infty)$,*

$$\int_{S^{N-1}} (V(rx + x_0) - m) \, \mathrm{d}\sigma(x) \leqslant 0,$$

where $\mathrm{d}\sigma$ is the standard volume element on the unit sphere S^{N-1}.

(iii) *When $N = 1$, it holds that $V - m \in L^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} (V(x) - m) \, \mathrm{d}x = 0,$$

$\tilde{V} - \tilde{m} \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} (\tilde{V}(x) - \tilde{m}) \, \mathrm{d}x \neq 0,$$

where

$$\tilde{V}(x) = \int_0^x (V(y) - L) \, \mathrm{d}y$$

and $\lim_{|x|\to\infty} \tilde{V}(x) = \tilde{m}$.

Proof. First note from the construction of ζ in proposition 3.1 that there exist $C, c > 0$, independent of $\varepsilon > 0$, satisfying $\zeta(t)(x) \leqslant C \exp(-c|x|)$ for $x \in \mathbb{R}^N$. Thus, $(V_\varepsilon - m)\zeta^2(t)(\cdot - \tilde{x_\varepsilon}) \in L^1(\mathbb{R}^N)$ for any $t \in (0, t_0)$.

(i) This case is obvious since $\zeta(t)(x) > 0$ for $t > 0$ and $x \in \mathbb{R}^N$.

(ii) Note that a function $\zeta(t)$ is radially symmetric for $t \in (0, t_0)$. Thus, we see that

$$\int_{\mathbb{R}^N} (V_\varepsilon(x) - m)\zeta^2(t)(x - \varepsilon x_0)\,\mathrm{d}x \leqslant 0$$

for any $t \in (0, t_0)$. This proves the claim with $\tilde{x_\varepsilon} = \varepsilon x_0$ in case (ii).

(iii) For $t \in (0, t_0)$, we denote $W(x) = \zeta(t)$. From the construction of ζ in proposition 3.1, we see that W is piecewise C^1, that for some $M > 0$, independent of $t \in (0, t_0)$, $\|W\|_{L^\infty} \leqslant M$, and that there exists $x_0 > 0$, independent of $t \in (0, t_0)$, satisfying $W'(x)x < 0$ for $|x| \geqslant x_0 - 1$. Moreover, we see that

$$\begin{aligned}
&\int_{\mathbb{R}} (V_\varepsilon(x) - m)W(x \pm x_0)\,\mathrm{d}x \\
&\quad = \varepsilon \int_{\mathbb{R}} (V(x) - m)W(\varepsilon x \pm x_0)\,\mathrm{d}x \\
&\quad = \varepsilon\left\{\tilde{V}(x)W(\varepsilon x \pm x_0)|_{-\infty}^{\infty} - \int_{\mathbb{R}} \tilde{V}(x)\frac{\mathrm{d}W(\varepsilon x \pm x_0)}{\mathrm{d}x}\,\mathrm{d}x\right\} \\
&\quad = -\varepsilon^2 \int_{\mathbb{R}} \tilde{V}(x)W'(\varepsilon x \pm x_0)\,\mathrm{d}x \\
&\quad = -\varepsilon^2\left\{\int_{\mathbb{R}} (\tilde{V}(x) - m_1)W'(\varepsilon x \pm x_0)\,\mathrm{d}x + m_1\int_{\mathbb{R}} W'(\varepsilon x \pm x_0)\,\mathrm{d}x\right\} \\
&\quad = -\varepsilon^2 \int_{\mathbb{R}} (\tilde{V}(x) - m_1)W'(\varepsilon x \pm x_0)\,\mathrm{d}x \\
&\quad = -\varepsilon^2\left\{\int_{\mathbb{R}} (\tilde{V}(x) - m_1)(W'(\varepsilon x \pm x_0) - W'(\pm x_0))\,\mathrm{d}x\right. \\
&\qquad\qquad \left. + W'(\pm x_0)\int_{\mathbb{R}} (\tilde{V}(x) - m_1)\,\mathrm{d}x\right\}.
\end{aligned} \tag{4.2}$$

As in the proof of proposition 2.2, we see that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} (\tilde{V}(x) - m_1)(W'(\varepsilon x \pm x_0) - W'(\pm x_0))\,\mathrm{d}x = 0.$$

Take one of the points $\pm x_0$ such that

$$W'(\pm x_0)\int_{\mathbb{R}} (\tilde{V}(x) - m_1)\,\mathrm{d}x > 0.$$

Then, it follows that, for small $\varepsilon > 0$,

$$\int_{\mathbb{R}} (V_\varepsilon(x) - m)W(x + x_0)\,\mathrm{d}x < 0 \quad \text{or} \quad \int_{\mathbb{R}} (V_\varepsilon(x) - m)W(x - x_0)\,\mathrm{d}x < 0.$$

This proves the claim.

□

Condition (iii) in the above proposition was introduced by Ambrosetti and Badiale in [2], where they proved that if (iii) holds, then (1.4) has two distinct families of solutions bifurcating from the trivial solutions for small $\varepsilon > 0$ when $f(t) = t^p$, $p \in (1, 5)$.

Proof of theorem 4.1. Note that

$$D_\varepsilon = \max_{t \in [0,t_0]} \Gamma_\varepsilon(\zeta(t)(\cdot - \tilde{x_\varepsilon})) \leqslant E_m.$$

Now we consider the two following cases.

CASE 1. If there exists a critical point u_ε of Γ_ε on the path $\zeta(t)(\cdot - \tilde{x_\varepsilon}) \in X^d$, we get the decay property of u_ε in a similar way as for the proof of proposition 3.7.

CASE 2. Suppose that there exist no critical points of Γ_ε on a set

$$\{\zeta(t)(\cdot - \tilde{x_\varepsilon}) \mid t \in [0, t_0)\} \cap X^d.$$

By considering a pseudo-gradient vector field on a neighbourhood Z_ε of

$$\{\zeta(t)(\cdot - \tilde{x_\varepsilon}) \mid t \in [0, t_0]\} \cap X^d \quad \text{for } \Gamma_\varepsilon,$$

we can deform a part of the curve $\{\zeta(t)(\cdot - \tilde{x_\varepsilon}) \mid t \in [0, t_0]\}$ inside X^d into a continuous curve $\zeta_\varepsilon \colon [0, t_0] \to H^1(\mathbb{R}^N)$ such that

$$\Gamma_\varepsilon(\zeta_\varepsilon(t)) < E_m \quad \text{for any } t \in [0, t_0].$$

Then, setting $D'_\varepsilon = \max_{t \in [0,t_0]} \Gamma_\varepsilon(\zeta_\varepsilon(t))$, we see that $D'_\varepsilon < E_m$ for sufficiently small $\varepsilon > 0$.

Now we note that, in the proofs of propositions 3.2 and 3.3, the same arguments hold with (V1) and (V2) but not with (V3). Then, as for the proof of proposition 3.5, we obtain a sequence $\{u_n\}_n$ in $X^d \cap \Gamma_\varepsilon^{D'_\varepsilon}$ for fixed, sufficiently small $\varepsilon > 0$ such that $\lim_{n \to \infty} \Gamma'_\varepsilon(u_n) = 0$. To get a strong convergence of $\{u_n\}_n$ to some u_ε in $H^1(\mathbb{R}^N)$, as in proposition 3.6, we only need a property $\limsup_{n \to \infty} \Gamma_\varepsilon(u_n) < E_m$, which follows from $D'_\varepsilon < E_m$ and $\{u_n\}_n \subset X^d \cap \Gamma_\varepsilon^{D'_\varepsilon}$. Finally, we get the decay property of u_ε in a similar way as in the proof of proposition 3.7. This proves the claim. □

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References

1 R. A. Adams. *Sobolev spaces* (Academic Press, 1975).

2 A. Ambrosetti and M. Badiale. Variational perturbative methods and bifurcation of bound states from the essential spectrum. *Proc. R. Soc. Edinb.* A **128** (1998), 1131–1161.

3 A. Ambrosetti, M. Badiale and S. Cingolani. Semiclassical states of nonlinear Schrödinger equations. *Arch. Ration. Mech. Analysis* **140** (1997), 285–300.

4 M. Badiale. A note on bifurcation from the essential spectrum. *Adv. Nonlin. Studies* **3** (2003), 261–272.

5 M. Badiale and A. Pomponio. Bifurcation results for semilinear elliptic problems in $\mathbb{R}^N$. *Proc. R. Soc. Edinb.* A **134** (2004), 11–32.

6 V. Benci and D. Fortunato. Does bifurcation from the essential spectrum occur? *Commun. PDEs* **6** (1981), 249–272.

7 H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. I. *Arch. Ration. Mech. Analysis* **82** (1983), 313–346.

8 J. Byeon. Singularly perturbed nonlinear Neumann problems with a general nonlinearity. *J. Diff. Eqns* **244** (2008), 2473–2497.

9 J. Byeon. Singularly perturbed nonlinear Dirichlet problems with a general nonlinearity. *Trans. Am. Math. Soc.* **362** (2010), 1981–2001.

10 J. Byeon and L. Jeanjean. Standing waves for nonlinear Schrödinger equations with a general nonlinearity. *Arch. Ration. Mech. Analysis* **185** (2007), 185–200.

11 J. Byeon and L. Jeanjean. Multi-peak standing waves for nonlinear Schrödinger equations with general nonlinearlity. *Discrete Contin. Dynam. Syst.* **19** (2007), 255–269.

12 J. Byeon, L. Jeanjean and K. Tanaka. Standing waves for nonlinear Schrödinger equations with a general nonlinearity: one and two dimensional cases. *Commun. PDEs* **33** (2008), 1113–1136.

13 J. Byeon, L. Jeanjean and M. Maris. Symmetry and monotonicity of least energy solutions. *Calc. Var. PDEs* **36** (2009), 481–492.

14 M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Funct. Analysis* **8** (1971), 321–340.

15 E. N. Dancer and S. Yan. On the existence of multipeak solutions for nonlinear field equations on $\mathbb{R}^N$. *Discrete Contin. Dynam. Syst.* **6** (2000), 39–50.

16 M. Del Pino and P. L. Felmer. Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. PDEs* **4** (1996), 121–137.

17 M. Del Pino and P. L. Felmer. Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. *Indiana Univ. Math. J.* **48** (1999), 883–898.

18 M. Del Pino and P. L. Felmer. Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. *Math. Ann.* **324** (2002), 1–32.

19 A. Floer and A. Weinstein. Non spreading wave packets for the cubic Schrödinger equations with a bounded potential. *J. Funct. Analysis* **69** (1986), 397–408.

20 D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, 2nd edn, Grundlehren der Mathematischen Wissenschaften, vol. 224 (Springer, 1983).

21 C. Gui. Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method. *Commun. PDEs* **21** (1996), 787–820.

22 L. Jeanjean and K. Tanaka. A remark on least energy solutions in R^N. *Proc. Am. Math. Soc.* **131** (2003), 2399–2408.

23 L. Jeanjean and K. Tanaka. A note on a mountain pass characterization of least energy solutions. *Ad. Nonlin. Studies* **3** (2003), 461–471.

24 T. Kupper and C. A. Stuart. Bifurcation into gaps in the essential spectrum. *J. Reine Angew. Math.* **409** (1990), 1–34.

25 Y. Y. Li. On a singular perturbed elliptic equation. *Adv. Diff. Eqns* **2** (1997), 955–980.

26 W. M. Ni and I. Takagi. On the shape of least-energy solutions to a semilinear Neumann problem. *Commun. Pure Appl. Math.* **44** (1991), 819–851.

27 W. M. Ni and I. Takagi. Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.* **70** (1993), 247–281.

28 W. M. Ni and J. Wei. On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. *Commun. Pure Appl. Math.* **48** (1995), 731–768.

29 P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Annales Inst. H. Poincaré* **1** (1984), 223–283.

30 P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, Regional Conference Series in Mathematics, vol. 65 (Providence, RI: American Mathematical Society, 1986).

31 P. H. Rabinowitz. On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* **43** (1992), 270–291.

32 M. Struwe. *Variational methods: application to nonlinear partial differential equations and Hamiltonian systems* (Springer, 1990).

33 C. A. Stuart. Bifurcation for Dirichlet problems without eigenvalues. *Proc. Lond. Math. Soc.* **45** (1982), 169–192.

34 C. A. Stuart. Bifurcation in $L^p(R^N)$ for a semilinear elliptic equation. *Proc. Lond. Math. Soc.* **57** (1988), 511–541.

35 C. A. Stuart. Bifurcation of homoclinic orbits and bifurcation from the essential spectrum. *SIAM J. Math. Analysis* **20** (1989), 1145–1171.

36 C. A. Stuart. Bifurcation from the essential spectrum. In *Topological nonlinear analysis, II (Frascati, 1995)*, Progress in Nonlinear Differential Equations and Their Applications, vol. 27, pp. 397–443 (Boston, MA: Birkhäuser, 1997).

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